Leibniz algebras of nilindex $n-3$ with characteristic sequence $(n-3,\ 2,\ 1)$

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A B S T R A C T

In this paper we present the classification of a subclass of naturally graded Leibniz algebras. This subclass has the nilindex $n-3$ and the characteristic sequence $(n-3,\ 2,\ 1)$, where n is the dimension of the algebra. In fact, this result completes the classification of naturally graded Leibniz algebras of nilindex $n-3$.

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1. Introduction

The theory of Lie algebras is one of the main fields of the modern algebra. From the classical theory of Lie algebras it is known that the study of finite dimensional Lie algebras is reduced to the nilpotent ones. From the investigation of nilpotent Lie algebras devoted many works [3,9,13], and others.

Since the description of nilpotent Lie algebras is an unsolvable problem they should be investigated adding some restrictions like restrictions to the index of nilpotency of the algebra, gradation, characteristic sequence and others. Firstly, Vergne studied Lie algebras with maximal nilindex (such algebras are called filiform algebras). In fact, she classified the naturally graded filiform Lie algebras and described filiform algebras in terms of 2-cohomologies. Further, in the work [8] the classification

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of the n-dimensional naturally graded Lie algebras of nilindex $n-2$ is obtained. However, the next stage, the classification of the naturally graded Lie algebras of nilindex $n-3$ is not established.

Intensive investigation of Lie algebras leads to the appearance of new algebraic object – Leibniz algebras. Recall, the Leibniz algebras introduced by Loday in [10] are a "non commutative" algebras analogue of Lie algebras. Several authors have studied the structural properties of Leibniz algebras, [1,7,11,12]. In the context of this work we mention the paper [1], where the authors described up to isomorphism, the naturally graded Leibniz algebras of nilindex greater than $n-1$. Concerning Leibniz algebras of nilindex $n-2$ we have the difference with the case of Lie algebras. Due to inherent property of n-dimensional nilpotent Lie algebras of nilindex $n-2$, they have characteristic sequence equal to $(n-2, 1, 1)$, but the Leibniz algebras have also another characteristic sequence, $(n-2, 2)$. The classification of such naturally graded Leibniz algebras is completed in [4,5].

The last possible stage – the classification of complex n-dimensional naturally graded Leibniz algebras of nilindex $n-3$ is a difficult problem and it should be divided into three cases. Namely, it is necessary to consider the possibilities of the characteristic sequence of such algebras: $(n-3, 3)$, $(n-3, 1, 1, 1)$ and $(n-3, 2, 1)$. To the classification of the complex naturally graded Leibniz algebras of nilindex $n-3$ with characteristic sequence equal to $(n-3, 3)$ and $(n-3, 1, 1, 1)$ devoted works [3,2,6].

The knowledge of naturally graded algebras of a certain family offers significant information about their structural properties. In this sense, in [11,12], the authors give an isomorphism criteria to obtain the classification of filiform Leibniz algebras arising from naturally graded complex filiform Leibniz algebras. The isomorphism criteria are given in terms of invariant functions.

In this paper we obtain the classification of naturally graded Leibniz algebras of nilindex $n-3$ with characteristic sequence $(n-3, 2, 1)$. Thus, we complete this study for the $n-3$ case. Remark that this classification includes Lie algebras of nilindex $n-3$, which were open until this work. All the spaces and the algebras are considered over the field of the complex numbers. We omit the products which are equal to zero, for convenience.

Throughout all the work we use the software *Mathematica* (see [4]) to compute the Leibniz identity in low dimensions and to formulate the generalizations of the calculations, which are proved for arbitrary dimension. Moreover, the program allows us to construct now bases using some general transformation of the generators of the algebra. Some examples of the programs for different types of Leibniz algebras classes can be found in http://personal.us.es/jrgomez.

2. Preliminaries

In this section we give the necessary definitions and notions.

Definition 2.1. An algebra L over the field F is called a Leibniz algebra if the Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y],$$

holds for any $x, y, z \in L$, where $[-, -]$ is the multiplication in L.

Note that in the case where the identity $[x, x] = 0$ is satisfied, the Leibniz identity can be easily reduced to the Jacobi identity. Thus, Leibniz algebras are "non commutative" algebras analogues of Lie algebras.

For an arbitrary algebra L, we define the following sequence:

$$L^1 = L, \; L^{k+1} = [L^k, L], \; k \geqslant 1.$$

Definition 2.2. A Leibniz algebra L is called nilpotent, if there exists $s \in \mathbb{N}$ such that $L^s \neq 0$ and $L^{s+1} = 0$. The minimal number s verifying this property is called nilindex of the algebra L.

Let L be a nilpotent Leibniz algebra and let x be an arbitrary element of L. We define the right multiplication operator $R_x : L \to L$ as $R_x(y) = [y, x]$.

For the operator of right multiplication R_x denote by $C(x)$ the descending sequence of its Jordan blocks dimensions with x an arbitrary element from the set $L \setminus [L, L]$. Consider the lexicographical order on the set of such sequence, i.e. $C(x) = (n_1, n_2, \dots, n_k) \leqslant C(y) = (m_1, m_2, \dots, m_s)$ if and only if there exists $i \in N$ such that $n_j = m_j$ for any $j < i$ and $n_j < m_j$.

Definition 2.3. The sequence $C(L) = \max_{x \in L \setminus [L,L]} C(x)$ is said to be the characteristic sequence of the Leibniz algebra L.

Definition 2.4. The set $\mathcal{R}(L) = \{x \in L : [y, x] = 0 \text{ for any } y \in L\}$ is called the right annihilator of the algebra L.

Let L be a finite dimensional nilpotent Leibniz algebra. Put $L_i := L^i/L^{i+1}$ with $1 \leqslant i \leqslant s-1$, where s is the nilindex of the algebra L and denote $gr(L) = L_1 \oplus L_2 \oplus \cdots \oplus L_{s-1}$. The graded Leibniz algebra, $gr(L)$, is obtained where $[L_i, L_j] \subseteq L_{i+j}$.

An algebra L is called naturally graded if $L \cong gr(L)$.

Let L be a naturally graded Leibniz algebra with characteristic sequence $(n-3, 2, 1)$. By definition of characteristic sequence there exist a basis $\{e_1, e_2, \dots, e_n\}$ in the algebra L such that the operator R_{e_1} has one block J_{n-3} of size $n-3$, one block J_2 of size 2 and one block J_1 of size one.

Note that there will be six possibilities for the operators R_{e_1}. By a change of basis it is easy to prove that the six cases can be reduced to the following three cases:

$$\begin{pmatrix} J_{n-3} & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_1 \end{pmatrix}, \quad \begin{pmatrix} J_2 & 0 & 0 \\ 0 & J_{n-3} & 0 \\ 0 & 0 & J_1 \end{pmatrix}, \quad \begin{pmatrix} J_1 & 0 & 0 \\ 0 & J_{n-3} & 0 \\ 0 & 0 & J_2 \end{pmatrix}.$$

Definition 2.5. A Leibniz algebra L is called either of the first type (type I) or the second type (type II) or the third type (type III) if the operator R_{e_1} has the form:

$$\begin{pmatrix} J_{n-3} & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_1 \end{pmatrix}, \quad \begin{pmatrix} J_2 & 0 & 0 \\ 0 & J_{n-3} & 0 \\ 0 & 0 & J_1 \end{pmatrix}, \quad \begin{pmatrix} J_1 & 0 & 0 \\ 0 & J_{n-3} & 0 \\ 0 & 0 & J_2 \end{pmatrix},$$

respectively.

3. The main result

3.1. Type I

In this subsection we consider the algebras of type I. So, we have the following brackets:

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leqslant i \leqslant n-4, \\ [e_{n-3}, e_1] = 0, \\ [e_{n-2}, e_1] = e_{n-1}, \\ [e_{n-1}, e_1] = 0, \\ [e_n, e_1] = 0. \end{cases}$$

From the above brackets it is easy to see that $\{e_2, e_3, \dots, e_{n-3}\} \in R(L)$ and

$$L_1 \supseteq \langle e_1 \rangle, \; L_2 \supseteq \langle e_2 \rangle, \; L_3 \supseteq \langle e_3 \rangle, \dots, L_{n-3} \supseteq \langle e_{n-3} \rangle.$$

Let us assume that $e_{n-2} \in L_{r_1}$ and $e_n \in L_{r_2}$, then $e_{n-1} \in L_{r_1+1}$.

We can distinguish the following cases:

Case 1. If $r_1 = r_2 = 1$.

Then we have

$$L_1 = \langle e_1, e_{n-2}, e_n \rangle,\ L_2 = \langle e_2, e_{n-1} \rangle,\ L_3 = \langle e_3 \rangle,\ \ldots,\ L_{n-3} = \langle e_{n-3} \rangle.$$

Case 1.1. Let us consider $e_{n-1} \notin R(L)$.

Since the elements e_2, and $[e_1, e_{n-2}] + [e_{n-2}, e_1]$ belong to the operator $R(L)$, then the equality $[e_1, e_{n-2}] = \alpha_1 e_2 - e_{n-1}$ is achieved. Similarly, considering the embedding of the elements $[e_{n-2}, e_{n-2}]$, $[e_n, e_n]$, $[e_n, e_{n-2}] + [e_{n-2}, e_n]$, and $[e_1, e_n] + [e_n, e_1]$ in $R(L)$ we get $[e_{n-2}, e_{n-2}]$, $[e_n, e_n]$, $[e_1, e_n]$ are in $\langle e_2 \rangle$. Moreover, $[e_{n-2}, e_n] = \beta_{n-2} e_2 - \gamma e_{n-1}$ and $[e_n, e_{n-2}] = \alpha_n e_2 + \gamma e_{n-1}$.

Taking this into account we obtain a first version of the family of the algebras:

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leqslant i \leqslant n-4, \\ [e_{n-2}, e_1] = e_{n-1}, \\ [e_1, e_{n-2}] = \alpha_1 e_2 - e_{n-1}, \\ [e_i, e_{n-2}] = \alpha_i e_{i+1}, & 2 \leqslant i \leqslant n-4, \\ [e_{n-2}, e_{n-2}] = \alpha_{n-2} e_2, \\ [e_n, e_{n-2}] = \alpha_n e_2 + \gamma e_{n-1}, \\ [e_i, e_n] = \beta_i e_{i+1}, & 1 \leqslant i \leqslant n-4, \\ [e_{n-2}, e_n] = \beta_{n-2} e_2 - \gamma e_{n-1}, \\ [e_n, e_n] = \beta_n e_2. \end{cases}$$

By the Leibniz identity, it is easy to prove that $[e_i, e_{n-1}] = 0$ for $1 \leqslant i \leqslant n-3$, which is a contradiction because $e_{n-1} \in R(L)$ for $n \geqslant 8$.

Case 1.2. Let us consider $e_{n-1} \in R(L)$.

In this case using the Leibniz identity we obtain the law of the family ($n \geqslant 7$):

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leqslant i \leqslant n-4, \\ [e_{n-2}, e_1] = e_{n-1}, \\ [e_1, e_{n-2}] = \alpha e_2 + \delta_1 e_{n-1}, \\ [e_i, e_{n-2}] = \alpha e_{i+1}, & 2 \leqslant i \leqslant n-4, \\ [e_{n-2}, e_{n-2}] = \delta_2 e_{n-1}, \\ [e_n, e_{n-2}] = \delta_3 e_{n-1}, \\ [e_1, e_n] = \beta e_2 + \gamma_1 e_{n-1}, \\ [e_i, e_n] = \beta e_{i+1}, & 2 \leqslant i \leqslant n-4, \\ [e_{n-2}, e_n] = \gamma_2 e_{n-1}, \\ [e_n, e_n] = \gamma_3 e_{n-1}. \end{cases}$$

It should be noted that $\{e_{n-3},\ e_{n-1}\} \in Center(L)$. Further we compute the Leibniz identity with aid of the software *Mathematica*.

Due to the property of natural gradation of the algebra it is enough to consider the following change of generators:

$$\begin{aligned}
e'_1 &= P_1 e_1 + P_{n-2} e_{n-2} + P_n e_n,\\
e'_{n-2} &= Q_{n-2} e_{n-2} + Q_n e_n,\\
e'_n &= R_{n-2} e_{n-2} + R_n e_n.
\end{aligned}$$

Case 1.2.1. Let us consider $e_{n-2} \in R(L)$.

Then it implies $\alpha = \delta_1 = \delta_2 = \delta_3 = 0$.

We have the family of algebras denoted by

$$L(\beta, \gamma_1, \gamma_2, \gamma_3) : \begin{cases}
[e_i, e_1] = e_{i+1}, & 1 \leqslant i \leqslant n-4,\\
[e_{n-2}, e_1] = e_{n-1}, & \\
[e_1, e_n] = \beta e_2 + \gamma_1 e_{n-1}, & \\
[e_i, e_n] = \beta e_{i+1}, & 2 \leqslant i \leqslant n-4,\\
[e_{n-2}, e_n] = \gamma_2 e_{n-1}, & \\
[e_n, e_n] = \gamma_3 e_{n-1}. &
\end{cases}$$

Lemma 3.1. *An arbitrary Leibniz algebra of the family $L(\beta, \gamma_1, \gamma_2, \gamma_3)$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\begin{aligned}
&L_1(0,1,0,0), \quad L_2(0,0,1,0), \quad L_3(1,0,0,0), \quad L_4(1,0,-1,0), \quad L_5(1,0,1,0), \quad L_6(1,1,1,0),\\
&L_7(0,0,0,1), \quad L_8(0,0,1,1), \quad L_9(1,0,0,1), \quad L_{10}(1,0,1,1), \quad L_{11}(1,1,1,1),\\
&L_{12}(1,0,-1,1).
\end{aligned}$$

Proof. Making the general change of basis in the family $L(\beta, \gamma_1, \gamma_2, \gamma_3)$, we derive the expressions of the new parameters in the new basis:

$$\begin{aligned}
\beta' &= \frac{\beta R_n}{P_1 + \beta P_n},\\
\gamma'_1 &= \frac{(\gamma_1 P_1 - \beta P_{n-2} + \gamma_2 P_{n-2} + \gamma_3 P_n) P_1 R_n}{(P_1 + \beta P_n)(P_1 Q_{n-2} + \gamma_2 P_n Q_{n-2} + \gamma_3 P_n Q_n)},\\
\gamma'_2 &= \frac{(\gamma_2 Q_{n-2} + \gamma_3 Q_n) R_n}{P_1 Q_{n-2} + \gamma_2 P_n Q_{n-2} + \gamma_3 P_n Q_n},\\
\gamma'_3 &= \frac{(\gamma_2 R_{n-2} + \gamma_3 R_n) R_n}{P_1 Q_{n-2} + \gamma_2 P_n Q_{n-2} + \gamma_3 P_n Q_n},
\end{aligned}$$

with the following restrictions:

$$\begin{aligned}
&\beta Q_n = 0,\\
&Q_n(\gamma_2 Q_{n-2} + \gamma_3 Q_n) = 0,\\
&Q_n(\gamma_2 R_{n-2} + \gamma_3 R_n) = 0,\\
&\gamma_1 P_1 - \beta P_{n-2} + \gamma_2 P_{n-2} + \gamma_3 P_n = 0,\\
&P_1(Q_{n-2} R_n - Q_n R_{n-2}) \neq 0.
\end{aligned}$$

If $Q_n \neq 0$, then $\beta = 0$ and we get the homogeneous linear system,

$$\gamma_2 Q_{n-2} + \gamma_3 Q_n = 0,$$

$$\gamma_2 R_{n-2} + \gamma_3 R_n = 0,$$

whose solutions are $\gamma_2 = \gamma_3 = 0$. Moreover, from $\gamma_1 P_1 - \beta P_{n-2} + \gamma_2 P_{n-2} + \gamma_3 P_n = 0$ we get $\gamma_1 P_1 = 0$ yielding $\gamma_1 = 0$. That leads to a split algebra.

Thus we will consider $Q_n = 0$. That implies $P_1 + \gamma_2 P_n \neq 0$ and hence $R_{n-2} = -\dfrac{\gamma_3 P_n R_n}{(P_1 + \gamma_2 P_n)}$.

We can distinguish two cases:

(a) When $e_n \in R(L)$.
Then $\beta = \gamma_1 = \gamma_2 = \gamma_3 = 0$, and we again obtain a split algebra.

(b) When $e_n \notin R(L)$.
It is necessary to distinguish the following subcases:

(b.1) $\gamma_3 = 0$.

Then we have $R_{n-2} = 0$ and let us observe that the nullities of β' and γ_2' are invariant.

- If $\beta = 0$ and $\gamma_2 = 0$, then $\gamma_1 \neq 0$. Choosing $e_n' = \frac{1}{\gamma_1} e_n$, we can assume $\gamma_1 = 1$ and we obtain the algebra $L_1(0, 1, 0, 0)$.
- If $\beta = 0$ and $\gamma_2 \neq 0$, then putting $P_{n-2} = -\frac{\gamma_1 P_1}{\gamma_2}$ and making $e_n' = \frac{1}{\gamma_2} e_n$, we have the algebra $L_2(0, 0, 1, 0)$.
- If $\beta \neq 0$ and $\gamma_2 = 0$, then choosing $e_n' = \frac{1}{\beta} e_n$ and $P_{n-2} = \frac{\gamma_1 P_1}{\beta}$, we get the algebra $L_3(1, 0, 0, 0)$.
- If $\beta \neq 0$ and $\gamma_2 \neq 0$, then it is not difficult to check the validity of the following expression:

$$\beta' - \gamma_2' = \frac{(\beta - \gamma_2) P_1 R_n}{(P_1 + \beta P_n)(P_1 + \gamma_2 P_n)}.$$

Let us distinguish the following cases:

(1) $\beta - \gamma_2 \neq 0$.

If we choose $P_{n-2} = \frac{\gamma_1 P_1}{\beta - \gamma_2}$, we obtain $\gamma_1' = 0$. Moreover, $\beta' = 1$ if and only if $R_n = \frac{P_1 + \beta P_n}{\beta}$ and $\gamma_2' = -1$ if and only if $P_n = -\frac{P_1(\gamma_2 + \beta)}{2\gamma_2 \beta}$.

The restrictions $P_1 + \gamma_2 P_n \neq 0$ and $P_1 + \beta P_n \neq 0$ are also verified. Indeed,

$$P_1 + \gamma_2 \left(-\frac{P_1(\gamma_2 + \beta)}{2\gamma_2 \beta} \right) = \frac{(\beta - \gamma_2) P_1}{2\beta} \neq 0$$

and

$$P_1 + \beta \left(-\frac{P_1(\gamma_2 + \beta)}{2\gamma_2 \beta} \right) = P_1 \frac{\gamma_2 - \beta}{2\gamma_2} \neq 0.$$

Thus, we obtain the algebra $L_4(1, 0, -1, 0)$.

(2) $\beta - \gamma_2 = 0$.

Then $\beta' = \gamma_2' = \frac{\beta R_n}{P_1+\beta P_n}$, and choosing $R_n = \frac{P_1+\beta P_n}{\beta}$ we yield the equalities $\beta' = \gamma_2' = 1$. Moreover, the nullity of γ_1 is invariant, because $\gamma_1' = \frac{P_1^2\gamma_1}{\beta(P_1+\beta P_n)Q_{n-2}}$.

(a) If $\gamma_1 = 0$, then we get $L_5(1, 0, 1, 0)$.

(b) If $\gamma_1 \neq 0$, then choosing $Q_{n-2} = \frac{\beta(P_1+\beta P_n)}{P_1^2\gamma_1}$, we can assume $\gamma_1' = 1$ and the algebra $L_6(1, 1, 1, 0)$ holds.

(b.2) $\gamma_3 \neq 0$.

Then we have

$$\beta' = \frac{\beta R_n}{P_1 + \beta P_n}, \quad \gamma_1' = \frac{(\gamma_1 P_1 - \beta P_{n-2} + \gamma_2 P_{n-2} + \gamma_3 P_n)P_1 R_n}{(P_1 + \beta P_n)(P_1 + \gamma_2 P_n)Q_{n-2}},$$

$$\gamma_2' = \frac{\gamma_2 R_n}{P_1 + \gamma_2 P_n}, \quad \gamma_3' = \frac{\gamma_3 P_1 R_n^2}{(P_1 + \gamma_2 P_n)^2 Q_{n-2}}.$$

Let us distinguish the following cases.

- If $\beta = 0$ and $\gamma_2 = 0$.

  Choosing $P_n = -\frac{\gamma_1 P_1}{\gamma_3}$ and $Q_{n-2} = \frac{\gamma_3 R_n^2}{P_1}$ we can suppose $\gamma_1' = 0$, and $\gamma_3' = 1$, so the algebra $L_7(0, 0, 0, 1)$ is obtained.
- If $\beta = 0$ and $\gamma_2 \neq 0$.

  Choosing $P_{n-2} = -\frac{\gamma_1 P_1+\gamma_3 P_n}{\gamma_2}$, $R_n = \frac{P_1+\gamma_2 P_n}{\gamma_2}$ and $Q_{n-2} = \frac{\gamma_3 P_1}{\gamma_2^2}$ we obtain that $\gamma_1' = 0$, $\gamma_2' = \gamma_3' = 1$. Thus, $L_8(0, 0, 1, 1)$.
- If $\beta \neq 0$ and $\gamma_2 = 0$.

  Choosing $R_n = \frac{P_1+\beta P_n}{\beta}$, $P_{n-2} = \frac{\gamma_1 P_1+\gamma_3 P_n}{\beta}$, and $Q_{n-2} = \frac{\gamma_3(P_1+\beta P_n)^2}{P_1\beta^2}$ we obtain $\beta' = 1$, $\gamma_1' = 0$, $\gamma_3' = 1$. Therefore we have the algebra $L_9(1, 0, 0, 1)$.
- If $\beta \neq 0$ and $\gamma_2 \neq 0$.

If $\beta - \gamma_2 = 0$, we have nullity invariant as a form:

$$\gamma_3' - \beta\gamma_1' = \frac{(\gamma_3 - \beta\gamma_1)R_n^2 P_1^2}{Q_{n-2}(P_1 + \beta P_n)^3}.$$

Thus,

(1) $\beta - \gamma_2 = 0$.

- If $\gamma_3 - \beta\gamma_1 \neq 0$, taking $P_n = -\frac{\gamma_1 P_1}{\gamma_3}$, $R_n = -\frac{P_1+\beta P_n}{\beta}$ and $Q_{n-2} = \frac{\gamma_3 R_n^2 P_1}{(P_1+\beta P_n)^2}$, we obtain $\beta' = 1, \gamma_1' = 0, \gamma_2' = 1, \gamma_3' = 1$ and the algebra $L_{10}(1, 0, 1, 1)$.
- If $\gamma_3 - \beta\gamma_1 = 0$, analogously to the above case, choosing the appropriate values of R_n and Q_{n-2} we can assume $\beta' = 1, \gamma_3' = 1$ and $\gamma_1' = 1$. Thus, we get $L_{11}(1, 1, 1, 1)$.

(2) $\beta - \gamma_2 \neq 0$.

Choosing the appropriated values of R_n, P_{n-2}, P_n and Q_{n-2} we obtain $\beta' = 1, \gamma_1' = 0, \gamma_2 = -1$ and $\gamma_3' = 1$, respectively. Therefore, the algebra $L_{12}(1, 0, -1, 1)$ is obtained. □

Case 1.2.2. Let us consider $e_{n-2} \notin R(L)$.

In this case if $e_n \in R(L)$, then by the following change $e_{n-2}' = e_n,\ e_n' = e_{n-2}$, we come to the above case 1.2.1.

Therefore we assert that $e_n \notin R(L)$. Then we derive the family

$$M(\alpha,\beta,\gamma_1,\gamma_2,\gamma_3,\delta_1,\delta_2,\delta_3):\begin{cases}[e_i,e_1]=e_{i+1}, & 1\leqslant i\leqslant n-4,\\ [e_{n-2},e_1]=e_{n-1}, & \\ [e_1,e_{n-2}]=\alpha e_2+\delta_1 e_{n-1}, & \\ [e_i,e_{n-2}]=\alpha e_{i+1}, & 2\leqslant i\leqslant n-4,\\ [e_{n-2},e_{n-2}]=\delta_2 e_{n-1}, & \\ [e_n,e_{n-2}]=\delta_3 e_{n-1}, & \\ [e_1,e_n]=\beta e_2+\gamma_1 e_{n-1}, & \\ [e_i,e_n]=\beta e_{i+1}, & 2\leqslant i\leqslant n-4,\\ [e_{n-2},e_n]=\gamma_2 e_{n-1}, & \\ [e_n,e_n]=\gamma_3 e_{n-1}. & \end{cases}$$

Since $e_{n-1}\in Center(L)$ we can consider the factor algebra $\widetilde{L}=L/\langle e_{n-1}\rangle$, which is naturally graded 2-filiform and by [5] it leads the nullity of that α is an invariant. Hence, we distinguish two cases: $\alpha\neq 0$ and $\alpha=0$. The next result shows the classification for the case $\alpha=0$.

Lemma 3.2. *An arbitrary algebra of the family $M(\alpha,\beta,\gamma_1,\gamma_2,\gamma_3,\delta_1,\delta_2,\delta_3)$ with $\alpha\neq 0$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

$M_1(1,0,0,0,1,\delta_1,\delta_2,\delta_3)$, $M_2(1,0,0,\gamma_2,0,0,0,1)$ $\gamma_2\neq 0$,

$M_3(1,0,0,-1,0,0,\delta_2,1)$, $M_4(1,0,0,\gamma_2,0,\frac{1}{\gamma_2},0,1)$, $\gamma_2\neq 0$,

$M_5(1,0,0,-1,0,-1,\delta_2,1)$, $\delta_2\in\{0,1\}$ $M_6(1,0,\gamma_1,0,0,0,0,1)$, $\gamma_1\neq 0$,

$M_7(1,0,1,0,0,\delta_1,0,1)$,

$M_8(1,0,\gamma_1,\gamma_2,0,0,0,1)$,

$\gamma_1\notin\{0,1\}$, $\gamma_2\notin\{0,-1\}$,

$M_9(1,0,1,\gamma_2,0,0,\delta_2,1)$,

$\gamma_2\notin\{0,-1\}$, $\delta_2\in\{0,1\}$,

$M_{10}(1,0,\gamma_1,-1,0,0,\delta_2,1)$,

$\gamma_1\notin\{0,1\}$, $\delta_2\in\{0,1\}$,

$M_{11}(1,0,1,-1,0,0,\delta_2,1)$, $\delta_2\in\{0,1\}$,

$M_{12}\left(1,0,\gamma_1,\gamma_2,0,0,-\frac{(1-\gamma_1)(1+\gamma_2)}{\gamma_1\gamma_2},1\right)$,

$\gamma_1\notin\{0,1\}$, $\gamma_2\notin\{0,-1\}$, $M_{13}(1,0,1,-1,0,1,0,1)$,

$M_{14}(1, 0, \gamma_1, -1, 0, 0, 0, 1),\ \gamma_1 \notin \{0, 1\},\ M_{15}(1, 0, 1, \gamma_2, 0, 0, 0, 1),\ \gamma_2 \notin \{0, -1\},$

$M_{16}(1, 0, 0, 0, 0, 2, -1, 0)$ $\quad M_{17}(1, 0, 0, 1, 0, \delta_1, 0, 0),\ \delta_1 \in \{0, 1\},$

$M_{18}(1, 0, 0, 1, 0, 0, 0, 0),$ $\quad M_{19}(1, 0, 1, 1, 0, -1, 0, 0),$

$M_{20}(1, 0, 1, 0, 0, 0, -1, 0),$ $\quad M_{21}(1, 0, 1, 0, 0, 0, 1, 0),$

$M_{22}(1, 0, 1, 0, 0, 0, 0, 0),$ $\quad M_{23}(1, 0, 0, 0, 0, 0, \delta_2, 0), \delta_2 \neq 0,$

$M_{24}(1, 0, 0, 0, 0, \delta_1, 1, 0),\ \delta_1 \in \{0, 1\},$ $\quad M_{25}(1, 0, 0, 0, 0, -2, -1, 0).$

Proof. This proof is similar to the proof of Lemma 3.1. □

Now, we consider the case where $\alpha = 0$. Thus, $Q_n = 0$ and $\beta' = \dfrac{\beta R_n}{P_1 + \beta P_n}$, i.e., β is nullity invariant.

Lemma 3.3. *An arbitrary algebra of the family $M(0, 0, \gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2, \delta_3)$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

$M_{26}(0, 0, 0, 0, 1, 0, 1, 0),$ $\quad M_{27}(0, 0, 0, 1, \gamma_3, 0, 1, 0), \gamma_3 \neq 0,$

$M_{28}(0, 0, 1, \gamma_2, \gamma_3, 0, 1, 0),\ \gamma_3 \neq 0$ $\quad M_{29}(0, 0, \gamma_1, 1, 0, 0, 1, 0),\ \gamma_1 \in \{0, 1\}$

$M_{30}(0, 0, 1, 1, 0, 0, 1, 0),$ $\quad M_{31}(0, 0, 0, 0, 1, \delta_1, 0, 0),\ \delta_1 \neq 1,$

$M_{32}(0, 0, \gamma_1, 0, 1, 1, 0, 0),\ \gamma_1 \in \{0, 1\},$ $\quad M_{33}(0, 0, 0, 1, \gamma_3, \delta_1, 0, 0),\ \gamma_3, \delta_1 \in \{0, 1\},$

$M_{34}(0, 0, 1, 0, 0, 0, 0, 1),$ $\quad M_{35}(0, 0, \gamma_1, \gamma_2, 0, 0, 0, 1),\ \gamma_1 \in \{0, 1\}, \gamma_2 \neq 0,$

$M_{36}(0, 0, 0, -1, \gamma_3, 0, 0, 1),\ \gamma_3 \in \{0, 1\},$ $\quad M_{37}(0, 0, \gamma_1, \gamma_2, 0, \frac{1}{\gamma_2}, 0, 1),\ \gamma_1 \in \{0, 1\}, \gamma_2 \neq 0,$

$M_{38}(0, 0, \gamma_1, -1, 0, -1, 0, 1),\ \gamma_1 \in \{0, 1\},\ M_{39}(0, 0, \gamma_1, -1, 1, -1, 0, 1),\ \gamma_1 \in \{0, -2\},$

$M_{40}(0, 0, 1, 1, 0, \delta_1, 0, 0).$

Proof. The proof is similar to the proof of Lemma 3.1. □

Lemma 3.4. *An arbitrary algebra of the family $M(0, \beta, \gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2, \delta_3)$, with $\beta \neq 0$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

$M_{41}(0, 1, 0, 0, 0, \delta_1, 0, 0,),\ \delta_1 \notin \{0, -1\},$ $\quad M_{42}(0, 1, \gamma_1, 0, 0, -1, 0, 0), \gamma_1 \in \{0, 1\},$

$M_{43}(0, 1, 0, 0, 1, \delta_1, 0, 0),\ \delta_1 \notin \{0, -1\},$ $\quad M_{44}(0, 1, \gamma_1, 0, 1, -1, 0, 0), \gamma_1 \in \{0, 1\},$

$M_{45}(0, 1, 0, -1, 1, \delta_1, 0, 0),\ \delta_1 \neq 0,$ $\quad M_{46}(0, 1, 0, 1, 1, 1, 0, 0),$

$M_{47}(0, 1, 0, -1, 0, \delta_1, 0, 0),\ \delta_1 \neq 0,$ $\quad M_{48}(0, 1, \gamma_1, -1, \gamma_3, -2, 0, 0), \gamma_1, \gamma_3 \in \{0, 1\},$

$M_{49}(0, 1, \gamma_1, 0, 1, \delta_1, 0, 1),\ \gamma_1 \in \{0, 1\},$

$M_{50}(0, 1, \gamma_1, -1, 0, \delta_1, 0, \delta_3),\ \gamma_1 \in \{0, 1\},\ \delta_3 \notin \{0, 1\},$

$M_{51}(0, 1, 0, -1, \gamma_3, \delta_1, 0, 1),\ \gamma_3 \in \{0, 1\},\ \delta_1 \neq 1,$

$M_{52}(0, 1, \gamma_1, 1, 1, 0, 0, -1),$ $M_{53}(0, 1, \gamma_1, -2, 3, 0, 1, -1)$

$M_{54}(0, 1, \gamma_1, 1, 0, 0, 0, \delta_3),\ \gamma_1 \in \{0, 1\},\ \delta_3 \neq 0,$ $M_{55}(0, 1, 1, 1, \gamma_3, 0, 0, -1),$

$M_{56}(0, 1, \gamma_1, 1, \gamma_3, -1, 0, -1),\ \gamma_1,\ \gamma_3 \in \{0, 1\},$ $M_{57}(0, 1, 0, 1, \gamma_3, 0, 0, -1),$ $\gamma_3 \in \{0, 1\},$

$M_{58}(0, 1, \gamma_1, 1, 0, \delta_1, 0, \delta_1),\ \gamma_1 \in \{0, 1\},\ \delta_1 \notin \{0, -1\},$ $M_{59}(0, 1, \gamma_1, \gamma_2, \gamma_3, 0, 1, 0,),$

$M_{60}(0, 1, \gamma_1, 0, 1, 0, 1, 1),$ $M_{61}(0, 1, \gamma_1, 0, 1, 0, 1, -1),$ $\gamma_1 \in \{0, 1\},$

$M_{62}(0, 1, \gamma_1, \gamma_2, 1, 0, 1, -1),\ \gamma_2 \neq 0,$ $M_{63}(0, 1, \gamma_1, 0, \gamma_3, 0, 1, -\gamma_3),$ $\gamma_3 \neq 0,$

$M_{64}(0, 1, \gamma_1, 1, 0, 0, 1, 1),$ $M_{65}(0, 1, \gamma_1, 0, 1, 0, 1, -1),$ $\gamma_1 \in \{0, 1\},$

$M_{65}(0, 1, 1, 1, 0, 1, 0, 0)$ $M_{66}(0, 1, \gamma_1, 1, 1, 0, 0, -1),$ $\gamma_1 \in \{0, 1\}.$

Proof. The proof of this Lemma is similar to the proof of Lemma 3.1. □

Consider now the following cases.

Case 2. If $r_2 = 1$ and $r_1 > 1$.

We also need to consider the following family of algebras

$$N(\alpha, \beta) : \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leqslant i \leqslant n-4, \\ [e_{n-2}, e_1] = e_{n-1}, & \\ [e_1, e_n] = \alpha e_2 + e_{n-2}, & \\ [e_2, e_n] = \alpha e_3 + e_{n-1}, & \\ [e_i, e_n] = \alpha e_{i+1}, & 3 \leqslant i \leqslant n-4, \\ [e_{n-2}, e_n] = \beta e_{n-1}. & \end{cases}$$

Theorem 3.5. *Let L be an n-dimensional Leibniz algebra with characteristic sequence $(n-3, 2, 1)$ of type I, $r_2 = 1$ and $r_1 > 1$. Then the algebra L is isomorphic to one of the following pairwise non-isomorphic algebras:*

$N_1(0, \beta),\ \beta \in \{0, 1\},$

$N_2(1, \beta),\ \beta \in \{0, -1\}.$

Proof. When we consider $r_2 = 1$ and $r_1 \geqslant 3$, we obtain the gradation $L_1 = \langle e_1, e_n\rangle,\ L_2 = \langle e_2\rangle,\ \ldots,$ $L_{r_1-1} = \langle e_{r-1}\rangle,\ L_{r_1} = \langle e_{r_1}, e_{n-2}\rangle, \ldots, L_{r_1+1} = \langle e_{r_1+1}, e_{n-1}\rangle, \ldots, L_{n-3} = \langle e_{n-3}\rangle$, so we can write $[e_1, e_n] = \alpha e_2$.

From the chain of equalities

$$[e_2, e_n] = [[e_1, e_1], e_n] = [e_1, [e_1, e_n]] + [[e_1, e_n], e_1] = \alpha[e_1, e_2] + \alpha[e_2, e_1] = \alpha e_3,$$

we have $[e_2, e_n] = \alpha e_3$.

Similarly we obtain $[e_i, e_n] = \alpha e_{i+1}$ for $1 \leqslant i \leqslant n-4$. Therefore,

$$[e_{r_1-1}, e_1] = e_{r_1},$$
$$[e_{r_1-1}, e_n] = \alpha e_{r_1},$$

and we conclude that it is impossible to obtain the basis element e_{n-2}, yielding contradiction.

Let us consider the case $r_2 = 1$ and $r_1 = 2$.

Then $L_1 = \langle e_1, e_n\rangle$, $L_2 = \langle e_2, e_{n-2}\rangle$, $L_3 = \langle e_3, e_{n-1}\rangle, \ldots, L_{n-3} = \langle e_{n-3}\rangle$ and we denote $[e_1, e_n] = \alpha_1 e_2 + \alpha_2 e_{n-2}$, $[e_1, e_n] = \beta_1 e_2 + \beta_2 e_{n-2}$.

The equation $[[e_n, e_n], e_1] = [e_n, [e_n, e_1]] + [[e_n, e_1], e_n]$ implies $\beta_1 = \beta_2 = 0$.

Since $e_{n-2} \in L_2$, then $\alpha_2 \neq 0$ and we can assume $\alpha_2 = 1$. Moreover, since $[e_1, e_n] + [e_n, e_1] \in R(L)$ and $\alpha_1 e_2 + e_{n-2} \in R(L)$, we have $e_{n-2} \in R(L)$ and $e_{n-1} \in R(L)$.

Using the induction method it can be proved that $[e_i, e_n] = \alpha_1 e_{i+1}$ for $2 \leqslant i \leqslant n-4$, and applying the Leibniz identity we come to the family $N(\alpha, \beta)$.

Taking the general change of basis in $N(\alpha, \beta)$ we obtain

$$\alpha' = \frac{\alpha Q_n}{P_1 + \alpha P_n}, \qquad \beta' = \frac{\beta Q_n}{P_1 + \beta P_n}.$$

The study of nullities of α, β leads to the algebras of Theorem. $\square$

Case 3. If $r_1 = 1$, $r_2 \geqslant 2$.

We obtain the following two families:

$$K(\alpha, \beta_1, \beta_2),\ n \geqslant 8: \quad \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leqslant i \leqslant n-4, \\ [e_{n-2}, e_1] = e_{n-1}, \\ [e_1, e_{n-2}] = \alpha e_2 + e_n, \\ [e_i, e_{n-2}] = \alpha e_{i+1}, \\ [e_{n-2}, e_{n-2}] = \beta_1 e_{n-1} + \beta_2 e_n. \end{cases}$$

$$T(\alpha_1, \alpha_2, \alpha_3),\ n \geqslant 10: \quad \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leqslant i \leqslant n-4, \\ [e_{n-2}, e_1] = e_{n-1}, \\ [e_1, e_{n-2}] = \alpha_1 e_2 - e_{n-1}, \\ [e_2, e_{n-2}] = \alpha_1 e_3 + \alpha_2 e_n, \\ [e_i, e_{n-2}] = \alpha_1 e_{i+1}, & 3 \leqslant i \leqslant n-4, \\ [e_{n-1}, e_{n-2}] = \alpha_3 e_n, \\ [e_1, e_{n-1}] = -\alpha_2 e_n, \\ [e_{n-2}, e_{n-1}] = -\alpha_3 e_n. \end{cases}$$

where $(\alpha_2, \alpha_3) \neq (0, 0)$.

If we study these two families of Leibniz algebras we can prove the following theorem.

Theorem 3.6. *Let L be an n-dimensional Leibniz algebra with characteristic sequence $(n-3, 2, 1)$ of type I, $r_1 = 1$ and $r_2 \geqslant 2$. Then the algebra L is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$K_1(0, \beta_1, 0),\ \beta_1 \in \{0, 1\};\ K_2(1, \beta_1, \beta_2),\ \beta_1 \in \{1, -1\},\ \beta_2 \in \{0, 1\};\ K_3(0, 1, \beta_2),\ \beta_2 \neq 0;$$
$$K_4(1, \beta_1, 0),\ \beta_1 \neq 0; \qquad K_5(1, 1, -1).$$

$$T_1(1, 0, 1);\ T_2(1, 1, \alpha_3),\ \alpha_3 \in \{0, 1\};\ T_3(0, 1, \alpha_3),\ \alpha_3 \in \{0, 1\};\ T_4(0, 0, 1).$$

Proof. Analogous to Theorem 3.5. □

3.2. Type II

As in type I we have that $L_1 \supseteq \langle e_1 \rangle,\ L_2 \supseteq \langle e_2 \rangle$ and we suppose $e_3 \in L_{r_1}, e_n \in L_{r_2}$. Analogously, we have the following brackets:

$$\begin{cases} [e_1, e_1] = e_2, \\ [e_i, e_1] = e_{i+1},\ 3 \leqslant i \leqslant n-2. \end{cases}$$

In the case $r_1 = 1, r_2 = 1$ and $e_4 \in R(L)$, if we make a change of basis, we can see that this case becomes type I.

If $r_1 = 1, r_2 = 1$ and $e_4 \notin R(L)$ with n odd, we have the following family of Leibniz algebras denoted by

$$L(\alpha_1, \alpha_3, \alpha_n, \beta_1, \beta_3, \beta_4, \beta_n) : \begin{cases} [e_1, e_1] = e_2, & \\ [e_i, e_1] = e_{i+1}, & 3 \leqslant i \leqslant n-2, \\ [e_1, e_i] = -e_{i+1}, & 4 \leqslant i \leqslant n-2, \\ [e_n, e_1] = -\beta_4 e_2, & \\ [e_1, e_3] = \alpha_1 e_2 - e_4, & \\ [e_3, e_3] = \alpha_3 e_2, & \\ [e_n, e_3] = \alpha_n e_2, & \\ [e_1, e_n] = \beta_1 e_2, & \\ [e_3, e_n] = \beta_3 e_2, & \\ [e_n, e_n] = \beta_n e_2. & \end{cases}$$

Theorem 3.7. *Let L be an n-dimensional Leibniz algebra with characteristic sequence $(n-3, 2, 1)$ of type II, $r_1 = r_2 = 1$ and n odd. Then the algebra L is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\begin{array}{ll} L_1(\alpha_1, \alpha_3, 1, 0, 0, 0, 1), & L_2(0, \alpha_3, 1, 0, 0, \beta_4, 1), \\ L_3(\pm 2\sqrt{\alpha_3}, \alpha_3, 1, 0, 0, \mp\frac{1}{\sqrt{\alpha_3}}, 1),\ \alpha_3 \neq 0, & L_4(1, 0, 1, 0, 0, -1, 1), \\ L_5(0, 1, 0, 0, 0, \beta_4, 1), & L_6(\pm 2, 1, 0, 0, 0, \beta_4, 1),\ \beta_4 \in \{0, 1\} \\ L_7(1, 0, 0, 0, 0, \beta_4, 1),\ \beta_4 \in \{0, 1\} & L_8(0, 0, 0, 0, 0, \beta_4, 1),\ \beta_4 \in \{0, 1\}, \end{array}$$

$L_9(\alpha_1, 0, \alpha_n, 0, 1, 1, 0),\ \alpha_1 \in \{0, -1\},\ \alpha_n \neq -1,$ $L_{10}(0, 0, \alpha_n, 0, 1, 0, 0),$
$L_{11}(1, 0, \alpha_n, 1, 1, -\alpha_n, 0),\ \alpha_n \neq -1,$ $L_{12}(0, \alpha_3, -1, 1, 1, 0, 0),\ \alpha_3 \in \{0, 1\},$
$L_{13}(\alpha_1, 1, -1, 0, 1, 0, 0),$ $L_{14}(\alpha_1, 0, -1, 0, 1, 0, 0),\ \alpha_1 \in \{0, 1\},$
$L_{15}(0, \alpha_3, -1, 1, 1, 1, 0),\ \alpha_1 \in \{0, 2\},\ \alpha_3 \in \{-1, 1\},$ $L_{16}(1, 0, -1, 1, 1, 1, 0),$
$L_{17}(\alpha_1, 0, 1, 1, 0, 0, 0),\ \alpha_1 \in \{0, 1\},$ $L_{18}(0, 0, 1, 0, 0, 0, 0),$
$L_{19}(-1, 0, 1, 0, 0, 1, 0),$ $L_{20}(0, \alpha_3, 0, 1, 0, \beta_4, 0),\ \alpha_3 \in \{0, 1\},\ \beta_4 \neq 1,$
$L_{21}(0, \alpha_3, 0, 0, 0, 1, 0),\ \alpha_3 \in \{0, 1\},$ $L_{22}(0, 1, 0, \beta_1, 0, 1, 0),\ \beta_1 \in \{0, 1\},$
$L_{23}(\pm 2, 1, 0, \beta_1, 0, 1, 0),\ \beta_1 \in \{0, 1\},$ $L_{24}(1, 0, 0, 1, 0, 1, 0),$
$L_{25}(2, 1, -1, 1, 1, 1, 0),$ $L_{26}(0, 0, 0, 1, 0, 1, 0),$

Proof. Analogous to Theorem 3.5. □

If $r_1 = 1, r_2 = 1, e_4 \notin R(L)$ and n even, then we get the following family:

$$M(\alpha, \alpha_1, \alpha_3, \alpha_n, \beta_1, \beta_3, \beta_4, \beta_n) : \begin{cases} [e_1, e_1] = e_2, & \\ [e_i, e_1] = e_{i+1}, & 3 \leqslant i \leqslant n-2, \\ [e_1, e_i] = -e_{i+1}, & 4 \leqslant i \leqslant n-2, \\ [e_1, e_3] = \alpha_1 e_2 - e_4, & \\ [e_3, e_3] = \alpha_3 e_2, & \\ [e_i, e_3] = \alpha_4 e_{i+1}, & 4 \leqslant i \leqslant n-3, \\ [e_n, e_3] = \alpha_n e_2 - \beta_4 e_4, & \\ [e_1, e_n] = \beta_1 e_2, & \\ [e_3, e_n] = \beta_3 e_2 + \beta_4 e_4, & \\ [e_i, e_n] = \beta_4 e_{i+1}, & 4 \leqslant i \leqslant n-2, \\ [e_n, e_n] = \beta_n e_2, & \\ [e_n, e_i] = -\beta_4 e_{i+1}, & 4 \leqslant i \leqslant n-2, \\ [e_3, e_i] = -\alpha_4 e_{i+1}, & 4 \leqslant i \leqslant n-3, \\ [e_i, e_{n+1-i}] = (-1)^i \alpha_{n-2} e_{n-1}, & 3 \leqslant i \leqslant n-2. \end{cases}$$

Theorem 3.8. *Let L be an n-dimensional Leibniz algebra with characteristic sequence $(n-3, 2, 1)$ of type II, $r_1 = r_2 = 1$ and n even. Then the algebra L is isomorphic to one of the following pairwise non-isomorphic algebras:*

$M_1(1, \alpha_1, \alpha_2, \alpha_n, 0, 0, 1, 0),\ \alpha_n \neq 0,$ $M_2(1, 0, \alpha_2, \alpha_n, 0, 0, \alpha_n, 1),\ \alpha_3 \notin \{0, -1\},\ \alpha_n \neq 0,$
$M_3(1, 1, \frac{1}{4}, \alpha_n, 0, 0, \alpha_n, 1),\ \alpha_n \neq 0$ $M_4(1, -2, -1, \alpha_n, 0, 0, \alpha_n, 1),\ \alpha_n \neq 0$
$M_5(1, -1, 0, \alpha_n, 0, 0, \alpha_n, 1),\ \alpha_n \neq 0,$ $M_6(1, 0, 0, 1, 0, 0, 1, 1),$
$M_7(1, 0, -(\frac{\alpha_n}{\beta_4})^2, \alpha_n, 0, 0, \beta_4, 1),\ \alpha_n, \beta_4 \neq 0,$ $M_8(-2, -1, (1 \pm \sqrt{2}), 0, 0, \beta_4, 1),\ \beta_4 \neq 0,$
$M_9(1, 1, \frac{1}{4}, -\frac{\beta_4}{2}, 0, 0, \beta_4, 1),\ \beta_4 \neq 0,$ $M_{10}(1, -1, 0, \beta_4, 0, 0, \beta_4, 1),\ \beta_4 \neq 0,$

$M_{11}(1, \alpha_1, 0, 0, 0, 0, 1, 1),\ \alpha_1 \neq 0,$

$M_{12}(1, 0, 0, 0, 0, 0, \beta_4, 1),\ \beta_4 \neq 0,$

$M_{13}(1, \alpha_1, \alpha_3, \alpha_n, 0, 0, \alpha_n, 1),\ \alpha_n \neq 0,$

$(\alpha_1, \alpha_3) \in \{(0, -1), (2, 1)\},$

$M_{14}(1, 1, 0, 1, 0, 0, 1, 1),$

$M_{15}(1, 0, \alpha_1, 0, 0, 1, \beta_4, 1),\ \alpha_1 \neq 0,$

$M_{16}(1, -2, -1, 0, 0, 1, \beta_4, 1),\ \beta_4 \neq 0,$

$M_{17}(1, 1, \frac{1}{4}, 0, 0, 1, \beta_4, 1),$

$M_{18}(1, 2, 1, 0, 0, 1, \beta_4, 1),\ \beta_4 \in \{0, 1\},$

$M_{19}(1, -1, 0, 0, 0, 1, \beta_4, 1),$

$M_{20}(1, 1, 0, 0, 0, 1, \beta_4, 1),\ \beta_4 \in \{0, 1\},$

$M_{21}(1, 0, 0, 1, \beta_1, \beta_3, 0, 0),\ \beta_1 \neq 0,\ \beta_3 \neq -1,$

$M_{22}(1, \frac{\beta_3+1}{\beta_1}, 0, 1, \beta_1, \beta_3, 0, 0),\ \beta_1, \beta_3 \neq 0,$

$M_{23}(1, 0, 0, 1, 0, \beta_3, -1, 0),\ \beta_3 \notin \{0, -1\},$

$M_{24}(1, 0, 0, 1, \beta_1, \beta_1, -1, 0),\ \beta_1 \notin \{0, -1\},$

$M_{25}(1, 0, 0, 1, \beta_1, 0, -1, 0),\ \beta_1 \in \{0, 1\},$

$M_{26}(1, (1+\beta_3), 0, 1, 0, \beta_3, -1, 0),\ \beta_3 \notin \{0, -1\}$

$M_{27}(1, 1, 0, 1, \beta_1, \beta_1, -1, 0),\ \beta_1 \notin \{0, -1\},$

$M_{28}(1, \frac{1}{2}, 0, 1, 1, 0, -1, 0),$

$M_{29}(1, 1, 0, 1, 0, 0, 1, 0),$

$M_{30}(1, 0, 0, 1, 0, \beta_3, 0, 0),\ \beta_3 \neq -1,$

$M_{31}(1, -1, 0, 1, -\beta_3, \beta_3, 1, 0),\ \beta_3 \neq -1$

$M_{32}(1, \alpha_1, 0, 1, \beta_1, \beta_1, -1, 0),$

$\alpha_1 \in \{0, 1\}, \beta_1 \neq -1,$

$M_{33}(1, 0, \alpha_3, 1, 0, -1, \beta_4, 0),\ \alpha_3 \in \{0, 1\}, \beta_4 \neq 0,$

$M_{34}(1, 0, \alpha_3, 1, -1, -1, 0, 0),\ \alpha_3 \in \{0, 1\},$

$M_{35}(1, \alpha_1, \alpha_3, 1, 0, -1, 0, 0),$

$M_{36}(1, 0, \alpha_3, 1, -1, -1, -1, 0),\ \alpha_3 \neq 0,$

$M_{37}(1, 1, \frac{1}{4}, 1, -1, -1, -1, 0),$

$M_{38}(1, \alpha_1, \alpha_3, 1, -1, -1, -1, 0),$

$(\alpha_1, \alpha_3) \in \{(-2, -1), (2, 1)\},$

$M_{39}(1, \alpha_1, 0, 1, -1, -1, -1, 0),\ \alpha_1 \in \{-1, 0, 1\},$

$M_{40}(1, -(1+\alpha_3), \alpha_3, 1, 1, -1, 1, 0),$

$M_{41}(1, -\frac{5}{2}, 1, 1, 2, -1, 2, 0),$

$M_{42}(1, (1+\alpha_3), \alpha_3, 1, -1, -1, -1, 0),$

$\alpha_3 \in \{-1, 0, 1\},$

$M_{43}(1, 0, 0, 0, 0, 1, \beta_4, 0),\ \beta_4 \neq 0,$

$M_{44}(1, -\frac{1}{\beta_4}, 0, 0, 0, 1, \beta_4, 0), \beta_4 \neq 0,$

$M_{45}(1, 0, 0, 1, 1, 1, 1, 0),$

$M_{46}(1, \frac{1}{2}, 0, 0, 1, 1, -1, 0),$

$M_{47}(1, 0, 0, 0, \beta_1, 1, 0, 0),\ \beta_1 \in \{0, 1\},$

$M_{48}(1, 1, 0, 0, 1, 1, 0, 0),$

$M_{49}(1, -1, 0, 0, 0, 1, 0, 0),$

$M_{50}(1, 1, 0, 0, 1, 1, 0, 0),$

$M_{51}(1, 0, \alpha_3, 0, \beta_1, 0, 1, 0),\ \alpha_3 \in \{0, 1\},\ \beta_1 \neq 1,$

$M_{52}(1, 0, \alpha_3, 0, 1, 0, 0, 0),\ \alpha_3 \in \{0, 1\},$

$M_{53}(1, 0, \alpha_3, 0, 1, 0, 1, 0),\ \alpha_3 \neq 0,$

$M_{54}(1, 1, \frac{1}{4}, 0, 1, 0, 1, 0),$

$M_{55}(1, 1, \frac{1}{2}, 0, 1, 0, 1, 0),$

$M_{56}(1, 2, 1, 0, 1, 0, 1, 0),$

$M_{57}(1, -1, 0, 0, 1, 0, 1, 0),$

$M_{58}(1, 1, 0, 0, 1, 0, 1, 0),$

$M_{59}(1, 0, 0, 0, 1, 0, 1, 0),$

$M_{60}(1, -3, 0, 0, 0, 0, 2, 1).$

Proof. Analogous to Theorem 3.5. □

Let us investigate the case $r_1 = 1$ and $r_2 = 2$. In this case, if $e_4 \in R(L)$, the algebra is of type I. If $e_4 \notin R(L)$, when n is odd we have the family:

$$K(\alpha_1, \alpha_3, \mu_1, \mu_2):\begin{cases}[e_1, e_1] = e_2, & \\ [e_i, e_1] = e_{i+1} & 3 \leqslant i \leqslant n-2 \\ [e_1, e_i] = -e_{i+1} & 4 \leqslant i \leqslant n-2 \\ [e_1, e_3] = \alpha_1 e_2 - e_4 + \mu_1 e_n, & \\ [e_3, e_3] = \alpha_3 e_2 + \mu_2 e_n, & (\mu_1, \mu_2) \neq (0, 0).\end{cases}$$

Theorem 3.9. *Let L be an n-dimensional Leibniz algebra with characteristic sequence $(n-3, 2, 1)$ of type II, $r_1 = 1$, $r_2 = 2$ and n odd. Then the algebra L is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$K_1(0,0,1,0),\quad K_2(0,0,0,1),\ K_3(1,0,1,0),\ K_4(1,0,0,1),\ K_5(1,0,1,1),$$
$$K_6(0,1,1,\mu_2),\ K_7(0,1,0,1),\ K_8(1,\tfrac{1}{4},0,1),\ K_9(1,\tfrac{1}{4},1,\tfrac{1}{2}),\ K_{10}(1,\tfrac{1}{4},1,\tfrac{1}{4}).$$

Proof. Analogous to Theorem 3.5. □

On the other hand, if $e_4 \notin R(L)$ and n is even we have the following family:

$$P(\alpha_1,\alpha_3,\mu_1,\mu_2):\begin{cases}[e_1,e_1]=e_2, & \\ [e_i,e_1]=e_{i+1}, & 3\leqslant i\leqslant n-2,\\ [e_1,e_i]=-e_{i+1}, & 4\leqslant i\leqslant n-2,\\ [e_1,e_3]=\alpha_1e_2-e_4+\mu_1e_n, & \\ [e_3,e_3]=\alpha_3e_2+\mu_2e_n, & (\mu_1,\mu_2)\neq(0,0)\\ [e_i,e_{n+1-i}]=(-1)^ie_{n-1}, & 3\leqslant i\leqslant n-2.\end{cases}$$

Theorem 3.10. *Let L be an n-dimensional Leibniz algebra with characteristic sequence $(n-3, 2, 1)$ of type II, $r_1 = 1$, $r_2 = 2$ and n even. Then the algebra L is isomorphic to one of the following pairwise non-isomorphic algebras:*

$P_1(0,0,1,0)$, $P_2(0,0,1,1)$, $P_3(0,0,2,1)$, $P_4(1,0,1,0)$,
$P_5(-1,0,1,0)$, $P_6(-1,0,\mu_1,1)$, $P_7(1,0,1,1)$, $P_8(1,0,0,1)$,
$P_9(2,1,2,1)$, $P_{10}(2,1,1,0)$, $P_{11}(2,1,1,1)$, $P_{12}(-2,1,2,1)$,
$P_{13}(2\alpha_3,\alpha_3,1,0),\ \alpha_3\neq0$, $P_{14}(\frac{-2}{3},\frac{-1}{3},-1,1)$, $P_{15}(-2,-1,1,1)$, $P_{16}(\alpha_1,\frac{\alpha_1^2}{4},1,0),\ \alpha_1\notin\{0,2\}$,
$P_{17}(1,\frac{1}{4},1,\frac{1}{2})$, $P_{18}(1,\frac{1}{4},0,1)$, $P_{19}(2,1,1,1)$, $P_{20}(1,\frac{1}{4},4,1)$,
$P_{21}(0,\alpha_3,\mu_1,1),\ \alpha_3\neq0$ $P_{22}(0,\alpha_3,1,0),\ \alpha_3\neq0$.

Proof. Analogous to Theorem 3.5. □

If $3\leqslant r_2\leqslant n-5$ and r_2 is even, we derive a contradiction, i.e. this case is impossible.
If $3\leqslant r_2\leqslant n-6$ and r_2 is odd, the obtained family is defined by:

$$S(\alpha_1,\alpha_3):\begin{cases}[e_1,e_1]=e_2, & \\ [e_i,e_1]=e_{i+1} & 3\leqslant i\leqslant n-2\\ [e_1,e_i]=-e_{i+1} & 4\leqslant i\leqslant n-2\\ [e_1,e_3]=\alpha_1e_2-e_4, & \\ [e_3,e_3]=\alpha_3e_2, & \\ [e_{r_2+4-j},e_j]=(-1)^{j+1}e_n,\ 3\leqslant j\leqslant r_2+1. & \end{cases}$$

Theorem 3.11. *Let L be an n-dimensional Leibniz algebra with characteristic sequence $(n-3, 2, 1)$ of type II, $r_1 = 1$, $3\leqslant r_2\leqslant n-6$ and r_2 odd. Then the algebra L is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$S_1(1,\alpha_3),\quad S_2(0,1),\quad S_3(0,0).$$

Proof. Analogous to Theorem 3.5. □

There is only left the case when $r_2 = n-5$ and $n \geqslant 12$ with n even. In this case, the got family is: $W(\alpha_1, \alpha_3, \alpha_{n-4}-\alpha_4, \alpha_{n-2}-\alpha_4)$:

$$\begin{cases}
[e_1,e_1]=e_2, & \\
[e_i,e_1]=e_{i+1} & 3\leqslant i\leqslant n-2\\
[e_1,e_i]=-e_{i+1} & 4\leqslant i\leqslant n-2\\
[e_1,e_3]=\alpha_1e_2-e_4, & \\
[e_3,e_3]=\alpha_3e_2, & \\
[e_{n-4},e_3]=(\alpha_{n-4}-\alpha_4)e_{n-3}+e_n, & \\
[e_{n-3},e_3]=\dfrac{n-6}{2}(\alpha_{n-4}-\alpha_4)e_{n-2}, & \\
[e_{n-2},e_3]=(\alpha_{n-2}-\alpha_4)e_{n-1}, & \\
[e_n,e_i]=-\dfrac{n-6}{2}(\alpha_4-\alpha_{n-4})^2e_{n-5+i}, & 3\leqslant i\leqslant 4\\
[e_i,e_n]=\dfrac{n-6}{2}(\alpha_4-\alpha_{n-4})^2e_{n-5+i}, & 3\leqslant i\leqslant 4\\
[e_3,e_{n-4}]=-(\alpha_{n-4}-\alpha_4)e_{n-3}-e_n, & \\
[e_3,e_{n-3}]=-\dfrac{n-6}{2}(\alpha_{n-4}-\alpha_4)e_{n-2}, & \\
[e_3,e_{n-2}]=(\alpha_4-\alpha_{n-2})e_{n-1}, & \\
[e_{n-1-j},e_j]=(-1)^j(\alpha_4-\alpha_{n-4})e_{n-3}+(-1)^{j+1}e_n, & 4\leqslant j\leqslant n-5\\
[e_{n-j},e_j]=(-1)^j\left(\dfrac{n}{2}-j\right)(\alpha_4-\alpha_{n-4})e_{n-2}, & 4\leqslant j\leqslant \dfrac{n-2}{2}\\
[e_j,e_{n-j}]=-[e_{n-j},e_j], & \\
[e_{n+1-j},e_j]=(-1)^{j+1}\left(\left((j-3)\left(\dfrac{n}{2}-4\right)-\dfrac{(j-4)(j-5)}{2}\right)\alpha_4-\right. & \\
\left.-\left((j-3)\left(\dfrac{n}{2}-4\right)-\dfrac{(j-4)(j-5)}{2}+1\right)\alpha_{n-4}+\alpha_{n-2}\right)e_{n-1}, & 4\leqslant j\leqslant \dfrac{n}{2}\\
[e_j,e_{n+1-j}]=-[e_{n+1-j},e_j], & 4\leqslant j\leqslant \dfrac{n}{2}.
\end{cases}$$

Theorem 3.12. *Let L be an n-dimensional Leibniz algebra with characteristic sequence $(n-3,2,1)$ of type II, $r_1=1$, $r_2=n-5$, r_2 even and $n\geqslant 12$. Then the algebra L is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$W_1(1,\alpha_3,0,0),\quad W_2(0,1,0,0),\quad W_3(0,0,0,0),\quad W_4(1,\alpha_3,\alpha_{n-4}-\alpha_4,0),$$
$$W_5(0,0,1,0),\quad W_6(1,0,1,0),\quad W_7\left(\tfrac{n-4}{2},0,1,0\right),\quad W_8(0,\alpha_3,1,0),\ \alpha_3\neq 0,$$
$$W_9\left(\tfrac{4}{n-4},1,1,0\right),\ W_{10}\left(1,\tfrac{1}{4},0,1\right),\ W_{11}\left(n-4,\left(\tfrac{n-4}{2}\right)^2,1,0\right).$$

Proof. Analogous to Theorem 3.5. □

On the one hand if $r_2 = n-4$, $n \geqslant 9$ and n even, we have a contradiction. On the other hand, if n is odd, we have the family of algebras denoted by

$$Y(\alpha,\alpha_1,\alpha_3,\beta):\begin{cases}[e_1,e_1]=e_2, & \\ [e_i,e_1]=e_{i+1} & 3\leqslant i\leqslant n-2\\ [e_1,e_i]=-e_{i+1} & 4\leqslant i\leqslant n-2\\ [e_1,e_3]=\alpha_1 e_2-e_4, & \\ [e_3,e_3]=\alpha_3 e_2, & \\ [e_{n-3},e_3]=\alpha e_{n-2}+e_n, & \\ [e_{n-2},e_3]=\beta e_{n-1}, & \\ [e_n,e_3]=-\alpha\beta e_{n-1}, & \\ [e_3,e_{n-3}]=-\alpha e_{n-2}-e_n, & \\ [e_3,e_{n-2}]=(-(n-5)\alpha+\beta)e_{n-1}, & \\ [e_{n-j},e_j]=(-1)^{j+1}(\alpha e_{n-2}+e_n), & 4\leqslant j\leqslant n-4\\ [e_{n+1-j},e_j]=(-1)^j((j-3)\alpha-\beta)e_{n-1}, & 4\leqslant i\leqslant n-3,\\ [e_1,e_n]=((n-5)\alpha-2\beta)e_{n-1}, & \\ [e_3,e_n]=\alpha((n-5)\alpha-\beta)e_{n-1}. & \end{cases}$$

Theorem 3.13. *An arbitrary Leibniz algebra of the family $Y(\alpha,\alpha_1,\alpha_3,\beta)$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

$Y_1(0,0,0,0)$, $Y_2(0,1,0,0)$, $Y_3(0,0,1,0)$,
$Y_4\left(0,1,\frac{1}{4},0\right)$, $Y_5(\alpha,0,\alpha_3,1)$, $\alpha_3\neq 0$, $Y_6\left(\alpha,1,\frac{1}{4},1\right)$, $\alpha\neq\frac{1}{2}$,
$Y_7\left(\frac{1}{2},-1,\frac{1}{4},1\right)$, $Y_8(\alpha,2,(\alpha+1),1)$, $\alpha\neq -1$, $Y_9(\alpha,2(\alpha+1),(\alpha+1)^2,1)$, $\alpha\neq -1$,
$Y_{10}(1,1,0,1)$, $Y_{11}(\alpha,(\alpha+1),0,1)$, $\alpha\neq -1$, $Y_{12}(\alpha,0,0,1)$,
$Y_{13}(1,0,\alpha_3,0)$, $\alpha_3\neq 0$, $Y_{14}\left(1,1,\frac{1}{4},0\right)$, $Y_{15}\left(1,1,\frac{1}{2},0\right)$,
$Y_{16}(1,2,1,0)$, $Y_{17}(1,-1,0,0)$, $Y_{18}(1,1,0,0)$,
$Y_{19}(1,0,0,0)$.

Proof. Analogous to Theorem 3.5. □

The case $r_2 = n-3$ can be included it in the general case n even and r_2 odd.

Let us introduce the case $r_2 = 1$ and $r_1 \geqslant 2$. It is clear that if $r_1 \geqslant 2$, we have $e_3 \in L^{r_1}$, yielding $e_{n-1} \in L^{r_1+n-4}$. Since $r_1 \geqslant 2$, then $r_1+n-4 \geqslant n-2$, i.e $L^{n-2} \neq 0$, but that is a contradiction with the supposed nilindex. Therefore the possible value for r_1 is 1.

3.3. Type III

In this subsection we investigate the Leibniz algebras of Type III, which have the brackets

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 2 \leqslant i \leqslant n-3, \\ [e_{n-1}, e_1] = e_n. \end{cases}$$

and the gradation

$$\langle e_1, e_2 \rangle \subseteq L_1, \quad \langle e_3 \rangle \subseteq L_2, \quad \langle e_4 \rangle \subseteq L_3, \ldots, \langle e_{n-2} \rangle \subseteq L_{n-3}.$$

Let us suppose that $e_{n-1} \in L_r$, then $[e_{n-1}, e_1] = e_n \in L_{r+1}$, where $1 \leqslant r \leqslant n-4$.
The following theorem is true.

Theorem 3.14. *Let L be a naturally graded n-dimensional Leibniz algebra with characteristic sequence $(n-3, 2, 1)$ of type III and $r = 1$. Then there exists a basis $\{e_1, e_2, \ldots, e_n\}$ of L such that the table of multiplication of the algebra has the following form:*

If n is even ($n \geqslant 8$)

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 2 \leqslant i \leqslant n-3, \\ [e_{n-1}, e_1] = e_n, & \\ [e_1, e_i] = -e_{i+1}, & 3 \leqslant i \leqslant n-3, \\ [e_1, e_2] = -e_3 + \gamma_1 e_n, & \\ [e_2, e_2] = \gamma_2 e_n, & \\ [e_{n-1}, e_2] = \gamma_3 e_n, & \\ [e_{n-3}, e_2] = \alpha e_{n-2}, & \\ [e_{n-1-j}, e_j] = (-1)^j \alpha e_{n-2}, & 3 \leqslant j \leqslant n-4, \\ [e_2, e_{n-3}] = -\alpha e_{n-2}, & \\ [e_1, e_{n-1}] = \mu_1 e_n, & \\ [e_2, e_{n-1}] = \mu_2 e_n, & \\ [e_{n-1}, e_{n-1}] = \mu_3 e_n. & \end{cases}$$

If n is odd ($n \geqslant 9$)

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 2 \leqslant i \leqslant n-3, \\ [e_{n-1}, e_1] = e_n, & \\ [e_1, e_i] = -e_{i+1}, & 3 \leqslant i \leqslant n-3, \\ [e_1, e_2] = -e_3 + \gamma_1 e_n, & \\ [e_2, e_2] = \gamma_2 e_n, & \\ [e_{n-1}, e_2] = \gamma_3 e_n, & \\ [e_1, e_{n-1}] = \mu_1 e_n, & \\ [e_2, e_{n-1}] = \mu_2 e_n, & \\ [e_{n-1}, e_{n-1}] = \mu_3 e_n. & \end{cases}$$

(omitted products are equal to zero)

Let us note that when n is odd, the obtained family is a particular case of the first family (taking $\alpha = 0$). We denote the first family of the above theorem by $L(\alpha, \gamma_1, \gamma_2, \gamma_3, \mu_1, \mu_2, \mu_3)$.

Theorem 3.15. *Let L be a naturally graded n-dimensional Leibniz algebra with characteristic sequence $(n-3, 2, 1)$ of type III and $r = 1$. Let n be even ($n \geqslant 8$), then L is isomorphic to one of the following pairwise non-isomorphic algebras:*

$L_1(1, \gamma_1, \gamma_2, \gamma_3, \mu_1, 0, 1)$, $L_2(0, 1, \gamma_2, \gamma_3, \mu_1, 0, 1)$,

$L_3(0, 0, 1, \gamma_3, \mu_1, 0, 1)$, $L_4(0, 0, 0, \gamma_3, \mu_1, 0, 1),\ \gamma_3 \in \{0, 1\}$,

$L_5(1, \gamma_1, 0, \gamma_3, \mu_1, \mu_2, 0),\ \gamma_1 \in \{0, 1\}$ $L_6(0, 0, 0, 1, \mu_1, 0, 0)$,

$L_7(0, 0, 0, 0, \mu_1, 1, 0)$, $L_8(0, 0, 0, 0, \mu_1, 0, 0)$,

$L_9(1, 0, \gamma_2, -\mu_2, \mu_1, \mu_2, 0)$, $\gamma_2 \in \{0, 1\},\ \mu_1 \neq -1$, $L_{10}(0, 0, \gamma_2, -1, \mu_1, 1, 0),\ \mu_1 \neq -1$,

$L_{11}(0, 0, \gamma_2, 0, \mu_1, 0, 0),\ \gamma_2 \in \{0, 1\},\ \mu_1 \neq -1$, $L_{12}(1, 1, \gamma_2, -\mu_2, -1, \mu_2, 0)$,

$L_{13}(1, 0, \gamma_2, -\mu_2, -1, \mu_2, 0),\ \gamma_2 \in \{0, 1\}$ $L_{14}(0, \gamma_1, \gamma_2, -1, -1, 1, 0)$, $\gamma_1, \gamma_2 \in \{0, 1\}$,

$L_{15}(0, 0, \gamma_2, -1, -1, 1, 0),\ \gamma_2 \in \{0, 1\}$ $L_{16}(0, \gamma_1, \gamma_2, 0, -1, 0, 0),\ \gamma_1, \gamma_2 \in \{0, 1\}$.

Proof. Analogous to Theorem 3.5. □

Sice we said above, when n is odd we obtain the algebras $L_2 - L_4, L_6 - L_8, L_{10} - L_{11}$ and $L_{14} - L_{16}$.

In the case $2 \leqslant r \leqslant n-4$ and r even we derive a contradiction.

Theorem 3.16. *Let L be a naturally graded n-dimensional Leibniz algebra with characteristic sequence $(n-3, 2, 1)$ of type III, $3 \leqslant r \leqslant n-4$ and r odd. Then L is a Lie algebra.*

Proof. Analogous to the Theorem 3.5. □

All the obtained algebras in this work lead us to the classification of n-dimensional naturally graded Leibniz algebras of nilindex $n-3$ and with characteristic sequence equal to $(n-3, 2, 1)$.

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