Naturally graded quasi-filiform Leibniz algebras ☆

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A B S T R A C T

The classification of naturally graded quasi-filiform Lie algebras is known; they have the characteristic sequence $(n-2,1,1)$ where n is the dimension of the algebra. In the present paper we deal with naturally graded quasi-filiform non-Lie–Leibniz algebras which are described by the characteristic sequence $C(\mathcal{L})=(n-2,1,1)$ or $C(\mathcal{L})=(n-2,2)$. The first case has been studied in [Camacho, L.M., Gómez, J.R., González, A.J., Omirov, B.A., 2006. Naturally graded 2-filiform Leibniz Algebra and its applications, preprint, MA1-04-XI06] and now, we complete the classification of naturally graded quasi-filiform Leibniz algebras. For this purpose we use the software *Mathematica* (the program used is explained in the last section).

1. Introduction

The knowledge of naturally graded algebras for a family of non-associative algebras is relevant because it contributes to obtaining information about the structure of the family, its irreducible components and some cohomological problems.

Leibniz algebras appear as a generalization of Lie algebras (Loday, 1993), so it is expected that the naturally graded algebras will play a similar important role in the study of the Leibniz algebras as in the Lie algebra case. The cases of 0-filiform and 1-filiform Leibniz algebras were studied in Ayupov and Omirov (2001) and naturally graded p-filiform Leibniz algebras in Camacho et al. (2006) and Gómez and Jiménez-Merchán (2002). Let $\mathcal{L}$ be a graded n-dimensional quasi-filiform non-Lie–Leibniz algebra,

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then it is clear that either $C(\mathcal{L}) = (n-2, 1, 1)$ or $C(\mathcal{L}) = (n-2, 2)$. The first case (the 2-filiform case) has been studied in Camacho et al. (2006). In this work, we consider the second case, that is, the algebras with $C(\mathcal{L}) = (n-2, 2)$.

In the theory of nilpotent Lie algebras powerful techniques have been generated in naturally graded Lie algebras (for example, in cohomology description and structural properties, see Goze and Khakimdjanov (1996)) which have been applied to non-graded algebras (Cabezas and Pastor, 2005; Gómez and Jiménez-Merchán, 2002). Since finding naturally graded Leibniz algebras is always possible for nilpotent algebras, these techniques are always applicable. They are more effective when the number of non-zero subspaces of the gradation is big enough. The works (Gómez and Jiménez-Merchán, 2002; Vergne, 1970) deal with naturally graded filiform and quasi-filiform Lie algebras up to isomorphism. The main aim of this paper is to extend the classification of naturally graded quasi-filiform Lie algebras to Leibniz algebras.

Definition 1. An algebra $\mathcal{L}$ over a field F is called the Leibniz algebra if it verifies the Leibniz identity: $[x, [y, z]] = [[x, y], z] - [[x, z], y]$ for any elements $x, y, z \in \mathcal{L}$ and where $[,]$ is the multiplication in $\mathcal{L}$.

Note that if in $\mathcal{L}$ the identity $[x, x] = 0$ holds, then the Leibniz identity coincides with the Jacobi identity. Thus, Leibniz algebras are a generalization of Lie algebras.

For a given Leibniz algebra $\mathcal{L}$ we define the following sequence: $\mathcal{L}^1 = \mathcal{L}$ and $\mathcal{L}^{k+1} = [\mathcal{L}^k, \mathcal{L}^1]$.

In this paper, we will work over $\mathbb{C}$. Let $\mathcal{L}$ be a nilpotent Leibniz algebra for which the index of nilpotency is k+1. Let us define the natural gradation of algebra $\mathcal{L}$ as follows, $\mathcal{L}_i = \mathcal{L}^i/\mathcal{L}^{i+1}$, then $\mathcal{L} \approx \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_k$. Using $[\mathcal{L}^i, \mathcal{L}^j] \subseteq \mathcal{L}^{i+j}$, it is easy to establish that $[\mathcal{L}_i, \mathcal{L}_j] \subseteq \mathcal{L}_{i+j}$. So, we have the gradation. The above constructed gradation is called *natural gradation.*

Let x be a nilpotent element of the set $\mathcal{L} \setminus \mathcal{L}^2$. For the nilpotent operator of right multiplication R_x we define a decreasing sequence $C(x) = (n_1, n_2, \ldots, n_k)$, which consists of the dimensions of Jordan blocks of the operator R_x. In the set of such sequences we consider the lexicographic order, that is, $C(x) = (n_1, n_2, \ldots, n_k) \le C(y) = (m_1, m_2, \ldots, m_s) \Leftrightarrow$ there exists $i \in \mathbf{N}$ such that $n_j = m_j$ for any $j < i$ and $n_i < m_i$.

Definition 2. The sequence $C(\mathcal{L}) = \max \mathrm{C(x)}_{x \in \mathcal{L} \setminus \mathcal{L}^2}$ is called characteristic sequence of the algebra $\mathcal{L}$.

Example 1. If $C(\mathcal{L}) = (1, 1, \ldots, 1)$, then evidently, the algebra $\mathcal{L}$ is abelian.

Definition 3. A Lie algebra $\mathfrak{g}$ is said to be *quasi-filiform* if $\mathfrak{g}^{n-2} \neq \{0\}$ and $\mathfrak{g}^{n-1} = \{0\}$, where $n = \dim(\mathfrak{g})$.

The set $R(\mathcal{L}) = \{x \in \mathcal{L} : [y, x] = 0 \ \forall y \in \mathcal{L}\}$ is said to be *the right annihilator of* $\mathcal{L}$. Note that for any $x, y \in \mathcal{L}$ the elements $[x, x]$ and $[x, y] + [y, x]$ are in $R(\mathcal{L})$.

2. Naturally graded quasi-filiform Lie algebras.

The following theorem describes the classification of naturally graded quasi-filiform Lie algebras.

Theorem 4 (*Gómez and Jiménez-Merchán, 2002*). *Let $\mathfrak{g}$ be a complex n-dimensional non-split naturally graded quasi-filiform Lie algebra. Then there exists a basis $\{x_0, x_1, \ldots, x_{n-2}, y\}$ of $\mathfrak{g}$, such that the multiplication in the algebra has the following form:*

$L(n, r)$ $(n \ge 5,\ 3 \le r \le 2\lfloor \frac{n-1}{2} \rfloor - 1,\ r\ odd)$:

$$\begin{cases} [x_0, x_i] = x_{i+1}, & 1 \le i \le n-3 \\ [x_i, x_{r-i}] = (-1)^{i-1} y, & 1 \le i \le \frac{r-1}{2} \end{cases}$$

$Q(n, r)$ $(n \ge 7,\ n\ odd, 3 \le r \le n-4,\ r\ odd)$:

$$\begin{cases} [x_0, x_i] = x_{i+1}, & 1 \le i \le n-3 \\ [x_i, x_{r-i}] = (-1)^{i-1} y, & 1 \le i \le \frac{r-1}{2} \\ [x_i, x_{n-2-i}] = (-1)^{i-1} x_{n-2}, & 1 \le i \le \frac{n-3}{2} \end{cases}$$

$\tau(n, n-3)$ $(n \geq 6,\ n\ even)$:

$$\begin{cases} [x_0, x_i] = x_{i+1}, \ 1 \leq i \leq n-3 \\ [x_{n-1}, x_1] = \frac{(n-4)}{2} x_{n-2}, \\ [x_i, x_{n-3-i}] = (-1)^{i-1}(x_{n-3} + x_{n-1}), \ 1 \leq i \leq \frac{n-4}{2} \\ [x_i, x_{n-2-i}] = (-1)^{i-1} \frac{(n-2-2i)}{2} x_{n-2}, \ 1 \leq i \leq \frac{n-4}{2} \end{cases}$$

$\tau(n, n-4)$ $(n \geq 7,\ n\ odd)$:

$$\begin{cases} [x_0, x_i] = x_{i+1}, \ 1 \leq i \leq n-3 \\ [x_{n-1}, x_i] = \frac{(n-5)}{2} x_{n-4+i}, \ 1 \leq i \leq 2 \\ [x_i, x_{n-4-i}] = (-1)^{i-1}(x_{n-4} + x_{n-1}), \ 1 \leq i \leq \frac{n-5}{2} \\ [x_i, x_{n-3-i}] = (-1)^{i-1} \frac{(n-3-2i)}{2} x_{n-3}, \ 1 \leq i \leq \frac{n-5}{2} \\ [x_i, x_{n-2-i}] = (-1)^{i}(i-1) \frac{(n-3-i)}{2} x_{n-2}, \ 2 \leq i \leq \frac{n-3}{2} \end{cases}$$

$\varepsilon(7, 3)$:

$$\begin{cases} [x_0, x_i] = x_{i+1}, & 1 \leq i \leq 3 \\ [y, x_i] = x_{i+3}, & 1 \leq i \leq 2 \\ [x_1, x_2] = x_3 + y, & \\ [x_1, x_i] = x_{i+1}, & 3 \leq i \leq 4 \end{cases}$$

$\varepsilon^1(9, 5)$:

$$\begin{cases} [x_0, x_i] = x_{i+1}, & 1 \leq i \leq 5 \\ [y, x_i] = 2x_{i+5}, & 1 \leq i \leq 2 \\ [x_1, x_4] = x_5 + y, & \\ [x_1, x_5] = 2x_6, & \\ [x_1, x_6] = 3x_7, & \\ [x_2, x_3] = -x_5 - y, & \\ [x_2, x_4] = -x_6, & \\ [x_2, x_5] = -x_7. & \end{cases}$$

$\varepsilon^2(9, 5)$:

$$\begin{cases} [x_0, x_i] = x_{i+1}, & 1 \leq i \leq 5 \\ [y, x_i] = 2x_{i+5}, & 1 \leq i \leq 2 \\ [x_1, x_4] = x_5 + y, & \\ [x_1, x_5] = 2x_6, & \\ [x_1, x_6] = x_7, & \\ [x_2, x_3] = -x_5 - y, & \\ [x_2, x_4] = -x_6, & \\ [x_2, x_5] = x_7, & \\ [x_3, x_4] = -2x_7. & \end{cases}$$

3. Naturally graded quasi-filiform Leibniz algebras

For Lie algebras the notions of 2-filiform and quasi-filiform coincide. However, for Leibniz algebras these notions are not equal and are the following:

Definition 5. A Leibniz algebra $\mathcal{L}$ is said to be 2-filiform if $C(\mathcal{L}) = (n-2, 1, 1)$.

Theorem 6 (Camacho et al., 2006). *Let $\mathcal{L}$ be an n-dimensional $(n \geq 6)$ graded 2-filiform non-split non-Lie–Leibniz algebra. Then $\mathcal{L}$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-3 \\ [e_1, e_{n-1}] = e_2 + e_n, & \\ [e_i, e_{n-1}] = e_{i+1}, & 2 \leq i \leq n-3 \end{cases} \qquad \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-3 \\ [e_1, e_{n-1}] = e_n. & \end{cases}$$

Note that the classification of such algebras of 2-filiform Leibniz algebras of dimension less that 6 can be found in the papers Albeverio et al. (2005) and Camacho et al. (2006).

Definition 7. A Leibniz algebra $\mathcal{L}$ is called a quasi-filiform Leibniz algebra if $\mathcal{L}^{n-2} \neq 0$ and $\mathcal{L}^{n-1} = 0$, where $\dim \mathcal{L} = n$.

Note that Definitions 3 and 7 hold.

Let $\mathcal{L}$ be a graded quasi-filiform n-dimensional non-Lie–Leibniz algebra. It is not difficult to see that either $C(\mathcal{L}) = (n-2, 1, 1)$ or $C(\mathcal{L}) = (n-2, 2)$. Since the case $C(\mathcal{L}) = (n-2, 1, 1)$ was classified in Theorem 6, we now consider the case $C(\mathcal{L}) = (n-2, 2)$.

From the definition of the characteristic sequence, $C(L) = (n-2, 2)$, it follows the existence of a basic element $e_1 \in \mathcal{L} \backslash \mathcal{L}^2$ and a basis $\{e_1, e_2, \ldots, e_n\}$ such that the operator of right multiplication R_{e_1} has one of the following forms:

$$\begin{pmatrix} J_{n-2} & 0 \\ 0 & J_2 \end{pmatrix}, \quad \begin{pmatrix} J_2 & 0 \\ 0 & J_{n-2} \end{pmatrix}.$$

Definition 8. A quasi-filiform non-Lie–Leibniz algebra $\mathcal{L}$ is called an algebra of first type if there exists a basic element $e_1 \in \mathcal{L}\backslash\mathcal{L}^2$ such that the operator R_{e_1} has the form: $\begin{pmatrix} J_{n-2} & 0 \\ 0 & J_2 \end{pmatrix}$; if R_{e_1} has the other form, it is called an algebra of second type.

Theorem 9. *Let $\mathcal{L}$ be a naturally graded Leibniz algebra of first type. Then it is isomorphic to one of the following pairwise non-isomorphic algebras:*

$\mathcal{L}^{1,\lambda}$:

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \le i \le n-3 \\ [e_{n-1}, e_1] = e_n, \\ [e_1, e_{n-1}] = \lambda e_n, & \lambda \in \mathbf{C} \end{cases}$$

$\mathcal{L}^{2,\lambda}$:

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \le i \le n-3 \\ [e_{n-1}, e_1] = e_n, \\ [e_1, e_{n-1}] = \lambda e_n, & \lambda \in \{0, 1\} \\ [e_{n-1}, e_{n-1}] = e_n \end{cases}$$

$\mathcal{L}^{3,\lambda}$:

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \le i \le n-3 \\ [e_{n-1}, e_1] = e_n + e_2, \\ [e_1, e_{n-1}] = \lambda e_n, & \lambda \in \{-1, 0, 1\} \end{cases}$$

$\mathcal{L}^{4,\lambda}$:

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \le i \le n-3 \\ [e_{n-1}, e_1] = e_n + e_2, \\ [e_{n-1}, e_{n-1}] = \lambda e_n, & \lambda \neq 0 \end{cases}$$

$\mathcal{L}^{5,\lambda,\mu}$:

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \le i \le n-3 \\ [e_{n-1}, e_1] = e_n + e_2, \\ [e_1, e_{n-1}] = \lambda e_n, & (\lambda, \mu) = (1, 1) \text{ or } (2, 4) \\ [e_{n-1}, e_{n-1}] = \mu e_n, \end{cases}$$

$\mathcal{L}^{6}$:

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \le i \le n-3 \\ [e_{n-1}, e_1] = e_n, \\ [e_1, e_{n-1}] = -e_n, \\ [e_{n-1}, e_{n-1}] = e_2, \\ [e_{n-1}, e_n] = e_3. \end{cases}$$

Proof. From the condition $\begin{pmatrix} J_{n-2} & 0 \\ 0 & J_2 \end{pmatrix}$ we have the following multiplication:

$$[e_i, e_1] = e_{i+1}, \quad 1 \le i \le n-3 \quad [e_{n-2}, e_1] = 0, \; [e_{n-1}, e_1] = e_n, \; [e_n, e_1] = 0.$$

It is easily seen that $\mathcal{L}_1 =< e_1, e_{n-1} >$, $\mathcal{L}_2 =< e_2, e_n >$, $\mathcal{L}_i =< e_i >$ for $3 \le i \le n-2$ and $\{e_2, e_3, \dots, e_{n-2}\} \subseteq R(\mathcal{L})$. Therefore, for defining the multiplication of $\mathcal{L}_1$ it is enough to study the multiplication of the element e_{n-1} on the right side.

Let $[e_1, e_{n-1}] = \alpha_1 e_2 + \alpha_2 e_n$, $[e_{n-1}, e_{n-1}] = \beta_1 e_2 + \beta_2 e_n$.

Considering the Leibniz identity for the basic elements:

$$[e_i, [e_j, e_k]] = [[e_i, e_j], e_k] - [[e_i, e_k], e_j]$$

for $j, k \in \{2, 3, \dots, n-2\}$ and $j = k \in \{1, n\}$ we do not obtain any restrictions. Therefore, the consideration of the Leibniz identity is reduced to the consideration of the cases:

$$j = 1 \text{ and } k = n-1, n; \quad j = n-1 \text{ and } k = 1, n-1, n; \quad j = n \text{ and } k = 1, n-1.$$

These cases lead to the following equalities for $1 \le i \le n$:

(1) $\alpha_2[e_i, e_n] = [[e_i, e_1], e_{n-1}] - [[e_i, e_{n-1}], e_1]$
(2) $[e_i, e_n] = [[e_i, e_{n-1}], e_1] - [[e_i, e_1], e_{n-1}]$
(3) $[[e_i, e_1], e_n] = [[e_i, e_n], e_1]$
(4) $[[e_i, e_n], e_{n-1}] = [[e_i, e_{n-1}], e_n]$
(5) $\beta_2[e_i, e_n] = 0.$

From Eqs. (1) and (2) we have $\alpha_2[e_i, e_n] = -[e_i, e_n]$.

Consider the following cases.

Let us suppose $\boxed{e_n \in R(\mathcal{L})}$.

Then from Eq. (2) for $i = 1, 2, \dots, n$ we have that $\beta_1 = 0$ and

$$\begin{array}{ll} [e_i, e_{n-1}] = \alpha_1 e_{i+1}, & 2 \le i \le n-3, \\ [e_{n-2}, e_{n-1}] = 0, & \\ [e_{n-1}, e_{n-1}] = \beta_2 e_n, & \\ [e_n, e_{n-1}] = 0. & \end{array}$$

Making the following change: $e'_{n-1} = e_{n-1} - \alpha_1 e_1$, $e'_i = e_i$ for $i \neq n-1$ and renaming the parameters, we obtain that the multiplication of $\mathcal{L}_1$ is

$$\begin{array}{lll} [e_i, e_1] = e_{i+1}, & 1 \le i \le n-3, & [e_1, e_{n-1}] = \beta e_n, \\ [e_{n-1}, e_1] = e_n + \alpha e_2, & & [e_{n-1}, e_{n-1}] = \gamma e_n. \end{array}$$

Let us consider the general change of the generators of basis:

$$e'_1 = \sum_{i=1}^{n} A_i e_i, \qquad e'_{n-1} = \sum_{i=1}^{n} B_i e_i.$$

We express the new basic elements $\{e'_1, e'_2, \dots, e'_{n-1}, e'_n\}$ via the basic elements $\{e_1, e_2, \dots, e_{n-1}, e_n\}$. Computing all the products we obtain that the parameters are the following:

$$(6)\quad \alpha' = \frac{B_{n-1}}{A_1 + A_{n-1}\alpha}\alpha, \quad \beta' = \frac{(A_1 + A_{n-1}\alpha)(A_1\beta + A_{n-1}\gamma)}{A_1(A_1 + A_{n-1}(\gamma - \alpha\beta))}, \quad \gamma' = \frac{B_{n-1}(A_1 + A_{n-1}\alpha)}{A_1(A_1 + A_{n-1}(\gamma - \alpha\beta))}\gamma$$

and the restrictions

$$(7)\quad \begin{array}{l} A_1 B_{n-1}(A_1 + A_{n-1}\alpha)(A_1 + A_{n-1}(\gamma - \alpha\beta)) \neq 0 \\ \beta'(B_{n-3}(A_1 + A_{n-1}\alpha) - A_{n-3}B_{n-1}\alpha) = 0 \\ \gamma'(B_{n-3}(A_1 + A_{n-1}\alpha) - A_{n-3}B_{n-1}\alpha) = 0 \\ B_i - A_i\alpha' = 0 \qquad 2 \le i \le n-4 \\ B_1 = 0. \end{array}$$

Note that coefficients A_{n-3}, B_{n-3} do not participate in the expressions for the parameters α', β', γ'. Hence, we may assume they are equal to zero.

At this moment, we consider the possible cases.

(a) $\alpha = \gamma = 0$. Then, from (6) and (7) we have:

$$A_1 B_{n-1} \neq 0, \qquad \beta' = \beta.$$

Since $A_1 \neq 0$, then $B_i = 0$ where $1 \le i \le n-3$ and we obtain the algebra $\mathcal{L}^{1,\lambda}$.

(b) $\alpha = 0$ and $\gamma \neq 0$. Then, from (6) and (7) we obtain that:

$$\alpha' = 0, \qquad \beta' = \frac{A_1\beta + A_{n-1}\gamma}{A_1 + A_{n-1}\gamma}, \qquad \gamma' = \frac{B_{n-1}}{A_1 + A_{n-1}\gamma}\gamma$$

$$B_{n-1}A_1(A_1 + A_{n-1}\gamma) \neq 0, \qquad B_i = 0 \quad 1 \le i \le n-3.$$

Note that the following equality:

$$\beta' - 1 = \frac{A_1}{A_1 + A_{n-1}\gamma}(\beta - 1)$$

holds.

Therefore for the expression $\beta - 1$ there exist two possible cases and in each of them we have non-isomorphic algebras, that is, the nullity of $\beta - 1$ is invariant.

b.1 $\beta = 1$. Then, $\beta' = 1$. Since $\gamma \neq 0$, then put $A_{n-1} = 0$, $B_{n-1} = \dfrac{A_1 + A_{n-1}\gamma}{\gamma}$, we obtain that $\gamma' = 1$ and in this case we have the algebra $\mathcal{L}^{2,\lambda}$ where $\lambda = 1$.

b.2 $\beta \neq 1$. Then, taking $A_{n-1} = -\frac{\beta}{\gamma}A_1$, $B_{n-1} = \frac{(1-\beta)}{\gamma}A_1$, we obtain that $\beta' = 0$, $\gamma' = 1$ and so we get $\mathcal{L}^{2,\lambda}$ where $\lambda = 0$.

(c) $\alpha \neq 0$ and $\gamma = 0$. Then, taking $B_{n-1} = \frac{A_1 + A_{n-1}\alpha}{\alpha}$, we obtain that:

$$\begin{aligned} &\alpha' = 1, \qquad \beta' = \frac{A_1 + A_{n-1}\alpha}{A_1 - \alpha\beta A_{n-1}}\beta, \qquad \gamma' = 0 \\ &B_i = A_i, \qquad 2 \leq i \leq n-3 \end{aligned}$$

Note that the following equality is true

$$\beta' + 1 = \frac{A_1}{A_1 - \alpha\beta A_{n-1}}(\beta + 1)$$

c.1 $\beta \neq 0$ and $\beta \neq -1$. Then, choosing $A_{n-1} = -\frac{\beta - 1}{2\alpha\beta}A_1$, we obtain $\beta' = 1$.

c.2 $\beta \neq 0$ and $\beta = -1$. Then, $\beta' = -1$.

c.3 $\beta = 0$. Then, $\beta' = 0$.

Note that restriction $A_1B_{n-1}(A_1 + A_{n-1}\alpha)(A_1 + (\gamma - \alpha\beta)A_{n-1} \neq 0$ in the cases c.1–c.3 also holds. Thus, in these cases we have the algebras $\mathcal{L}^{3,\lambda}$ where $\lambda \in \{-1, 0, 1\}$.

(d) $\alpha \neq 0$ and $\gamma \neq 0$. Using (6), we have the following equalities:

$$\begin{aligned} \gamma' - \alpha'\beta' &= \frac{B_{n-1}}{A_1 + A_{n-1}(\gamma - \alpha\beta)}(\gamma - \alpha\beta) \\ \alpha'\beta'^2 + \gamma' - \beta'\gamma' &= \frac{B_{n-1}(A_1 + A_{n-1}\alpha)}{(A_1 + A_{n-1}(\gamma - \alpha\beta))^2}(\alpha\beta^2 + \gamma - \beta\gamma) \\ \gamma' - \alpha'\beta' - \alpha' &= \frac{A_1B_{n-1}}{(A_1 + A_{n-1}\alpha)(A_1 + A_{n-1}(\gamma - \alpha\beta))}(\gamma - \alpha\beta - \alpha). \end{aligned}$$

Therefore, in the cases below, we obtain the algebras, pairwise non-isomorphic.

d.1 $\gamma - \alpha\beta \neq 0$ and $\alpha\beta^2 + \gamma - \beta\gamma \neq 0$. Then, taking $A_{n-1} = -\frac{A_1\beta}{\gamma}$, $B_{n-1} = \frac{A_1(\gamma - \alpha\beta)}{\alpha\gamma}$, we obtain that

$$\alpha' = 1, \quad \beta' = 0, \quad \gamma' = \frac{(\gamma - \alpha\beta)^2}{\alpha(\alpha\beta^2 + \gamma - \beta\gamma)} = \lambda,$$

that is, we obtain the one-parametric family of algebras $\mathcal{L}^{4,\lambda}$.

d.2 $\gamma - \alpha\beta \neq 0$ and $\alpha\beta^2 + \gamma - \beta\gamma = 0$. The case $\beta = 0$ is a contradiction. Let us suppose that $\beta = 1$, then $\alpha = 0$ which contradicts case (d). Therefore, $\beta \notin \{0, 1\}$.

Let us replace the expression $\alpha = \frac{(\beta - 1)\gamma}{\beta^2}$ in other expressions and we obtain

$$\alpha' = \frac{(\beta - 1)\gamma B_{n-1}}{\beta^2A_1 + (\beta - 1)\gamma A_{n-1}}, \qquad \beta' = \frac{\beta^2A_1 + (\beta - 1)\gamma A_{n-1}}{\beta A_1},$$
$$\gamma' = \frac{\gamma(\beta^2A_1 + (\beta - 1)\gamma A_{n-1})B_{n-1}}{\beta A_1(\beta A_1 + \gamma A_{n-1})}.$$

Putting $A_{n-1} = -\frac{(\beta - 2)\beta}{(\beta - 1)\gamma}A_1$, $B_{n-1} = \frac{2\beta}{(\beta - 1)\gamma}A_1$, we obtain that $\alpha' = 1$, $\beta' = 2$, $\gamma' = 4$. Note that in this case the restriction

$$A_1B_{n-1}(A_1 + A_{n-1}\alpha)(A_1 + A_{n-1}(\gamma - \alpha\beta)) = \frac{4A_1^4}{(\beta - 1)^2\gamma} \neq 0$$

is also verified. Thus, we have the algebra $\mathcal{L}^{5,\lambda,\mu}$ where $\lambda = 2,\ \mu = 4$.

d.3 $\gamma - \alpha\beta = 0$ and $\alpha\beta^2 + \gamma - \beta\gamma \neq 0$. Then, $\alpha\beta \neq 0$ and $\beta = \dfrac{\gamma}{\alpha}$. Taking $B_{n-1} = -\dfrac{A_1}{\sqrt{\alpha\gamma}}$ and $A_{n-1} = -\dfrac{\sqrt{\alpha}+\sqrt{\gamma}}{\alpha\sqrt{\gamma}}A_1$, we obtain that $\alpha' = \beta' = \gamma' = 1$, hence we have the algebra $\mathcal{L}^{5,\lambda,\mu}$ where $\lambda = \mu = 1$.

Since $\gamma \neq 0$, then the case $\gamma - \alpha\beta = 0$ and $\alpha\beta^2 + \gamma - \beta\gamma = 0$ is impossible.

Let us suppose $\boxed{e_n \notin R(\mathcal{L}).}$

Then from (1), (2) and (5) we have that $\alpha_2 = -1$ and $\beta_2 = 0$. Consider

$$[e_1, e_n] = \gamma e_3, \quad [e_{n-1}, e_n] = \mu e_3, \quad [e_n, e_n] = \lambda e_4.$$

From (3) we obtain $[e_i, e_n] = \gamma e_{i+2}$, $1 \le i \le n-4$.

Using equalities (2) and (4) we obtain the following restrictions:

$\lambda = \beta_1 - \mu$, $\gamma(\alpha_1 - 2\gamma) = \alpha_1\gamma - \lambda$, $\gamma(\alpha_1 - i\gamma) = \gamma(\alpha_1 - (i-2)\gamma)$, $3 \le i \le n-4$, $\mu(\alpha_1 - 2\gamma) = \beta_1\gamma$ from which we conclude that $\beta_1 = \mu$, $\gamma = \lambda = 0$ and $\alpha_1\mu = 0$. Since $e_n \notin R(\mathcal{L})$, then $\mu \neq 0$ and, therefore, $\alpha_1 = 0$. Making the change of basis: $e'_n = \dfrac{1}{\sqrt{\mu}}e_n$, $e'_{n-1} = \dfrac{1}{\sqrt{\mu}}e_{n-1}$ we obtain the algebra $\mathcal{L}^6$. □

The theorem below describes naturally graded quasi-filiform Leibniz algebras of the second type. The next theorem completes the classification of quasi-filiform non-Lie–Leibniz algebras.

Theorem 10. *Let $\mathcal{L}$ be a naturally graded quasi-filiform Leibniz algebra of second type. Then $\mathcal{L}$ is isomorphic to one of the following pairwise non-isomorphic algebras: n even*

$\mathcal{L}^1$:

$$\begin{cases} [e_1, e_1] = e_2 & \\ [e_i, e_1] = e_{i+1} & 3 \le i \le n-1 \\ [e_1, e_i] = -e_{i+1}, & 3 \le i \le n-1 \end{cases}$$

$\mathcal{L}^2$:

$$\begin{cases} [e_1, e_1] = e_2 & \\ [e_i, e_1] = e_{i+1} & 3 \le i \le n-1 \\ [e_1, e_3] = e_2 - e_4, & \\ [e_1, e_j] = -e_{j+1}, & 4 \le j \le n-1 \end{cases}$$

$\mathcal{L}^3$:

$$\begin{cases} [e_1, e_1] = e_2 & \\ [e_i, e_1] = e_{i+1} & 3 \le i \le n-1 \\ [e_3, e_3] = e_2, & \\ [e_1, e_i] = -e_{i+1}, & 3 \le i \le n-1 \end{cases}$$

$\mathcal{L}^4$:

$$\begin{cases} [e_1, e_1] = e_2 & \\ [e_i, e_1] = e_{i+1} & 3 \le i \le n-1 \\ [e_1, e_3] = 2e_2 - e_4, & \\ [e_3, e_3] = e_2, & \\ [e_1, e_j] = -e_{j+1}, & 4 \le j \le n-1 \end{cases}$$

n odd

$\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^3, \mathcal{L}^4, \mathcal{L}^5$:

$$\begin{cases} [e_1, e_1] = e_2 & \\ [e_i, e_1] = e_{i+1} & 3 \le i \le n-1 \\ [e_1, e_i] = -e_{i+1}, & 3 \le i \le n-1 \\ [e_i, e_{n+2-i}] = (-1)^i e_n, & 3 \le i \le n-1 \end{cases}$$

$\mathcal{L}^{6,\lambda}$:

$$\begin{cases} [e_1, e_1] = e_2 & \\ [e_i, e_1] = e_{i+1} & 3 \le i \le n-1 \\ [e_1, e_3] = \lambda e_2 - e_4, & \lambda \in \{1, 2\} \\ [e_1, e_j] = -e_{j+1}, & 4 \le j \le n-1 \\ [e_i, e_{n+2-i}] = (-1)^i e_n, & 3 \le i \le n-1 \end{cases}$$

$\mathcal{L}^{7,\lambda}$:

$$\begin{cases} [e_1, e_1] = e_2 & \\ [e_i, e_1] = e_{i+1} & 3 \le i \le n-1 \\ [e_3, e_3] = \lambda e_2, & \lambda \neq 0 \\ [e_1, e_i] = -e_{i+1}, & 3 \le i \le n-1 \\ [e_i, e_{n+2-i}] = (-1)^i e_n, & 3 \le i \le n-1 \end{cases}$$

$\mathcal{L}^{8,\lambda,\mu}$:

$$\begin{cases} [e_1, e_1] = e_2, & \\ [e_i, e_1] = e_{i+1}, & 3 \le i \le n-1 \\ [e_1, e_3] = \lambda e_2 - e_4, & \\ [e_3, e_3] = \mu e_2, & (\lambda, \mu) = (-2, 1),\ (2, 1) \text{ or } (4, 2) \\ [e_1, e_j] = -e_{j+1}, & 4 \le j \le n-1 \\ [e_i, e_{n+2-i}] = (-1)^i e_n, & 3 \le i \le n-1 \end{cases}$$

Proof. From the condition of the theorem we have the following multiplication of the basic element e_1 on the right side:

$$\begin{array}{ll} [e_1, e_1] = e_2, & [e_i, e_1] = e_{i+1}, \quad 3 \le i \le n-1 \\ [e_2, e_1] = 0, & [e_n, e_1] = 0. \end{array}$$

Note that $e_2 \in R(\mathcal{L})$ and that $\mathcal{L}_1 =< e_1, e_3 >$, $\mathcal{L}_2 =< e_2, e_4 >$, $\mathcal{L}_i =< e_{i+2} >$ for $3 \le i \le n-2$. Let us pose

$$\begin{array}{lll} [e_1, e_3] = \alpha_1 e_2 + \beta_1 e_4, & [e_2, e_3] = \beta_2 e_5, & [e_3, e_3] = \alpha_2 e_2 + \beta_3 e_4, \\ [e_i, e_3] = \beta_i e_{i+1}, \ 4 \le i \le n-1 & [e_n, e_3] = 0. & \end{array}$$

We consider the following equality

$$(8) \quad [e_i, [e_3, e_1]] = [[e_i, e_3], e_1] - [[e_i, e_1], e_3], \qquad 3 \le i \le n-2.$$

From (8) we have

$$\begin{array}{ll} [e_1, e_4] = (\beta_1 - \beta_2) e_5, & [e_2, e_4] = \beta_2 e_6, \\ [e_i, e_4] = (\beta_i - \beta_{i+1}) e_{i+2}, \quad 3 \le i \le n-2 & [e_{n-1}, e_4] = [e_n, e_4] = 0. \end{array}$$

The following equality holds.

$$(9) \quad [e_i, e_j] = \left(\sum_{k=0}^{j-3} (-1)^k \binom{j-3}{k} C_{j-3}^k \beta_{i+k} \right) e_{i+j-2}$$

where $i + j \le n + 2, \ \ 3 \le i \le n-1, \ \ 4 \le j \le n + 2 - i$.

This equality will be proved by induction on j for any value i. For $j = 4$ we have that $[e_i, e_4] = [e_i, [e_3, e_1]] = [[e_i, e_3], e_1] - [[e_i, e_1], e_3] = (\beta_i - \beta_{i+1}) e_{i+2}$ where $4 \le i \le n-2$.

By the induction hypothesis and the following chain of equalities:

$$\begin{aligned} [e_i, e_{j+1}] &= [e_i, [e_j, e_1]] = [[e_i, e_j], e_1] - [[e_i, e_1], e_j] \\ &= \left(\sum_{k=0}^{j-3} (-1)^k C_{j-3}^k \beta_{i+k} \right) e_{i+j-1} - \left(\sum_{k=0}^{j-3} (-1)^k C_{j-3}^k \beta_{i+k+1} \right) e_{i+j-1} \\ &= \left(\beta_i + \sum_{k=1}^{j-3} (-1)^k C_{j-3}^k \beta_{i+k} + \sum_{k=1}^{j-2} (-1)^k C_{j-3}^{k-1} \beta_{i+k} \right) e_{i+j-1} \\ &= \left(\beta_i + \sum_{k=1}^{j-3} (-1)^k (C_{j-3}^k + C_{j-3}^{k-1}) \beta_{i+k} + (-1)^{j-2} C_{j-3}^{j-3} \beta_{i+j-2} \right) e_{i+j-1} \\ &= \left(\sum_{k=0}^{j-2} (-1)^k C_{j-2}^k \beta_{i+k} \right) e_{i+j-1} \end{aligned}$$

we complete equality (9).

Thus, for the basic element e_4 there exist two possible cases.

Let us suppose $\boxed{e_4 \in R(\mathcal{L}).}$

Then we obtain that

$$\beta_1 = \beta_2 = 0 \quad \text{and} \quad \beta_i = \gamma \quad 3 \le i \le n-1.$$

Making the change of basis $e_3' = e_3 - \gamma e_1$, $e_4' = e_4 - \gamma e_2$, we obtain the algebra with the following multiplication:

$$\begin{array}{ll} [e_1, e_1] = e_2, & [e_1, e_3] = \alpha e_2, \\ [e_2, e_1] = 0, & [e_2, e_3] = 0, \\ [e_i, e_1] = e_{i+1}, \quad 3 \le i \le n-1 & [e_3, e_3] = \beta e_2, \\ & [e_i, e_3] = 0, \quad 4 \le i \le n. \end{array}$$

If we make the following change $e'_1 = Ae_1 + Be_3,\ e'_{n-1} = e_1$ where $AB(A + B\alpha) \neq 0$, we have that $R_{e'_1} = \begin{pmatrix} J_{n-2} & 0 \\ 0 & J_2 \end{pmatrix}$, that is, it contradicts that $\mathcal{L}$ is an algebra of second type.

Let us suppose $\boxed{e_4 \notin R(\mathcal{L}).}$

Then, from the equality $[e_i, [e_3, e_3]] = 0$, we have that $\beta_3[e_i, e_4] = 0$ for any $i \in \{1, 2, \dots, n\}$. As the following identity $[e_i, [e_3, e_1]] = -[e_i, [e_1, e_3]]$, then $[e_i, e_4] = -\beta_1[e_i, e_4]$, and therefore $\beta_1 = -1$, $\beta_3 = 0$ (otherwise $e_4 \in R(\mathcal{L})$).

Now, we consider the following equality:

$$(10)\quad [e_i, [e_4, e_1]] = [[e_i, e_4], e_1] - [[e_i, e_1], e_4].$$

From (10) we have that

$$[e_1, e_5] = -(1 + 2\beta_2)e_6,$$
$$[e_2, e_5] = \beta_2 e_7.$$

Suppose that $e_5 \in R(\mathcal{L})$, then $\beta_2 = -\frac{1}{2}$ and $\beta_2 = 0$, which is a contradiction. Therefore, $e_5 \notin R(\mathcal{L})$, then from $[e_4, e_1] + [e_1, e_4] = -\beta_2 e_5 \in R(\mathcal{L})$, we obtain that $\beta_2 = 0$. By induction on i, we can establish that $[e_2, e_i] = [e_i, e_2] = 0$ (that is, $e_2 \in R(\mathcal{L})$) and $[e_1, e_i] = -e_{i+1}$ with $4 \leq i \leq n - 1$.

From (9) and the following equality: $[2e_1, [e_3, e_{2j}]] = [[e_1, e_3], e_{2j}] - [[e_1, e_{2j}], e_3]$ we obtain that

$$(11)\quad 2\beta_{2j+1} = \beta_4 + \beta_{2j} + \sum_{k=1}^{2j-4}(-1)^k C_{2j-3}^k(\beta_{4+k} - \beta_{3+k}).$$

Now we prove that

$$(12)\quad \begin{cases} \beta_i = \beta_4, & 5 \leq i \leq n-1 \quad \text{for n even} \\ \beta_i = \beta_4, & 5 \leq i \leq n-2 \quad \text{for n odd.} \end{cases}$$

Firstly, consider the case <u>n even.</u>

According to (11) we have that

$$j = 2 \quad \Rightarrow \quad \beta_4 = \beta_5,$$
$$j = 3 \quad \Rightarrow \quad \beta_4 = \beta_5, \quad \beta_7 = -\beta_4 + 2\beta_6.$$

Replacing $\beta_4 = \beta_5$, $\beta_7 = -\beta_4 + 2\beta_6$ in the following equality:

$$[e_3, [e_4, e_5]] = [[e_3, e_4], e_5] - [[e_3, e_5], e_4]$$

we obtain that $\beta_4 = \beta_6 = \beta_7$. Thus, we have the basis for the induction.

Suppose that $\beta_i = \beta_4$ for any $i \leq 2j + 1$. Let us prove that $\beta_4 = \beta_{2j+2}$. From (11) and the induction hypothesis we obtain that

$$(13)\quad \beta_{2j+3} = -(j-1)\beta_4 + j\beta_{2j+2}.$$

From the induction hypothesis and the following equality:

$$[e_3, [e_4, e_{2j+1}]] = [[e_3, e_4], e_{2j+1}] - [[e_3, e_{2j+1}], e_4]$$

we have that $\beta_{2j+2} = \beta_4$. Then from (13) we obtain that $\beta_{2j+3} = \beta_4$. Hence, (12) is proved.

By induction and the equalities $[e_4, e_j] = 0$ with $4 \leq j \leq n - 2$, it is easy to show that

$$[e_3, e_j] = -\beta e_{j+1}, \qquad 4 \leq j \leq n-1.$$

Thus, we have the following family:

$$\begin{array}{lll} [e_1, e_1] = e_2, & [e_1, e_3] = \alpha_1 e_2 - e_4, & [e_1, e_i] = -e_{i+1},\ 4 \leq i \leq n-1 \\ [e_i, e_1] = e_{i+1},\ 3 \leq i \leq n-1 & [e_3, e_3] = \alpha_2 e_2, & [e_3, e_i] = -\beta e_{i+1},\ 4 \leq i \leq n-1 \\ & [e_i, e_3] = \beta e_{i+1},\ 4 \leq i \leq n-1 & \end{array}$$

(the rest are equal to zero).

Making the following change of basis: $e'_3 = e_3 - \beta e_1$, $\quad e'_i = e_i$, for $i \neq 3$ we have the family:

$$\begin{array}{lll} [e_1, e_1] = e_2 & [e_1, e_3] = \lambda e_2 - e_4, & [e_1, e_i] = -e_{i+1},\ 4 \leq i \leq n-1 \\ [e_i, e_1] = e_{i+1},\ 3 \leq i \leq n-1 & [e_3, e_3] = \mu e_2 & \end{array}$$

(the rest are equal to zero).

Let us make the general change of the generators of basis: $e'_1 = \sum_{i=1}^{n} A_i e_i,\ e'_3 = \sum_{i=1}^{n} B_i e_i$.

Then computing all the products we obtain the following restriction and the expressions of λ' and μ':

$$\lambda' = \frac{B_3(A_1\lambda + 2A_3\mu)}{A_1^2 + A_1A_3\lambda + A_3^2\mu}, \quad \mu' = \frac{B_3^2}{A_1^2 + A_1A_3\lambda + A_3^2\mu}\mu$$

$$A_1B_3(A_1^2 + A_1A_3\lambda + A_3^2\mu) \neq 0$$

Consider the following cases:

(**a**) $\mu = 0$. Then, $\mu' = 0$ and $\lambda' = \dfrac{B_3}{A_1 + A_3\lambda}\lambda$.

a.1 $\lambda = 0$. Then, $\lambda' = 0$ and we obtain the algebra $\mathcal{L}^1$.

a.2 $\lambda \neq 0$. Then, taking $B_3 = \dfrac{A_1 + A_3\lambda}{\lambda}$, we obtain $\lambda' = 1$, that is, we have the algebra $\mathcal{L}^2$.

(**b**) $\mu \neq 0$. Note that the equality $4\mu' - \lambda'^2 = \dfrac{A_1^2B_3^2}{(A_1^2 + A_1A_3\lambda + A_3^2\mu)^2}(4\mu - \lambda^2)$ holds.

b.1 $4\mu \neq \lambda^2$. Then, taking $A_3 = -\dfrac{\lambda}{2\mu}A_1$, $B_3 = \pm\dfrac{\sqrt{4\mu - \lambda^2}}{2\mu}A_1$, we obtain that $\lambda' = 0$, $\mu' = 1$. Moreover, $A_1B_3(A_1^2 + A_1A_3\lambda + A_3^2\mu) = \dfrac{A_1(\sqrt{4\mu - \lambda^2})^3}{\mu} \neq 0$. Thus, in this case we have the algebra $\mathcal{L}^3$.

b.2 $4\mu = \lambda^2$. Then, $\lambda \neq 0$ and $\mu = \dfrac{\lambda^2}{4}$. Replacing the value of μ and $B_3 = \dfrac{2A_1 + A_3\lambda}{\lambda}$ in the expressions λ', μ', we obtain that $\lambda' = 2$, $\mu' = 1$. Therefore we have the algebra $\mathcal{L}^4$.

Consider now the case when <u>n is odd.</u>

In this case we have the family:

$$\begin{array}{lll} [e_1, e_1] = e_2, & [e_1, e_3] = \alpha_1 e_2 - e_4, & [e_1, e_i] = -e_{i+1},\ 4 \le i \le n-1 \\ [e_i, e_1] = e_{i+1},\ 3 \le i \le n-1 & [e_3, e_3] = \alpha_2 e_2, & [e_3, e_i] = -\beta e_{i+1},\ 4 \le i \le n-2 \\ & [e_i, e_3] = \beta e_{i+1},\ 4 \le i \le n-2, & \\ & [e_{n-1}, e_3] = \beta_{n-1} e_n & \end{array}$$

$$[e_i, e_{n+2-i}] = (-1)^i(\beta_{n-1} - \beta)e_n,\ 4 \le i \le n-2, \qquad [e_3, e_{n-1}] = -\beta_{n-1}e_n.$$

(the rest are equal to zero).

If $\beta = \beta_{n-1}$, then the study of the above family is the same as for n even.

If $\beta \neq \beta_{n-1}$, then making the following change of basis: $e'_1 = (\beta_{n-1} - \beta)e_1,\ e'_3 = e_3 - \beta e_1$, we obtain the following family:

$$\begin{array}{llll} [e_1, e_1] = e_2, & [e_1, e_3] = \lambda e_2 - e_4, & [e_1, e_i] = -e_{i+1}, & 4 \le i \le n-1 \\ [e_i, e_1] = e_{i+1},\ 3 \le i \le n-1 & [e_3, e_3] = \mu e_2, & [e_i, e_{n+2-i}] = (-1)^i e_n, & 3 \le i \le n-1. \end{array}$$

(the rest are equal to zero).

Let us consider the general change of the generators of basis in the form:

$$e'_1 = \sum_{i=1}^{n} A_i e_i, \qquad e'_3 = \sum_{i=1}^{n} B_i e_i.$$

Then thinking as in the proof of Theorem 9, we obtain the following restriction and the expressions for the parameters λ', μ':

$$\lambda' = \frac{(A_1 + A_3)(\lambda A_1 + 2\mu A_3)}{A_1^2 + \lambda A_1A_3 + \mu A_3^2}, \qquad \mu' = \frac{(A_1 + A_3)^2}{A_1^2 + \lambda A_1A_3 + \mu A_3^2}\mu$$

and $A_1(A_1 + A_3)(A_1^2 + \lambda A_1A_3 + \mu A_3^2) \neq 0$.

(**a**) $\mu = 0$. Then, $\mu' = 0, \lambda' = \dfrac{A_1 + A_3}{A_1 + \lambda A_3}\lambda$. Note that $\lambda' - 1 = \dfrac{\lambda - 1}{A_1 + \lambda A_3} A_1$.

a.**1** $\lambda = 0$. Then, $\lambda' = 0$ and we obtain the algebra $\mathcal{L}^5$.

a.**2** $\lambda \neq 0$ and $\lambda = 1$. Then, $\lambda' = 1$ and we have $\mathcal{L}^{6,\lambda}$ with $\lambda = 1$.

a.**3** Let $\lambda \neq 0$ and $\lambda \neq 1$. Then taking $A_3 = \dfrac{\lambda - 2}{\lambda} A_1$, we obtain $\lambda' = 2$. Then, in this case, we obtain the algebra $\mathcal{L}^{6,\lambda}$, where $\lambda = 2$.

(**b**) $\mu \neq 0$. Note that the following equalities hold:

$$\lambda'^2 - 4\mu' = \frac{A_1^2 (A_1 + A_3)^2}{(A_1^2 + \lambda A_1 A_3 + \mu A_3^2)^2} (\lambda^2 - 4\mu),$$
$$\lambda' - 2\mu' = \frac{A_1 (A_1 + A_3)}{A_1^2 + \lambda A_1 A_3 + \mu A_3^2} (\lambda - 2\mu).$$

b.**1** $\lambda^2 - 4\mu \neq 0$ and $\lambda - 2\mu \neq 0$. Then, if $A_3 = -\dfrac{\lambda}{2\mu} A_1$, we obtain $\lambda' = 0$ and $\mu' = -\dfrac{(\lambda - 2\mu)^2}{\lambda^2 - 4\mu}$. The following restriction $A_1(A_1 + A_3)(A_1^2 + \lambda A_1 A_3 + \mu A_3^2) \neq 0$ is also verified and therefore we have the algebra $\mathcal{L}^{7,\lambda}$.

b.**2** $\lambda^2 - 4\mu \neq 0$ and $\lambda - 2\mu = 0$. Since $4\mu(\mu - 1) \neq 0$, then $\mu \neq 1$. Taking $A_3 = -\dfrac{(\sqrt{2(\mu - 1)} + \sqrt{\mu})}{\sqrt{\mu}} A_1$, we obtain that $\lambda' = 4$, $\mu' = 2$. We note that the restriction in this case $A_1(A_1 + A_3)(A_1^2 + \lambda A_1 A_3 + \mu A_3^2) \neq 0$ is also verified. Thus, in this case we have the algebra $\mathcal{L}^{8,\lambda,\mu}$ where $\lambda = 4$ and $\mu = 2$.

b.**3** $\lambda^2 - 4\mu = 0$ and $\lambda - 2\mu \neq 0$. Then, $\lambda' = \dfrac{2(A_1 + A_3)}{2A_1 + \lambda A_3}\lambda$, $\mu' = \dfrac{\lambda^2 (A_1 + A_3)^2}{(2A_1 + \lambda A_3)^2}$. Since $\lambda^2 - 2\lambda \neq 0$, then $\lambda \neq 0, 2$. Therefore, if $A_3 = -\dfrac{\lambda + 2}{2\lambda} A_1$, we have $\lambda' = -2$, $\mu' = 1$. We obtain the algebra $\mathcal{L}^{8,\lambda,\mu}$ with $\lambda = -2$, $\mu = 1$.

b.**4** $\lambda^2 - 4\mu = 0$ and $\lambda - 2\mu = 0$. Then $4\mu(\mu - 1) = 0$, that is, $\mu = 1$ and $\lambda = 2$. Hence, $\lambda' = 2$, $\mu' = 1$. We have $\mathcal{L}^{8,\lambda,\mu}$ where $\lambda = 2$, $\mu = 1$. □

To conclude, the classifications appearing in the above theorems complete the classification of naturally graded quasi-filiform Leibniz algebras.

4. Computational processing

In the final section, we describe the computational process. The classification is very complex because there are a lot of computations. So it is necessary to use a program to compute the Leibniz identity and then, by a process induction, to generalize the calculations for an arbitrary finite dimension. Some examples can be seen in the following Web site, http://www.personal.us.es/jrgomez.

As the first step, we introduce the necessary conditions of the Leibniz algebras and then write the brackets of the quasi-filiform Leibniz algebra family.

```
dim =12; base = Table[x[i], {i, 1, dim}];

mu[0, x_] := 0; mu[x_, 0] := 0;
mu[x_ + y_, z_] := Simplify[mu[x, z]+mu[y,z]];
mu[z_, x_ + y_] := Simplify[mu[z, x] + mu[z, y]];
mu[x_, a_ y_] := a mu[x, y];
mu[a_ x_, y_] := a mu[x, y];

mu[x[1], x[1]] = x[2];
Module[{s}, For[s = 3, s <= dim - 1, s++,
```

```
      mu[x[s], x[1]] = x[s + 1]]];

Module[{s}, For[s = 4, s <= dim - 1, s++,
      mu[x[1], x[s]] = -x[s + 1]]];
mu[x[1], x[3]] = \[Lambda] x[2] - x[4];
mu[x[3], x[3]] = \[Mu]  x[2];
mu[x[i_],x[j_]] := 0;
```

As the second step, we compute the Leibniz identity and we obtain the relations between the parameters of the initial family.

```
lei[i_Integer, j_Integer, k_Integer] := Collect[mu[x[i], mu[x[j],x[k]]] -
    mu[mu[x[i], x[j]], x[k]] + mu[mu[x[i], x[k]], x[j]], base];
RestLei := Module[{i, j, k},For[i = 1, i <= dim, i++, For[j = 1,j<=dim,
    j++,For[k = j, k <= dim, k++, If[lei[i, j, k]==0,
        Print["Leibniz(", i, j, k, ")->",lei[i, j, k]], {},
            Print["Leibniz(", i, j, k, ")->", lei[i, j, k]]]]]]];
list1 = Select[Flatten[Table[ Coefficient[lei[i, j, k], base], {i,1, dim},
    {j,1, dim},{k, j, dim}]], ! NumberQ[#] &]; Print["null list---->", list1]
```

As the third step, we make a general change of basis and we compute the new products in the new basis:

```
y[1] = sum_{k=1}^{dim} A[k] x[k];
y[2]=Collect[mu[y[1],y[1]],base,FullSimplify];
y[3]=sum_{k=1}^{dim} B[k] x[k];

Module[{i}, For[i = 4, i <= dim,i++,
   y[i] = Collect[mu[y[i - 1], y[1]], base, FullSimplify]]];

base1 = Table[y[i], {i, 1, dim}]; base2 = Table[Y[i], {i, 1, dim}];
eqn = Table[Y[i] == y[i], {i, 1, dim}];
 resolution =Solve[eqn, base][[1]];
For[u = 1, u <= dim, u++,
  For[v = 1, v <= dim, v++,
    product[u_, v_] :=
      Collect[Collect[mu[y[u], y[v]], base, Simplify] /. resolution,
       base2,FullSimplify]]]

Module[{u, v}, For[u = 1, u <= dim, u++,
 For[v = 1, v <= dim, v++,
  If[! NumberQ[product[u, v]], Print["[", T[u], ",", T[v], "]=",
          product[u, v]]]]]];
```

Finally, we compute the new parameter and then we discuss the nullity invariants.

The classification of nul-filiform naturally graded Leibniz algebras and filiform naturally graded Leibniz algebra is known and by this work we complete the classification of naturally graded quasi-filiform Leibniz algebras.

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