

Naturally graded $(n - 3)$ -filiform Leibniz algebras

L.M. Camacho ^{a,*}, J.R. Gómez ^a, B.A. Omirov ^b

^a *Dpto. Matemática Aplicada I, Universidad de Sevilla, Avda. Reina Mercedes, s/n, 41012 Sevilla, Spain*

^b *Institute of Mathematics and Information Technologies, Uzbekistan Academy of Science, F. Hodjaev str. 29, 100125 Tashkent, Uzbekistan*

A B S T R A C T

Keywords:

Lie algebra

Leibniz algebra

Nilpotence

Natural gradation

Characteristic sequence

p -Filiformlicity

Naturally graded nilpotent p -filiform Leibniz algebras are studied for $p \geq n - 4$, where n is the dimension of the algebra. Using linear algebra methods we describe the naturally graded $(n - 3)$ -filiform Leibniz algebras.

1. Introduction and preliminaries

In [11], Loday introduces the definition of Leibniz algebra as a vector space over a field with a multiplication $[-, -]$ which satisfies the Leibniz identity,

$$[x, [y, z]] = [[x, y], z] - [[x, z], y].$$

It should be noted that if a Leibniz algebra satisfies the identity $[x, x] = 0$, the Leibniz and the Jacobi identities coincide. Therefore, Leibniz algebras are a “non-antisymmetric” generalization of Lie algebras.

* Corresponding author.

E-mail addresses: lcamacho@us.es (L.M. Camacho), jrgomez@us.es (J.R. Gómez), omirovb@mail.ru (B.A. Omirov).

The natural gradations of nilpotent Lie and Leibniz algebras give some useful information about their properties in the general case, that is, without restriction on the gradation. Such a gradation is important for the study of cohomology groups for the algebras considered because it induces their corresponding gradation [10,13].

Naturally graded n -dimensional Lie algebras have been studied by several authors [2,9]. In n -dimensional Leibniz algebras, naturally graded 2-filiform and p -filiform with $p \geq n - 4$ are known (see [3,4]).

Several authors have studied the cohomological and structural properties of Leibniz algebras [1,5–8,12].

In the present paper all spaces and algebras are considered over the field of complex numbers. We omit the products which are equal to zero, for convenience. Also we shall not consider the algebras which are the direct sum of algebras of less dimension (such algebras are called split algebras).

Given an arbitrary Leibniz algebra L , we define the lower central series:

$$L^1 = L, \quad L^{k+1} = [L^k, L], \quad k \geq 1.$$

An algebra L is called *nilpotent* if there exists $s \in \mathbb{N}$ such that $L^s = 0$. The minimal number s is called the index of nilpotency or nilindex.

Given any element x of the Leibniz algebra L , we define the right multiplication operator $R_x : L \rightarrow L$ as $R_x(y) = [y, x]$.

Let x be a generator of the algebra L and R_x be the nilpotent right operator. Thanks to the complex number field we can use the fundamental result of linear algebra on Jordan form of linear transformations. We define the decreasing sequence $C(x) = (n_1, n_2, \dots, n_k)$ that consists of the dimensions of the Jordan blocks of R_x in lexicographic order, i.e. $C(x) = (n_1, n_2, \dots, n_k) \leq C(y) = (m_1, m_2, \dots, m_s) \Leftrightarrow$ there exists $i \in \mathbb{N}$ such that $n_j = m_j$ for any $j < i$ and $n_i < m_i$.

The sequence $C(L) = \max_{x \in L \setminus L^2} C(x)$ is called the *characteristic sequence* of the Leibniz algebra L .

According to the Lie algebras, the characteristic sequence is invariant by isomorphisms [1].

A Leibniz algebra L is called *p -filiform* if $C(L) = (n - p, \underbrace{1, \dots, 1}_p)$, where $p \geq 0$.

Note that the above definition agrees with the definition of p -filiform Lie algebras when $p > 0$ [2]. Since Lie algebras are not singly-generated algebras, the notion of 0-filiform algebra for Lie algebras has no sense.

The sets $\mathcal{R}(L) = \{x \in L : [y, x] = 0 \text{ for all } y \in L\}$ and $Center(L) = \{a \in L : [a, L] = [L, a] = 0\}$ are called the *right annihilator* and *center* of L , respectively.

Given an n -dimensional p -filiform Leibniz algebra L , put $L_i = L^i/L^{i+1}$, $1 \leq i \leq n - p$ and $grL = L_1 \oplus L_2 \oplus \dots \oplus L_{n-p}$. Then $[L_i, L_j] \subseteq L_{i+j}$ and we obtain the graded algebra grL . If grL and L are isomorphic, $grL \cong L$, we say that L is naturally graded.

We shall use the expression “graded algebra” instead of “naturally graded algebra” for convenience.

The list of graded p -filiform Lie algebras ($1 \leq p \leq n - 1$) can be found in [2]. For this reason, we only study the non Lie Leibniz algebras.

In this work, we focus our attention on graded $(n - 3)$ -filiform non Lie Leibniz algebras where n is the dimension of the algebra.

The graded p -filiform Leibniz algebras with $0 \leq p \leq 2$ are known (see [1,3]). In [4], the authors study the graded p -filiform Leibniz algebras with $p \geq n - 4$.

Since the description of all nilpotent Leibniz algebras is an unsolvable task (even in case of Lie algebras), we focus our attention on graded algebras with some restriction on their characteristic sequences.

2. Graded p -filiform Leibniz algebras

Let L be a graded p -filiform n -dimensional Leibniz algebra. Then, there exists a basis $\{e_1, e_2, \dots, e_{n-p}, f_1, \dots, f_p\}$ such that $e_1 \in L \setminus L^2$ and $C(e_1) = (n - p, \underbrace{1, \dots, 1}_{p\text{-times}})$. From the definition of characteristic

sequence, the operator R_{e_1} in the Jordan form has one block J_{n-p} of size $n-p$ and p blocks J_1 (where $J_1 = \{0\}$) of size one.

We can reduce the study to the following two possibilities of Jordan forms of the matrix R_{e_1} (see [4]):

$$\begin{pmatrix} J_{n-p} & 0 & 0 & \cdots & 0 \\ 0 & J_1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & J_1 \end{pmatrix}, \begin{pmatrix} J_1 & 0 & 0 & \cdots & 0 \\ 0 & J_{n-p} & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & J_1 \end{pmatrix}.$$

A p -filiform Leibniz algebra L is said to be of first type (type I) if the operator R_{e_1} has the first form and of second type (type II) in the other case.

It should be noted that a p -filiform Leibniz algebra of the first type is a non-Lie algebra. Indeed, if the Leibniz algebra is of the first type, then $[e_1, e_1] = e_2$, which contradicts the identity $[x, x] = 0$.

2.1. Classification of graded $(n-3)$ -filiform Leibniz algebras of type I

Let L be a $(n-3)$ -filiform Leibniz algebra of the first type. Then there exists a basis $\{e_1, e_2, e_3, f_1, \dots, f_{n-3}\}$ (so called adapted basis) such that

$$\begin{aligned} [e_i, e_1] &= e_{i+1}, & 1 \leq i \leq 2, \\ [f_j, e_1] &= 0, & 1 \leq j \leq n-3. \end{aligned}$$

From these products we have:

$$\langle e_1 \rangle \subseteq L_1, \quad \langle e_2 \rangle \subseteq L_2, \quad \langle e_3 \rangle \subseteq L_3.$$

However, we do not have information about the position of the elements $\{f_1, f_2, \dots, f_{n-3}\}$.

Let us denote by r_1, r_2, \dots, r_{n-3} the positions of the basic elements f_1, f_2, \dots, f_{n-3} in the natural gradation, respectively, i.e. $f_i \in L_{r_i}$ for $1 \leq i \leq n-3$. Without loss of generality one can suppose that $1 \leq r_1 \leq r_2 \leq \dots \leq r_{n-3} \leq 3$. It should be noted that $\langle e_2, e_3 \rangle \subseteq \mathcal{R}(L)$. For p -filiform Leibniz algebras, the following theorem holds.

Theorem 2.1 ([4]). *Let L be a graded p -filiform Leibniz algebra of type I. Then $r_s \leq s$ for any $s \in \{1, 2, \dots, p\}$.*

To prove the main result of this subsection we need the following lemmas.

Lemma 2.1 ([4]). *An arbitrary p -filiform Leibniz algebra of type I satisfying the property $r_i = 1$ for $1 \leq i \leq p$ is split.*

Lemma 2.2. *Let L be a n -dimensional graded $(n-3)$ -filiform Leibniz algebra of type I. If the condition $1 = r_1 = \dots = r_k < r_{k+1}$, $1 \leq k \leq n-4$ holds, we have that*

$$L_1 = \langle e_1, f_1, f_2, \dots, f_k \rangle, \quad L_2 = \langle e_2, f_{k+1}, f_{k+2}, \dots, f_{n-3} \rangle, \quad L_3 = \langle e_3 \rangle.$$

If n is odd then $k \geq \frac{n-3}{2}$ and $k \geq \frac{n-2}{2}$ if n is even.

Proof. Let L be a Leibniz algebra of type I which satisfies the condition of the lemma. Then $f_i \in L_1$ for $1 \leq i \leq k$ and

$$L_1 = \langle e_1, f_1, \dots, f_k \rangle, \quad L_2 \supset \langle e_2 \rangle, \quad L_3 \supset \langle e_3 \rangle.$$

Let us introduce the notations

$$[e_1, f_j] = \alpha_j e_2 + \sum_{s=1}^{n-3-k} \beta_{j,s} f_{k+s}, \quad 1 \leq j \leq k.$$

From the equalities $[[e_i, f_j], e_1] = [e_{i+1}, f_j]$, we get $[e_2, f_j] = \alpha_j e_3$, $1 \leq j \leq k$. By equalities $[[f_i, f_j], e_1] = 0$, we can conclude $[f_i, f_j] = \gamma_{ij}^1 f_{k+1} + \dots + \gamma_{ij}^{n-3-k} f_{n-3}$ for $1 \leq i, j \leq k$.

We may assume that $[f_i, f_1] = 0$ for $1 \leq i \leq k$. Indeed, if we consider the element $Ae_1 + Bf_1$ with a large enough value of A and $B \neq 0$, then $\text{rank}(R_{Ae_1+Bf_1}) > 3$. Therefore, $C(Ae_1 + Bf_1)$ would be larger than $(3, 1, \dots, 1)$, i.e., we get a contradiction. Applying similar arguments, we obtain $[f_i, f_j] = 0$ for $1 \leq i, j \leq k$.

Thus, we have the following products:

$$(*) \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq 2, \\ [e_1, f_j] = \alpha_j e_2 + \sum_{s=1}^{n-3-k} \beta_{j,s} f_{k+s}, & 1 \leq j \leq k, \\ [e_2, f_j] = \alpha_j e_3, & 1 \leq j \leq k, \\ [e_3, f_j] = [f_j, e_1] = 0, & 1 \leq j \leq n-3, \\ [f_i, f_j] = 0, & 1 \leq i, j \leq k. \end{cases}$$

Since $[f_i, f_j] = [f_i, e_1] = 0$ for $1 \leq i, j \leq k$, one can assume $L_2 = \langle [e_1, e_1], [e_1, f_i] \rangle$ that is, $\dim(L_2) \leq k+1$.

Let us suppose $f_{k+1}, f_{k+2}, \dots, f_{k+s}$ lie in L_2 for $1 \leq s \leq n-3-k$. Then applying the arguments used for $[f_i, f_1]$, we obtain $[f_{k+i}, f_j] = \gamma_{ij} e_3$ for $1 \leq i \leq s, 1 \leq j \leq k$. Now using $(*)$ we can conclude $L_3 = \langle e_3 \rangle$ and $k+s = n-3$.

Thus,

$$L_1 = \langle e_1, f_1, f_2, \dots, f_k \rangle, \quad L_2 = \langle e_2, f_{k+1}, f_{k+2}, \dots, f_{n-3} \rangle, \quad L_3 = \langle e_3 \rangle. \quad \square$$

The following theorem presents the classification of graded non-split $(n-3)$ -filiform Leibniz algebras of type I.

Theorem 2.2. Any n -dimensional graded non-split $(n-3)$ -filiform Leibniz algebra of type I is isomorphic to one of the following pairwise non-isomorphic algebras:
 n is odd:

$$M_t : \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq 2, \\ [e_1, f_j] = f_{\frac{n-3}{2}+j}, & 1 \leq j \leq t, \\ [e_1, f_{t+1}] = e_2 + f_{\frac{n-3}{2}+t+1}, \\ [e_1, f_j] = f_{\frac{n-3}{2}+j}, & t+2 \leq j \leq \frac{n-3}{2}, \\ [e_2, f_{t+1}] = e_3, \\ [f_{\frac{n-3}{2}+j}, f_j] = e_3, & 1 \leq j \leq t, 0 \leq t \leq \frac{n-3}{2}. \end{cases}$$

$$P_t^1(\alpha) : \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq 2, \\ [e_1, f_1] = \alpha e_2 + f_{\frac{n-3}{2}+1}, \\ [e_1, f_2] = \beta e_2 + f_{\frac{n-3}{2}+2}, \\ [e_1, f_j] = f_{\frac{n-3}{2}+j}, & 3 \leq j \leq \frac{n-3}{2}, \\ [e_2, f_1] = \alpha e_3, & \alpha \in \mathbb{C}, \\ [e_2, f_2] = \beta e_3, & \alpha^2 + \beta^2 = 0, \\ [f_{\frac{n-3}{2}+j}, f_j] = e_3, & 1 \leq j \leq t, 0 \leq t \leq \frac{n-3}{2}, \end{cases}$$

where $P_t^1(\alpha)$ is isomorphic to $P_t^1(-\alpha)$.

$$P_t^2(\alpha) : \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq 2, \\ [e_1, f_1] = \alpha e_2 + f_{\frac{n-3}{2}+1}, & \alpha \in \mathbb{C} \setminus \{0\}, \\ [e_1, f_j] = f_{\frac{n-3}{2}+j}, & 2 \leq j \leq \frac{n-3}{2}, \\ [e_2, f_1] = \alpha e_3, \\ [f_{\frac{n-3}{2}+j}, f_j] = e_3, & 1 \leq j \leq t, 0 \leq t \leq \frac{n-3}{2}, \end{cases}$$

where $P_t^2(\alpha)$ is isomorphic to $P_t^2(-\alpha)$.

n is even:

$$N_t : \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq 2, \\ [e_1, f_j] = f_{\lfloor \frac{n-3}{2} \rfloor + 1 + j}, & 1 \leq j \leq \lfloor \frac{n-3}{2} \rfloor, \\ [e_1, f_{\lfloor \frac{n-3}{2} \rfloor + 1}] = e_2, \\ [e_2, f_{\lfloor \frac{n-3}{2} \rfloor + 1}] = e_3, \\ [f_{\lfloor \frac{n-3}{2} \rfloor + 1 + i}, f_j] = e_3, & 1 \leq i \leq t, 0 \leq t \leq \lfloor \frac{n-3}{2} \rfloor. \end{cases}$$

Proof. Due to Lemmas 2.1 and 2.2 we have

$$L_1 = \langle e_1, f_1, f_2, \dots, f_k \rangle, \quad L_2 = \langle e_2, f_{k+1}, f_{k+2}, \dots, f_{n-3} \rangle, \quad L_3 = \langle e_3 \rangle$$

and

$$(**) \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq 2, \\ [e_1, f_j] = \alpha_j e_2 + \sum_{s=1}^{n-3-k} \beta_{j,s} f_{k+s}, & 1 \leq j \leq k, \\ [e_2, f_j] = \alpha_j e_3, & 1 \leq j \leq k, \\ [e_3, f_j] = [f_j, e_1] = 0, & 1 \leq j \leq n-3, \\ [f_i, f_j] = 0, & 1 \leq i, j \leq k, \\ [f_{k+i}, f_j] = \gamma_{ij} e_3, & 1 \leq i \leq n-3-k, 1 \leq j \leq k. \end{cases}$$

Consider **the case n odd.**

If $k > \frac{n-3}{2}$, we obtain $[e_1, f'_k] = 0$, putting $f'_k = a_1 f_1 + a_2 f_2 + \dots + a_k f_k$ for appropriate values of a_1, a_2, \dots, a_k , we get a split algebra. The same happens if $k < \frac{n-3}{2}$. Thus, we have $k = \frac{n-3}{2}$.

If $\beta_{j,1} = 0$ for $1 \leq j \leq \frac{n-3}{2}$, then $f_{\frac{n-3}{2}+1} \notin L_2$, which is a contradiction with the condition $L_2 = \langle e_2, f_{\frac{n-3}{2}+1}, f_{\frac{n-3}{2}+2}, \dots, f_{n-3} \rangle$. Hence, $\beta_{j_0,1} \neq 0$ for some $j_0 \in \{1, \dots, \frac{n-3}{2}\}$. Making the change of variables $f'_1 = f_{j_0}$, $f'_{j_0} = f_1$ and $f'_{\frac{n-3}{2}+1} = \sum_{s=1}^{\frac{n-3}{2}} \beta_{j_0,s} f_{\frac{n-3}{2}+s}$, we obtain

$$[e_1, f'_1] = \alpha'_1 e_2 + \sum_{s=1}^{\frac{n-3}{2}} \beta'_{1,s} f_{\frac{n-3}{2}+s} = \alpha'_1 e_2 + f'_{\frac{n-3}{2}+1},$$

where $\alpha'_1 = \alpha_{j_0}$, $\beta'_{1,s} = \beta_{j_0,s}$.

Accordingly, for f_j with $2 \leq j \leq \frac{n-3}{2}$, we can obtain

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq 2, \\ [e_1, f_j] = \alpha_j e_2 + f_{\frac{n-3}{2}+j}, & 1 \leq j \leq \frac{n-3}{2}, \\ [e_2, f_j] = \alpha_j e_3, & 1 \leq j \leq \frac{n-3}{2}, \\ [e_3, f_j] = [f_j, e_1] = 0, & 1 \leq j \leq n-3, \\ [f_i, f_j] = 0, & 1 \leq i, j \leq \frac{n-3}{2}, \\ [f_{\frac{n-3}{2}+i}, f_j] = \gamma_{ij} e_3, & 1 \leq i, j \leq \frac{n-3}{2}. \end{cases}$$

By the following products $[f_{\frac{n-3}{2}+i}, f_j] = [[e_1, f_i] - \alpha_i e_2, f_j] = [[e_1, f_i], f_j] - \alpha_i [e_2, f_j] = [e_1, [f_i, f_j]] + [[e_1, f_j], f_i] - \alpha_i \alpha_j e_3$ with $1 \leq i, j \leq \frac{n-3}{2}$ we obtain that $\gamma_{ij} = \gamma_{ji}$ with $1 \leq i, j \leq \frac{n-3}{2}$.

Now consider the matrix consisting of the element γ_{ij} , where $1 \leq i, j \leq \frac{n-3}{2}$. Let t be a natural number of rows of the matrix (γ_{ij}) in which there exists a non zero element. If all $\gamma_{ij} = 0$, then we assume $t = 0$. By corresponding renumbering of the basis elements $\{f_1, \dots, f_{\frac{n-3}{2}}\}$ we can suppose that the first t rows have non zero elements. It means that in the products of the algebra we can suppose that for each i from $0 \leq i \leq t$ there exists some j such that $\gamma_{ij} \neq 0$.

If $\gamma_{1,1} \neq 0$ we can make the following change of basis:

$$\begin{cases} f'_1 = \frac{1}{\sqrt{\gamma_{1,1}}}f_1, \\ f'_j = \gamma_{1,j}f_j - \gamma_{1,j}f_1, & 2 \leq j \leq \frac{n-3}{2}, \\ f'_{\frac{n-3}{2}+1} = \frac{1}{\sqrt{\gamma_{1,1}}}f_{\frac{n-3}{2}+1}, \\ f'_{\frac{n-3}{2}+i} = \gamma_{1,i}f_{\frac{n-3}{2}+i} - \gamma_{1,i}f_{\frac{n-3}{2}+1}, & 2 \leq i \leq \frac{n-3}{2}, \end{cases}$$

and we have $\gamma_{1,1} = 1$, $\gamma_{1,i} = 0$ if $2 \leq i \leq \frac{n-3}{2}$.

If $\gamma_{1,1} = 0$ and there exists i with $2 \leq i \leq \frac{n-3}{2}$ such that $\gamma_{1,i} \neq 0$ then we can obtain $\gamma_{1,1} = 1$ and $\gamma_{1,i} = 0$ with $2 \leq i \leq \frac{n-3}{2}$, from the following change

$$\begin{cases} f'_1 = Af_1 + f_i, \\ f'_{\frac{n-3}{2}+1} = Af_{\frac{n-3}{2}+1} + f_{\frac{n-3}{2}+i}, \\ f'_j = f_j, & 2 \leq j \leq n-3, j \notin \left\{1, \frac{n-3}{2} + 1\right\}, \end{cases}$$

with $A \neq -\frac{\gamma_{1,i}}{\gamma_{1,i}}$, $A \neq 0$.

Thus,

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq 2, \\ [e_1, f_j] = \alpha_j e_2 + f_{\frac{n-3}{2}+j}, & 1 \leq j \leq \frac{n-3}{2}, \\ [e_2, f_j] = \alpha_j e_3, & 1 \leq j \leq \frac{n-3}{2}, \\ [e_3, f_j] = [f_j, e_1] = 0, & 1 \leq j \leq n-3, \\ [f_i, f_j] = 0, & 1 \leq i, j \leq \frac{n-3}{2}, \\ \left[f_{\frac{n-3}{2}+1}, f_1 \right] = e_3, \\ \left[f_{\frac{n-3}{2}+1}, f_j \right] = 0, & 2 \leq j \leq \frac{n-3}{2}, \\ \left[f_{\frac{n-3}{2}+i}, f_j \right] = \gamma_{ij} e_3, & 2 \leq i \leq \frac{n-3}{2}, 1 \leq j \leq \frac{n-3}{2}. \end{cases}$$

If $\gamma_{2,2} \neq 0$, then we can obtain $\gamma_{2,2} = 1$ and $\gamma_{2,i} = 0$ with $3 \leq i \leq \frac{n-3}{2}$, from the following change of basis:

$$\begin{cases} f'_2 = \frac{1}{\sqrt{\gamma_{2,2}}}f_2, \\ f'_j = \gamma_{2,j}f_j - \gamma_{2,j}f_2, & 3 \leq j \leq \frac{n-3}{2}, \\ f'_{\frac{n-3}{2}+2} = \frac{1}{\sqrt{\gamma_{2,2}}}f_{\frac{n-3}{2}+2}, \\ f'_{\frac{n-3}{2}+i} = \gamma_{2,i}f_{\frac{n-3}{2}+i} - \gamma_{2,i}f_{\frac{n-3}{2}+2}, & 3 \leq i \leq \frac{n-3}{2}. \end{cases}$$

If $\gamma_{2,2} = 0$ and there exists i with $3 \leq i \leq \frac{n-3}{2}$ such that $\gamma_{2,i} \neq 0$, then we can obtain $\gamma_{2,2} = 1$ and $\gamma_{2,i} = 0$ with $3 \leq i \leq \frac{n-3}{2}$, from the following change

$$\begin{cases} f'_2 = Af_2 + f_i, \\ f'_{\frac{n-3}{2}+2} = Af_{\frac{n-3}{2}+2} + f_{\frac{n-3}{2}+i}, \\ f'_j = f_j, & 1 \leq j \leq n-3, j \notin \left\{2, \frac{n-3}{2} + 2\right\}, \end{cases}$$

with $A \neq -\frac{\gamma_{2,i}}{\gamma_{2,i}}$, $A \neq 0$.

If we repeat this process with $\gamma_{i,i}$, $i = 3, \dots, \frac{n-3}{2}$, finally we obtain the following family:

$$L(t) : \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq 2, \\ [e_1, f_j] = \alpha_j e_2 + f_{\frac{n-3}{2}+j}, & 1 \leq j \leq \frac{n-3}{2}, \\ [e_2, f_j] = \alpha_j e_3, & 1 \leq j \leq \frac{n-3}{2}, \\ \left[f_{\frac{n-3}{2}+j}, f_j \right] = e_3, & 1 \leq j \leq t, 0 \leq t \leq \frac{n-3}{2}. \end{cases}$$

If $\gamma_{ij} = 0$ with $1 \leq i, j \leq \frac{n-3}{2}$, then we have $L(0)$. Therefore, $L(t) \not\cong L(t')$ with $t \neq t'$ because $\dim(\text{Center}(L(t))) = \frac{n-3}{2} - t + 1$.

Let $\{\alpha_{t+1}, \alpha_{t+2}, \dots, \alpha_{\frac{n-3}{2}}\}$ be elements of \mathbb{C} . If there exists i_0 with $1 \leq i_0 \leq \frac{n-3}{2} - t$ such that $\alpha_{t+i_0} \neq 0$, then one can suppose that $\alpha_{t+1} = 1$ and $\alpha_{t+j} = 0$ with $2 \leq j \leq \frac{n-3}{2} - t$, by the following change of basis:

Case $i_0 \neq 1$

$$\begin{aligned} f'_{t+1} &= \frac{1}{\alpha_{t+i_0}} f_{t+i_0}, \\ f'_{\frac{n-3}{2}+t+1} &= \frac{1}{\alpha_{t+i_0}} f_{\frac{n-3}{2}+t+i_0}, \\ f'_{t+j} &= \alpha_{t+i_0} f_{t+j} - \alpha_{t+j} f_{t+i_0}, & 2 \leq j \leq \frac{n-3}{2} - t, j \neq i_0, \\ f'_{\frac{n-3}{2}+t+j} &= \alpha_{t+i_0} f_{\frac{n-3}{2}+t+j} - \alpha_{t+j} f_{\frac{n-3}{2}+t+i_0}, & 2 \leq j \leq \frac{n-3}{2} - t, j \neq i_0, \\ f'_{t+i_0} &= \alpha_{t+i_0} f_{t+1} - \alpha_{t+1} f_{t+i_0}, \\ f'_{\frac{n-3}{2}+t+i_0} &= \alpha_{t+i_0} f_{\frac{n-3}{2}+t+1} - \alpha_{t+1} f_{\frac{n-3}{2}+t+i_0}. \end{aligned}$$

Case $i_0 = 1$

$$\begin{aligned} f'_{t+1} &= \frac{1}{\alpha_{t+1}} f_{t+1}, \\ f'_{\frac{n-3}{2}+t+1} &= \frac{1}{\alpha_{t+1}} f_{\frac{n-3}{2}+t+1}, \\ f'_{t+j} &= \alpha_{t+1} f_{t+j} - \alpha_{t+j} f_{t+1}, & 2 \leq j \leq \frac{n-3}{2} - t, \\ f'_{\frac{n-3}{2}+t+j} &= \alpha_{t+1} f_{\frac{n-3}{2}+t+j} - \alpha_{t+j} f_{\frac{n-3}{2}+t+1}, & 2 \leq j \leq \frac{n-3}{2} - t \end{aligned}$$

and we obtain $L(t, 1)$. If $\alpha_{t+i_0} = 0$ with $1 \leq i_0 \leq \frac{n-3}{2} - t$, then we have $L(t, 0)$ and $L(0)$ if $t = 0$. These families do not intersect:

$$L(t, 1) : \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq 2, \\ [e_1, f_j] = \alpha_j e_2 + f_{\frac{n-3}{2}+j}, & 1 \leq j \leq t, \\ [e_1, f_{t+1}] = e_2 + f_{\frac{n-3}{2}+t+1}, \\ [e_1, f_j] = f_{\frac{n-3}{2}+j}, & t+2 \leq j \leq \frac{n-3}{2}, \\ [e_2, f_j] = \alpha_j e_3, & 1 \leq j \leq t, \\ [e_2, f_{t+1}] = e_3, \\ [e_2, f_j] = 0, & t+2 \leq j \leq \frac{n-3}{2}, \\ [f_{\frac{n-3}{2}+j}, f_j] = e_3, & 1 \leq j \leq t, 1 \leq t \leq \frac{n-3}{2}. \end{cases}$$

$$L(t, 0) : \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq 2, \\ [e_1, f_j] = \alpha_j e_2 + f_{\frac{n-3}{2}+j}, & 1 \leq j \leq t, \\ [e_1, f_j] = f_{\frac{n-3}{2}+j}, & t+1 \leq j \leq \frac{n-3}{2}, \\ [e_2, f_j] = \alpha_j e_3, & 1 \leq j \leq t, \\ [f_{\frac{n-3}{2}+j}, f_j] = e_3, & 1 \leq j \leq t, 1 \leq t \leq \frac{n-3}{2}. \end{cases}$$

We have that $\dim(\text{Center}(L(t, 0)^2)) = \dim(\text{Center}(L(t, 1)^2)) + 1$.

Case 1: If we consider the first family, $L(t, 1)$, then it is easy to check that the following change of basis leads to the algebra M_t .

$$\begin{aligned} f'_i &= \alpha_i f_{t+1} - f_i, & 1 \leq i \leq t, \\ f'_{\frac{n-3}{2}+i} &= \alpha_i f_{\frac{n-3}{2}+t+1} - f_{\frac{n-3}{2}+i}, & 1 \leq i \leq t. \end{aligned}$$

Case 2: If we consider the second family, $L(t, 0)$, then we make the general change of basis (it is sufficient to change the generators):

$$e'_1 = A_1 e_1 + A_2 e_2 + A_3 e_3 + \sum_{j=1}^{n-3} B_j f_j,$$

$$f'_i = C_{1i} e_1 + C_{2i} e_2 + C_{3i} e_3 + \sum_{j=1}^{n-3} D_{ij} f_j, \quad 1 \leq i \leq \frac{n-3}{2}.$$

Since the vectors $e_3, f_{\frac{n-3}{2}+t+1}, f_{\frac{n-3}{2}+t+2}, \dots, f_{n-3}$ are in the center of the algebra, we can suppose that $A_3 = C_{3i} = 0$ and $B_{\frac{n-3}{2}+j} = D_{i, \frac{n-3}{2}+j} = 0$ with $t+1 \leq j \leq \frac{n-3}{2}$ and $1 \leq i \leq \frac{n-3}{2}$. Thus, if we compute the products, then we obtain:

$$e'_1 = A_1 e_1 + A_2 e_2 + \sum_{j=1}^{\frac{n-3}{2}+t} B_j f_j,$$

$$e'_2 = A_1 \left(A_1 + \sum_{j=1}^t B_j \alpha_j \right) e_2 + \left[A_2 \left(A_1 + \sum_{j=1}^t B_j \alpha_j \right) + \sum_{j=1}^t B_{\frac{n-3}{2}+j} B_j \right] e_3 + A_1 \sum_{j=1}^{\frac{n-3}{2}} B_j f_{\frac{n-3}{2}+j},$$

$$e'_3 = A_1 \left[\left(A_1 + \sum_{j=1}^t B_j \alpha_j \right)^2 + \sum_{j=1}^t B_j^2 \right] e_3,$$

$$f'_i = C_{1i} e_1 + C_{2i} e_2 + \sum_{j=1}^{\frac{n-3}{2}+t} D_{ij} f_j, \quad 1 \leq i \leq \frac{n-3}{2},$$

$$f'_{\frac{n-3}{2}+i} = A_1 \left[C_{1i} + \sum_{j=1}^t D_{ij} \alpha_j - \alpha'_i \left(A_1 + \sum_{j=1}^t B_j \alpha_j \right) \right] e_2 + \left[A_2 C_{1i} + A_2 \sum_{j=1}^t D_{ij} \alpha_j \right. \\ \left. + \sum_{j=1}^t B_{\frac{n-3}{2}+j} D_{ij} - \alpha'_i \left(A_2 \left(A_1 + \sum_{j=1}^t B_j \alpha_j \right) + \sum_{j=1}^t B_{\frac{n-3}{2}+j} B_j \right) \right] e_3 \\ + A_1 \sum_{j=1}^{\frac{n-3}{2}} (D_{ij} - \alpha'_i B_j) f_{\frac{n-3}{2}+j}, \quad 1 \leq i \leq t,$$

$$f'_{\frac{n-3}{2}+i} = \left[A_1 C_{1i} + A_1 \sum_{j=1}^t D_{ij} \alpha_j \right] e_2 + \left[A_2 C_{1i} + A_2 \sum_{j=1}^t D_{ij} \alpha_j + \sum_{j=1}^t B_{\frac{n-3}{2}+j} D_{ij} \right] e_3 \\ + A_1 \sum_{j=1}^{\frac{n-3}{2}} D_{ij} f_{\frac{n-3}{2}+j}, \quad t+1 \leq i \leq \frac{n-3}{2}$$

and the following restrictions:

1. $C_{1i} = 0, 1 \leq i \leq \frac{n-3}{2},$
2. $\left(A_1 + \sum_{j=1}^t B_j \alpha_j \right) \sum_{j=1}^t D_{ij} \alpha_j + \sum_{j=1}^t B_j D_{ij} = 0, t+1 \leq i \leq \frac{n-3}{2},$
3. $\left(A_1 + \sum_{j=1}^t B_j \alpha_j \right) \sum_{j=1}^t D_{ij} \alpha_j + \sum_{j=1}^t B_j D_{ij} = \alpha'_i \left[\left(A_1 + \sum_{j=1}^t B_j \alpha_j \right)^2 + \sum_{j=1}^t B_j^2 \right], 1 \leq i \leq t,$
4. $\left(\sum_{j=1}^t D_{ij} \alpha_j - \alpha'_i \left(A_1 + \sum_{j=1}^t B_j \alpha_j \right) \right) \sum_{j=1}^t D_{ij} \alpha_j + \sum_{j=1}^t (D_{ij} - \alpha'_i B_j) D_{ij} = \left(A_1 + \sum_{j=1}^t B_j \alpha_j \right)^2 + \sum_{j=1}^t B_j^2, 1 \leq i \leq t,$

5. $\left(\sum_{j=1}^t D_{ij}\right)^2 + \sum_{j=1}^t D_{ij}^2 = 0$, $t+1 \leq i \leq \frac{n-3}{2}$, $i \neq j$,
6. $C_{2i} \left(A_1 + \sum_{j=1}^t B_j \alpha_j\right) + \sum_{j=1}^t D_{i, \frac{n-3}{2}+j} B_j = 0$, $1 \leq i \leq \frac{n-3}{2}$,
7. $C_{2i} \sum_{m=1}^t D_{jm} \alpha_m + \sum_{m=1}^t D_{i, \frac{n-3}{2}+m} D_{jm} = 0$, $1 \leq i, j \leq \frac{n-3}{2}$,
8. $\left(\sum_{m=1}^t D_{im} \alpha_m - \alpha'_i (A_1 + \sum_{m=1}^t B_m \alpha_m)\right) \sum_{m=1}^t D_{jm} \alpha_m + \sum_{m=1}^t (D_{im} - \alpha'_i B_m) D_{jm} = 0$,
 $1 \leq i \leq t$, $1 \leq j \leq \frac{n-3}{2}$, $i \neq j$,
9. $\left(\sum_{m=1}^t D_{im} \alpha_m\right) \left(\sum_{m=1}^t D_{jm} \alpha_m\right) + \sum_{m=1}^t D_{im} D_{jm} = 0$, $t+1 \leq i \leq \frac{n-3}{2}$, $1 \leq j \leq \frac{n-3}{2}$, $i \neq j$.

Since A_2 , C_{2i} do not appear in the expressions of α'_i , we can suppose that $A_2 = C_{2i} = 0$, $1 \leq i \leq \frac{n-3}{2}$. Accordingly,

$$D_{ij} = \begin{cases} 0 & t+1 \leq i \leq \frac{n-3}{2}, \text{ or } t+1 \leq j \leq \frac{n-3}{2} \text{ and } i \neq j, \\ 1 & i = j \text{ and } t+1 \leq i \leq \frac{n-3}{2}. \end{cases}$$

Taking into account the above restrictions, we have the following expressions:

$$e'_1 = A_1 e_1 + \sum_{j=1}^{\frac{n-3}{2}+t} B_j f_j,$$

$$e'_2 = A_1 \left(A_1 + \sum_{j=1}^t B_j \alpha_j\right) e_2 + \left(\sum_{j=1}^t B_{\frac{n-3}{2}+j} B_j\right) e_3 + A_1 \sum_{j=1}^{\frac{n-3}{2}} B_j f_{\frac{n-3}{2}+j},$$

$$e'_3 = A_1 \left[\left(A_1 + \sum_{j=1}^t B_j \alpha_j\right)^2 + \sum_{j=1}^t B_j^2 \right] e_3,$$

$$f'_i = \sum_{j=1}^t D_{ij} f_j, \quad 1 \leq i \leq t,$$

$$f'_i = f_i, \quad t+1 \leq i \leq \frac{n-3}{2},$$

$$f'_{\frac{n-3}{2}+i} = A_1 \left[\sum_{j=1}^t D_{ij} \alpha_j - \alpha'_i \left(A_1 + \sum_{j=1}^t B_j \alpha_j\right) \right] e_2 + \left[\sum_{j=1}^t B_{\frac{n-3}{2}+j} D_{ij} - \alpha'_i \left(\sum_{j=1}^t B_{\frac{n-3}{2}+j} B_j\right) \right] e_3 \\ + A_1 \sum_{j=1}^t (D_{ij} - \alpha'_i B_j) f_{\frac{n-3}{2}+j}, \quad 1 \leq i \leq t,$$

$$f'_{\frac{n-3}{2}+i} = A_1 f_{\frac{n-3}{2}+i}, \quad t+1 \leq i \leq \frac{n-3}{2}$$

and the restrictions (1)–(9) can be rewritten:

- 1'. $\left(A_1 + \sum_{j=1}^t B_j \alpha_j\right) \sum_{j=1}^t D_{ij} \alpha_j + \sum_{j=1}^t B_j D_{ij} = \alpha'_i \left[\left(A_1 + \sum_{j=1}^t B_j \alpha_j\right)^2 + \sum_{j=1}^t B_j^2 \right]$, $1 \leq i \leq t$,
- 2'. $\left(\sum_{j=1}^t D_{ij} \alpha_j - \alpha'_i (A_1 + \sum_{j=1}^t B_j \alpha_j)\right) \sum_{j=1}^t D_{ij} \alpha_j + \sum_{j=1}^t (D_{ij} - \alpha'_i B_j) D_{ij} = \left(A_1 + \sum_{j=1}^t B_j \alpha_j\right)^2 + \sum_{j=1}^t B_j^2$, $1 \leq i \leq t$,
- 3'. $\left(\sum_{m=1}^t D_{im} \alpha_m - \alpha'_i (A_1 + \sum_{m=1}^t B_m \alpha_m)\right) \sum_{m=1}^t D_{jm} \alpha_m + \sum_{m=1}^t (D_{im} - \alpha'_i B_m) D_{jm} = 0$,
 $1 \leq i \leq t$, $1 \leq j \leq t$, $i \neq j$.

We only have to compute the restriction on the determinant of change of basis. Since we have that

$$e'_3 = A_1 \left[\left(A_1 + \sum_{j=1}^t B_j \alpha_j \right)^2 + \sum_{j=1}^t B_j^2 \right] e_3,$$

it implies that $A_1 \left[\left(A_1 + \sum_{j=1}^t B_j \alpha_j \right)^2 + \sum_{j=1}^t B_j^2 \right] \neq 0$.

If we observe the change of basis and using the properties of the determinant, it is easy to obtain:

$$\begin{aligned} & A_1^2 \left[\left(A_1 + \sum_{j=1}^t B_j \alpha_j \right)^2 + \sum_{j=1}^t B_j^2 \right] \begin{vmatrix} D_{11} & \cdots & D_{1t} \\ \vdots & & \vdots \\ D_{t1} & \cdots & D_{tt} \end{vmatrix} \\ & \times \begin{vmatrix} A_1 \left(A_1 + \sum_{j=1}^t B_j \alpha_j \right) & A_1 B_1 & \cdots \\ A_1 \left[\sum_{j=1}^t D_{1j} \alpha_j - \alpha'_1 \left(A_1 + \sum_{j=1}^t B_j \alpha_j \right) \right] & A_1 (D_{11} - \alpha'_1 B_1) & \cdots \\ \vdots & \vdots & \vdots \\ A_1 \left[\sum_{j=1}^t D_{tj} \alpha_j - \alpha'_t \left(A_1 + \sum_{j=1}^t B_j \alpha_j \right) \right] & A_1 (D_{t1} - \alpha'_t B_1) & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \vdots \\ 0 & 0 & \cdots \end{vmatrix} \\ & \begin{vmatrix} A_1 B_t & A_1 B_{t+1} & \cdots & A_1 B_{\frac{n-3}{2}} \\ A_1 (D_{1t} - \alpha'_1 B_t) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A_1 (D_{tt} - \alpha'_t B_t) & 0 & \cdots & 0 \\ 0 & A_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & A_1 \end{vmatrix} \\ & = A_1^{\frac{n-3}{2}+3} \left[\left(A_1 + \sum_{j=1}^t B_j \alpha_j \right)^2 + \sum_{j=1}^t B_j^2 \right] \begin{vmatrix} D_{11} & \cdots & D_{1t} \\ \vdots & & \vdots \\ D_{t1} & \cdots & D_{tt} \end{vmatrix} \\ & \times \begin{vmatrix} A_1 + \sum_{j=1}^t B_j \alpha_j & B_1 & \cdots & B_t \\ \sum_{j=1}^t D_{1j} \alpha_j - \alpha'_1 \left(A_1 + \sum_{j=1}^t B_j \alpha_j \right) & D_{11} - \alpha'_1 B_1 & \cdots & D_{1t} - \alpha'_1 B_t \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{j=1}^t D_{tj} \alpha_j - \alpha'_t \left(A_1 + \sum_{j=1}^t B_j \alpha_j \right) & D_{t1} - \alpha'_t B_1 & \cdots & D_{tt} - \alpha'_t B_t \end{vmatrix} \end{aligned}$$

$$= A_1^{\frac{n-3}{2}+3} \left[\left(A_1 + \sum_{j=1}^t B_j \alpha_j \right)^2 + \sum_{j=1}^t B_j^2 \right] \begin{vmatrix} D_{11} & \cdots & D_{1t} \\ \vdots & & \vdots \\ D_{t1} & \cdots & D_{tt} \end{vmatrix} \cdot X.$$

Applying the determinant properties, we compute the value of X.

$$X = \begin{vmatrix} A_1 + \sum_{j=1}^t B_j \alpha_j & B_1 & \cdots & B_t \\ \sum_{j=1}^t D_{1j} \alpha_j & D_{11} & \cdots & D_{1t} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^t D_{tj} \alpha_j & D_{t1} & \cdots & D_{tt} \end{vmatrix}$$

$$= \begin{vmatrix} A_1 & B_1 & \cdots & B_t \\ 0 & D_{11} & \cdots & D_{1t} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & D_{t1} & \cdots & D_{tt} \end{vmatrix} + \begin{vmatrix} \sum_{j=1}^t B_j \alpha_j & B_1 & \cdots & B_t \\ \sum_{j=1}^t D_{1j} \alpha_j & D_{11} & \cdots & D_{1t} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^t D_{tj} \alpha_j & D_{t1} & \cdots & D_{tt} \end{vmatrix}$$

$$= A_1 \cdot \begin{vmatrix} D_{11} & \cdots & D_{1t} \\ \vdots & & \vdots \\ D_{t1} & \cdots & D_{tt} \end{vmatrix}.$$

Thus, the determinant is equal to:

$$A_1^{\frac{n-3}{2}+4} \left[\left(A_1 + \sum_{j=1}^t B_j \alpha_j \right)^2 + \sum_{j=1}^t B_j^2 \right] \cdot \begin{vmatrix} D_{11} & \cdots & D_{1t} \\ \vdots & & \vdots \\ D_{t1} & \cdots & D_{tt} \end{vmatrix}^2 \neq 0.$$

The general change of basis can be reduced to two changes of basis:

Case A: Let $f'_i = \sum_{j=1}^{n-3} D_{ij} f_j$ with $1 \leq i \leq \frac{n-3}{2}$ and $e'_j = e_j$ with $1 \leq j \leq 3$. The same way that the general change of basis we can suppose that:

$$D_{ij} = \begin{cases} 0 & t+1 \leq i \leq \frac{n-3}{2}, \text{ or } t+1 \leq j \leq \frac{n-3}{2} \text{ and } i \neq j, \\ 1 & i = j \text{ and } t+1 \leq i \leq \frac{n-3}{2}. \end{cases}$$

In this case, we can rewrite the restrictions (1')–(3') as follows:

$$1'' \cdot \sum_{j=1}^t D_{ij} \alpha_j = \alpha'_i, \quad 1 \leq i \leq t.$$

$$2'' \cdot \sum_{j=1}^t D_{ij}^2 = 1, \quad 1 \leq i \leq t.$$

$$3'' \cdot \sum_{m=1}^t D_{im} D_{jm} = 0, \quad 1 \leq i \leq t, \quad 1 \leq j \leq t, \quad i \neq j.$$

We have that $1 \leq t \leq \frac{n-3}{2}$. In the matrix form, we can write the conditions to new parameters by:

$$\alpha' = \begin{pmatrix} \alpha'_1 \\ \vdots \\ \alpha'_t \end{pmatrix} = \begin{pmatrix} D_{11} & \dots & D_{t1} \\ \vdots & \ddots & \vdots \\ D_{1t} & \dots & D_{tt} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_t \end{pmatrix} \Leftrightarrow \alpha' = D\alpha.$$

In the matrix form, the restrictions (1'')–(3'') can be reformulated by:

$$D \cdot D^t = D^t \cdot D = I \Rightarrow |D| = \pm 1, \text{ where } I \text{ is the unitary matrix.}$$

It is easy to prove that $\alpha'_1{}^2 + \alpha'_2{}^2 + \dots + \alpha'_t{}^2 = \alpha_1^2 + \alpha_2^2 + \dots + \alpha_t^2$.

1. Let $\alpha_1^2 + \dots + \alpha_t^2 = 0$.

(a) If $\alpha_1 = \alpha_2 = \dots = \alpha_t = 0$, then $L_{(t,0)}$ is isomorphic to $P_t^1(0)$.

(b) If there exists j_0 such that $\alpha_{j_0} \neq 0$, it can be supposed that $\alpha_1 \neq 0$.

In this case one can obtain $\alpha'_3 = \dots = \alpha'_t = 0$ (choosing $D_{j_1} = -\frac{\sum_{i=2, j \neq i}^t D_{ji} \alpha_i}{\alpha_1}$ with $3 \leq j \leq t-1$).

We only have to prove that $D \cdot D^t = I$. We have t^2 free parameters and we use $t-2$ parameters. If $D \cdot D^t = I$, then we have $\frac{t(t+1)}{2}$ equations. In fact, there exist $t^2 - t + 2$ free parameters and $\frac{t(t+1)}{2}$ equations, since $t^2 - t + 2 > \frac{t(t+1)}{2}$, the system has a solution.

We have that $\alpha_1 \neq 0$, $\alpha_2 = \pm i \alpha_1$, $\alpha_i = 0$, $3 \leq i \leq t$ and by a change of basis, one cannot obtain $\alpha'_1 = 0$.

If $\alpha_2 = \pm i \alpha_1$, we obtain $P_t^1(\alpha)$ with $\alpha \in \mathbb{C} \setminus \{0\}$.

2. Let $\alpha_1^2 + \dots + \alpha_t^2 \neq 0$. Then there exists j_0 such that $\alpha_{j_0} \neq 0$. It can be supposed that $\alpha_1 \neq 0$. In

this case we obtain $\alpha'_2 = \dots = \alpha'_t = 0$ choosing $D_{j_1} = -\frac{\sum_{i=2, j \neq i}^t D_{ji} \alpha_i}{\alpha_1}$ with $2 \leq j \leq t$. Accordingly, the system $D \cdot D^t = I$ has a solution. In this case we have $P_t^2(\alpha)$.

Case B: Let $e'_1 = A_1 e_1 + \sum_{j=1}^{n-3} B_j f_j$ and $f'_j = f_j$ with $1 \leq j \leq \frac{n-3}{2}$. We can suppose $B_j = 0$ with $\frac{n-3}{2} + t \leq j \leq n-3$. We can obtain this change by the composition of the two following changes:

B.1: Let $e'_1 = A e_1$, then $B_i = 0$, $D_{ii} = 1$, $D_{ij} = 0$, $i \neq j$ and $A \neq 0$ (by the condition of the determinant). The restrictions are the following:

1. $\alpha_i = A \alpha'_i$, $1 \leq i \leq t$.
2. $A^2 = 1$.

We have $\alpha'_i = \pm \alpha_i$ with $1 \leq i \leq t$.

B.2: Let $e'_1 = e_1 + \sum_{j=1}^{n-3} B_j f_j$, but it is sufficient to consider $e'_1 = e_1 + B_s f_s$ with $1 \leq s \leq t$ and $B_s \neq 0$. The restrictions are the following:

1. $(1 + B_s \alpha_s) \alpha_i = \alpha'_i \left[(1 + B_s \alpha_s)^2 + B_s^2 \right]$, $i \neq s$.
2. $(1 + B_s \alpha_s) \alpha_s + B_s = \alpha'_s \left[(1 + B_s \alpha_s)^2 + B_s^2 \right]$.
3. $\alpha_i^2 + 1 = (\alpha_i'^2 + 1) \left[(1 + B_s \alpha_s)^2 + B_s^2 \right]$, $1 \leq i \leq t$.
4. $\alpha_i \alpha_j = \alpha'_i \alpha'_j \left[(1 + B_s \alpha_s)^2 + B_s^2 \right]$, $1 \leq i, j \leq t$, $i \neq j$.
5. $(1 + B_s)^2 + B_s^2 \neq 0$.

We have that the expression $\alpha_s^2 + 1$ is an invariant, i.e., if $\alpha_s^2 + 1$ is zero then $\alpha_s'^2 + 1$ is zero and if $\alpha_s^2 + 1$ is not zero then $\alpha_s'^2 + 1$ is also non-zero. Thus

(a) Let $\alpha_s^2 + 1 = 0 \Rightarrow \alpha_s'^2 + 1 = 0$.

Replacing in the restrictions we have that $B_s = 0$, which is a contradiction.

(b) Let $\alpha_s^2 + 1 \neq 0 \Rightarrow \alpha_s'^2 + 1 \neq 0$.

The product $\alpha_i \alpha_s$ is an invariant expression, i.e., if $\alpha_i \alpha_s \neq 0$ then $\alpha_i' \alpha_s' \neq 0$ and if $\alpha_i \alpha_s = 0$ then $\alpha_i' \alpha_s' = 0$.

If there exists i_0 with $1 \leq i_0 \leq t$ such that $\alpha_{i_0} \neq 0$ and $i_0 \neq s$ then we have two cases:

◇ Let $\alpha_{i_0} \alpha_s = 0$. It implies that $\alpha_s = 0$ and $\alpha_{i_0}' \alpha_s' = 0$. Replacing in the restriction we have that $\alpha_i' = \frac{\alpha_i}{1+B_s^2}$, and $\alpha_s' = \frac{B_s}{1+B_s^2}$. Since $\alpha_{i_0}' \alpha_s' = 0$ and $\alpha_{i_0}' \neq 0$ we have $\alpha_s' = 0 \Rightarrow B_s = 0$, which is a contradiction.

◇ Let $\alpha_{i_0} \alpha_s \neq 0$. This implies that $\alpha_s \neq 0$ and $\alpha_{i_0}' \alpha_s' \neq 0$. We can choose $B_s = -\frac{\alpha_s}{1+\alpha_s^2}$ and we obtain $\alpha_s' = 0$, which is a contradiction.

If $\alpha_i \neq 0$ with $1 \leq i \leq t$ and $i \neq s$, then we have that $\alpha_s' = \frac{\alpha_s + B_s(\alpha_s^2 + 1)}{(1+B_s\alpha_s)^2 + B_s^2}$ and $\alpha_i' = 0$ with $1 \leq i \leq t$ and $i_0 \neq s$. Choosing $B_s = -\frac{\alpha_s}{\alpha_s^2 + 1}$ we obtain $\alpha_s' = 0$.

In these cases we obtain the same algebras or families of algebras that in the case A.

Consider **the case n even**.

In a similar way to the case n odd, we can obtain

$$\left\{ \begin{array}{ll} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq 2, \\ [e_1, f_j] = \alpha_j e_2 + f_{\lfloor \frac{n-3}{2} \rfloor + 1 + j}, & 1 \leq j \leq \lfloor \frac{n-3}{2} \rfloor, \\ [e_2, f_j] = \alpha_j e_3, & 1 \leq j \leq \lfloor \frac{n-3}{2} \rfloor, \\ \left[e_1, f_{\lfloor \frac{n-3}{2} \rfloor + 1} \right] = e_2, \\ \left[e_2, f_{\lfloor \frac{n-3}{2} \rfloor + 1} \right] = e_3, \\ \left[f_{\lfloor \frac{n-3}{2} \rfloor + 1 + i}, f_j \right] = \gamma_{ij} e_3, & 1 \leq i \leq \lfloor \frac{n-3}{2} \rfloor, 1 \leq j \leq \lfloor \frac{n-3}{2} \rfloor + 1. \end{array} \right.$$

By the following expression $\left[f_{\lfloor \frac{n-3}{2} \rfloor + 1 + i}, f_j \right] = [[e_1, f_i] - \alpha_i e_2, f_j]$ with $1 \leq i, j \leq \lfloor \frac{n-3}{2} \rfloor$ we obtain that $\gamma_{ij} = \gamma_{j,i}$ with $1 \leq i, j \leq \lfloor \frac{n-3}{2} \rfloor$.

A similar process to the case n odd allows us to obtain:

$$N(t) : \left\{ \begin{array}{ll} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq 2, \\ [e_1, f_j] = \alpha_j e_2 + f_{\lfloor \frac{n-3}{2} \rfloor + 1 + j}, & 1 \leq j \leq \lfloor \frac{n-3}{2} \rfloor, \\ [e_2, f_j] = \alpha_j e_3, & 1 \leq j \leq \lfloor \frac{n-3}{2} \rfloor, \\ \left[e_1, f_{\lfloor \frac{n-3}{2} \rfloor + 1} \right] = e_2, \\ \left[e_2, f_{\lfloor \frac{n-3}{2} \rfloor + 1} \right] = e_3, \\ \left[f_{\lfloor \frac{n-3}{2} \rfloor + 1 + i}, f_j \right] = e_3, & 1 \leq i \leq t, 0 \leq t \leq \lfloor \frac{n-3}{2} \rfloor. \end{array} \right.$$

If we make the following change of basis:

$$f_j' = f_j - \alpha_j f_{\lfloor \frac{n-3}{2} \rfloor + 1}, \quad 1 \leq j \leq \lfloor \frac{n-3}{2} \rfloor,$$

we get the family N_t .

If we repeat the process with the family $L(0)$, then we obtain the same algebras for $t = 0$. \square

2.2. Classification of graded $(n - 3)$ -filiform Leibniz algebras of type II

Let L be an n -dimensional graded $(n - 3)$ -filiform Leibniz algebra of type II and let $\{e_1, e_2, e_3, f_1, \dots, f_{n-3}\}$ be an adapted basis.

Theorem 2.3 ([4]). *Let L be a complex n -dimensional graded $(n - 3)$ -filiform Leibniz algebra of type II. Then L is a Lie algebra.*

This theorem completes the classification of graded p -filiform Leibniz algebras in each dimension with $n - p \geq 3$. In particular, the classification of graded $(n - 3)$ -filiform non-Lie Leibniz algebras is present on the list of Theorem 2.2. The classification of graded p -filiform Leibniz algebras ($n \geq 4$) was studied in [4]. Moreover, it should be noted that the list of graded p -filiform non-Lie Leibniz algebras $n - p \geq 3$, is simpler than in the Lie algebras case, despite the complementary set of Lie algebras in the Leibniz algebras set forms a Zariski open set (it is well-known that the open set in Zariski topology is a big set).

Acknowledgments

This work is supported in part by the PAICYT, FQM143 of Junta de Andalucía (Spain). B.A. Omirov was supported by the grant NATO-Reintegration ref. CBPEAP.RIG.983169 and he would like to thank the Universidad de Sevilla for their hospitality.

References

- [1] Sh.A. Ayupov, B.A. Omirov, On some classes of nilpotent Leibniz algebras, *Sibirsk. Mat. Zh.* 42 (1) (2001) 18–29 (Russian) (translation in *Siberian Math. J.* 42 (1) (2001) 15–24).
- [2] J.M. Cabezas, E. Pastor, Naturally graded p -filiform Lie algebras in arbitrary finite dimension, *J. Lie Theory* 15 (2005) 379–391.
- [3] L.M. Camacho, J.R. Gómez, A.J. González, B.A. Omirov, Naturally graded 2-filiform Leibniz algebras, *Comm. Algebra* 38 (10) (2010), doi:10.1080/00927870903236160.
- [4] L.M. Camacho, J.R. Gómez, A.J. González, B.A. Omirov, The classification of naturally graded p -filiform Leibniz algebras, *Comm. Algebra* 39 (1) (2011), doi:10.1080/00927870903451900.
- [5] J.M. Casas, M. Ladra, Non-abelian tensor product of Leibniz algebras and an exact sequence in Leibniz homology, *Comm. Algebra* 31 (9) (2003) 4639–4646.
- [6] A.S. Dzhumadil'daev, Cohomologies of colour Leibniz algebras: pre-simplicial approach, *World Sci. Publ., River Edge, NJ*, 2000., pp. 124–136.
- [7] A.S. Dzhumadil'daev, A.A. Davydov, Factor-complex for Leibniz cohomology. Special issue dedicated to Alexei Ivanovich Kostrikin, *Comm. Algebra* 29 (9) (2001) 4197–4210.
- [8] A. Fialowski, A. Mandal, G. Mukherjee, Versal deformations of Leibniz algebras, *K-Theory* 3 (2) (2009) 327–358.
- [9] J.R. Gómez, A. Jiménez-Merchán, Naturally graded quasi-filiform Lie algebras, *J. Algebra* 256 (1) (2002) 211–228.
- [10] M. Goze, B. Hakimjanov You (Khakimdjano), *Nilpotent Lie Algebras*, Kluwer Academic Publishers, 1996.
- [11] J.L. Loday, Une version non commutative des algèbres de Lie: les algèbres de Leibniz, *Enseign. Math.* 39 (1993) 269–293.
- [12] J.L. Loday, T. Pirashvili, Leibniz representations of Leibniz algebras, *J. Algebra* 181 (1996) 414–425.
- [13] M. Vergne, Cohomologie des algèbres de Lie nilpotentes, Application à l'étude de la variété des algèbres de Lie nilpotentes, *Bull. Soc. Math. France* 98 (1970) 81–116.