LEIBNIZ ALGEBRAS ASSOCIATED TO EXTENSIONS OF sl_2

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In this work, we investigate the structure of Leibniz algebras whose associated Lie algebra is a direct sum of sl_2 and the solvable radical. In particular, we obtain the description of such algebras when: the ideal generated by the squares of elements of a Leibniz algebra is irreducible over sl_2 and when the dimension of the radical is equal to two.

Key Words: Irreducible module; Lie algebra; Leibniz algebra; Simple algebra; Solvability.

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1. INTRODUCTION

The notion of Leibniz algebra was introduced in 1993 by J.-L. Loday [6] as a generalization of Lie algebras. In the last 20 years, the theory of Leibniz algebras has been actively studied, and many results of the theory of Lie algebras have been extended to Leibniz algebras.

From the theory of Lie algebras it is known that every finite dimensional Lie algebra is decomposed into a semidirect sum of a semisimple subalgebra and solvable radical (Levi's Theorem) [5]. Moreover, thanks to Mal'cev's results [7] the study of solvable Lie algebras is reduced to the study of nilpotent algebras.

Recently, Barnes proved an analogue of Levi's Theorem for Leibniz algebras [4]. Namely, a Leibniz algebra is decomposed into a semidirect sum of its solvable radical and a semisimple Lie algebra.

Recall, an algebra L over a field F is called *Leibniz algebra* if for any elements $x, y, z \in L$ *the Leibniz identity* holds:

$$[x, [y, z]] = [[x, y], z] - [[x, z], y],$$

where $[-, -]$ is multiplication of L.

Let L be a Leibniz algebra and $I = ideal < [x, x] \mid x \in L >$ be the ideal of L generated by all squares. Then I is the minimal ideal with respect to the property that L/I is a Lie algebra. The natural epimorphism $\varphi : L \to L/I$ determines the *associated Lie algebra* L/I of the Leibniz algebra L.

The inherent properties of non-Lie Leibniz algebras imply that the subspace spanned by the squares of elements of the algebra is a nontrivial ideal (further denoted by I). Moreover, the ideal I is abelian and hence, it belongs to the solvable radical.

Due to the existence of the non-trivial ideal I in non-Lie Leibniz algebra we can not consider the notion of simplicity in ordinary sense. In [1], the adaptation of simplicity for Leibniz algebras was proposed. Namely, a Leibniz algebra L is said to be *simple* if it only has the following ideals: $\{0\}$, I, L and $[L, L] \neq I$. Obviously, this definition agrees with the definition of a simple Lie algebra.

In fact, Abdykassymova and Dzhumadil'daev in [2] suggested the following construction of Leibniz algebras.

Let G be a simple Lie algebra and M be an irreducible skew-symmetric G-module (i.e., $[x, m] = 0$ for all $x \in G, m \in M$). Then the vector space $Q = G + M$ equipped with the multiplication

$$[x + m, y + n] = [x, y] + [m, y],$$

where $m, n \in M$, $x, y \in G$ is a simple Leibniz algebra.

From the analogue of Levi's decomposition for Leibniz algebras (see [4]), we conclude that the above construction is universal for a simple Leibniz algebra assuming $G := L/I,\ M := I, Q := L$, that is, any simple Leibniz algebra is realized by this construction.

According to [5], in any three-dimensional simple Lie algebra there exits a basis $\{e, f, h\}$ such that the table of multiplication has the form

$$\begin{array}{lll} [e, h] = 2e, & [f, h] = -2f, & [e, f] = h, \\ [h, e] = -2e, & [h, f] = 2f, & [f, e] = -h. \end{array}$$

This Lie algebra is denoted by sl_2, and the basis $\{e, f, h\}$ is called the *canonical basis*.

Theorem 1.1 ([5]). *For each integer $m = 0, 1, 2, \dots,$ there exists one, up to isomorphism, irreducible sl_2-module M of dimension $m + 1$. The module M has a basis $\{x_0, x_1, ..., x_m\}$ such that the representing transformations E, F, and H corresponding to the canonical basis $\{e, f, h\}$ are given by*

$$\begin{array}{ll} H(x_k) = (m - 2k)x_k, & 0 \leq k \leq m, \\ F(x_m) = 0,\ F(x_k) = x_{k+1}, & 0 \leq k \leq m - 1, \\ E(x_0) = 0,\ E(x_k) = -k(m + 1 - k)x_{k-1}, & 1 \leq k \leq m. \end{array}$$

In [8], the authors described the complex finite-dimensional Leibniz algebras whose associated Lie algebra is isomorphic to sl_2. The crucial role in that classification is played by Theorem 1.1. Taking a similar approach, a property of Leibniz algebras with an associated Lie algebra $sl_2 \dot{+} R$, where R is a solvable radical, is obtained. In addition, in this article, we present the classification of such algebras for a two-dimensional radical R.

Throughout the article, we consider finite-dimensional algebras over the field of complex numbers. Moreover, in the multiplication table of an algebra we omit the null products.

2. COMPLEX LEIBNIZ ALGEBRAS WHOSE ASSOCIATED LIE ALGEBRAS ARE ISOMORPHIC TO $sl_2 \dot{+} R$.

In this section, we consider the Leibniz algebra L for which its corresponding Lie algebra is a semidirect sum of sl_2 and a two-dimensional solvable ideal R. In addition, we shall assume that I is a right irreducible module over sl_2.

Let L be a Leibniz algebra such that $L/I \simeq sl_2 \oplus R$, where R is a solvable Lie algebra and $\{\bar{e}, \bar{h}, \bar{f}\}$, $\{x_0, x_1, \ldots, x_m\}$, $\{\overline{y_1}, \overline{y_2}, \ldots, \overline{y_n}\}$ are the bases of sl_2, I, and R, respectively.

Let $\{e, h, f, x_0, x_1, \ldots, x_m, y_1, y_2, \ldots, y_n\}$ be a basis of the algebra L such that

$$\varphi(e) = \bar{e}, \qquad \varphi(h) = \bar{h}, \qquad \varphi(f) = \bar{f}, \qquad \varphi(y_i) = \overline{y_i}, \qquad 1 \le i \le n.$$

Then taking into account the Theorem 1.1, we have

$$\begin{array}{lll}
[e,h] = 2e + \sum\limits_{j=0}^{m} a_{eh}^{j} x_j, & [h,f] = 2f + \sum\limits_{j=0}^{m} a_{hf}^{j} x_j, & [e,f] = h + \sum\limits_{j=0}^{m} a_{ef}^{j} x_j, \\
[h,e] = -2e + \sum\limits_{j=0}^{m} a_{he}^{j} x_j & [f,h] = -2f + \sum\limits_{j=0}^{m} a_{fh}^{j} x_j, & [f,e] = -h + \sum\limits_{j=0}^{m} a_{fe}^{j} x_j, \\
[e,y_i] = \sum\limits_{j=0}^{m} \alpha_{ij} x_j & [f,y_i] = \sum\limits_{j=0}^{m} \beta_{ij} x_j, & [h,y_i] = \sum\limits_{j=0}^{m} \gamma_{ij} x_j, \\
[x_k,h] = (m-2k)x_k, & 0 \le k \le m, & \\
[x_k,f] = x_{k+1}, & 0 \le k \le m-1, & \\
[x_k,e] = -k(m+1-k)x_{k-1}, & 1 \le k \le m, &
\end{array}$$

where $1 \le i \le n$.

Similar as in the proof of Proposition 3.1 of the [8], one can get

$$\begin{array}{lll}
[e,h] = 2e, & [h,f] = 2f, & [e,f] = h, \\
[h,e] = -2e & [f,h] = -2f, & [f,e] = -h, \\
[e,e] = 0 & [f,f] = 0, & [h,h] = 0.
\end{array}$$

Let us denote the following vector spaces:

$$sl_2^{-1} = < e, h, f >, \qquad R^{-1} = < y_1, y_2, \ldots, y_n > .$$

Proposition 2.1. *Let L be a Leibniz algebra whose quotient $L/I \cong sl_2 \oplus R$, where R is a solvable ideal and I is a right irreducible module over sl_2 with $dim I \neq 3$. Then* $[sl_2^{-1}, R^{-1}] = 0$.

Proof. It is clearly sufficient to prove the equality for the basic elements of sl_2^{-1} and R^{-1}. Consider the Leibniz identity

$$[e,[e,y_i]] = [[e,e],y_i] - [[e,y_i],e] = -[[e,y_i],e]$$
$$= -\sum_{j=0}^{m} \alpha_{ij}[x_j, e] = \sum_{j=1}^{m} (-mj + j(j-1))\alpha_{ij} x_{j-1}, \qquad 1 \le i \le n.$$

On the other hand, we have that $[e,[e,y_i]] = [e, \sum_{j=0}^{m} \alpha_{ij}x_j] = 0$ for $1 \leq i \leq n$.

Comparing the coefficients of the basic elements, we obtain $\alpha_{ij} = 0$ for $1 \leq j \leq m$. Thus, $[e, y_i] = \alpha_{i,0}x_0$ with $1 \leq i \leq n$.

Consider the equalities

$$\begin{aligned} 0 &= [e, \sum_{j=0}^{m} \beta_{ij}x_j] = [e,[f,y_i]] = [[e,f],y_i] - [[e,y_i],f] \\ &= [h,y_i] - \alpha_{i,0}[x_0,f] = [h,y_i] - \alpha_{i,0}x_1. \end{aligned}$$

Then we have that $[h, y_i] = \alpha_{i,0}x_1$ with $1 \leq i \leq n$.

From the equalities

$$\begin{aligned} 0 &= [e,[h,y_i]] = [[e,h],y_i] - [[e,y_i],h] = 2[e,y_i] - \alpha_{i,0}[x_0,h] \\ &= 2\alpha_{i,0}x_0 - m\alpha_{i,0}x_0 = \alpha_{i,0}(2-m)x_0, \end{aligned}$$

it follows that $\alpha_{i,0} = 0$ for $1 \leq i \leq n$. Taking into account that $m \neq 2$ (because of $dim I \neq 3$), we get $[e, y_i] = [h, y_i] = 0$ with $1 \leq i \leq n$.

From the equalities

$$\begin{aligned} 0 &= [f,[e,y_i]] = [[f,e],y_i] - [[f,y_i],e] = [h,y_i] - [[f,y_i],e] = -[[f,y_i],e] \\ &= -\sum_{j=0}^{m} \beta_{ij}[x_j,e] = -\sum_{j=0}^{m}(-mj + j(j-1))\beta_{ij}x_{j-1}, \end{aligned}$$

we derive $\beta_{i,j} = 0$ for $1 \leq j \leq m$. Consequently, $[f, y_i] = \beta_{i,0}x_0$, for all $1 \leq i \leq n$.

Similarly, from

$$\begin{aligned} 0 &= [f,[f,y_i]] = [[f,f],y_i] - [[f,y_i],f] = [[f,y_i],f] \\ &= \beta_{i,0}[x_0,f] = \beta_{i,0}x_1, \end{aligned}$$

we obtain $[f, y_i] = 0$ for all $1 \leq i \leq n$.

Thus, we obtain $[e, y_i] = [f, y_i] = [h, y_i] = 0$ with $1 \leq i \leq n$. We complete the proof of the proposition. □

3. COMPLEX LEIBNIZ ALGEBRAS WHOSE ASSOCIATED LIE ALGEBRA IS ISOMORPHIC TO $sl_2 \dot{+} R$, $dim\, R = 2$

In this section, we consider the case when R is a two-dimensional solvable radical. From the classification of two-dimensional Lie algebras (see [5]) we know that in R there exists a basis $\{\overline{y_1}, \overline{y_2}\}$ in which the following table of multiplication of R has this form:

$$[\overline{y_1}, \overline{y_2}] = \overline{y_1}, \qquad [\overline{y_2}, \overline{y_1}] = -\overline{y_1}.$$

In the case when $dimI \neq 3$ and I a right irreducible module over sl_2, summarizing the results of the Proposition 2.1, we get the following table of multiplication:

$$
\begin{array}{lll}
[e,h]=2e, & [h,f]=2f, & [e,f]=h, \\
[h,e]=-2e, & [f,h]=-2f, & [f,e]=-h, \\
[x_k,h]=(m-2k)x_k, & 0\le k\le m, & \\
[x_k,f]=x_{k+1}, & 0\le k\le m-1, & \\
[x_k,e]=-k(m+1-k)x_{k-1}, & 1\le k\le m, & \\
[y_i,e]=\sum\limits_{j=0}^{m} a_{ie}^{j}x_j, & 1\le i\le 2, & \\
[y_i,f]=\sum\limits_{j=0}^{m} a_{if}^{j}x_j, & 1\le i\le 2, & \\
[y_i,h]=\sum\limits_{j=0}^{m} a_{ih}^{j}x_j, & 1\le i\le 2, & \\
[x_k,y_i]=\sum\limits_{j=0}^{m} a_{ij}^{k}x_j, & 0\le k\le m, & 1\le i\le 2, \\
[y_1,y_2]=y_1+\sum\limits_{j=0}^{m} b_{12}^{j}x_j, & [y_2,y_1]=-y_1, & \\
[y_1,y_1]=\sum\limits_{j=0}^{m} b_{1}^{j}x_j, & [y_2,y_2]=\sum\limits_{j=0}^{m} b_{2}^{j}x_j, &
\end{array}
\tag{1}
$$

where $\{e,h,f,x_0,x_1,\ldots,x_m,y_1,y_2\}$ is a basis of L.

Let us present the following theorem which describes the Leibniz algebras with the condition $L/I \cong sl_2 \oplus R$, where $dimI \neq 3$, $dimR = 2$, and I a right irreducible module over sl_2.

Theorem 3.1. *Let L be a Leibniz algebra whose quotient $L/I \cong sl_2 \oplus R$, where R is a two-dimensional solvable radical and I is a right irreducible module over sl_2 ($dimI \neq 3$). Then there exists a basis $\{e,h,f,x_0,x_1,\ldots,x_m,y_1,y_2\}$ of L such that the table of multiplication in L has the following form*:

$$
\begin{array}{lll}
[e,h]=2e, & [h,f]=2f, & [e,f]=h, \\
[h,e]=-2e, & [f,h]=-2f, & [f,e]=-h, \\
[x_k,h]=(m-2k)x_k, & 0\le k\le m, & \\
[x_k,f]=x_{k+1}, & 0\le k\le m-1, & \\
[x_k,e]=-k(m+1-k)x_{k-1}, & 1\le k\le m, & \\
[y_1,y_2]=y_1, & [y_2,y_1]=-y_1, & \\
[x_k,y_2]=ax_k, & 0\le k\le m, \quad a\in\mathbb{C}. &
\end{array}
$$

Proof. Let L be an algebra satisfying the conditions of the theorem, then we get the table of multiplication (1).

We consider the chain of the equalities

$$0 = [x_i, [h, y_1]] = [[x_i, h], y_1] - [[x_i, y_1], h] = (m-2i)[x_i, y_1] - \sum_{k=0}^{m} a_{1k}^i [x_k, h]$$
$$= (m-2i)\sum_{k=0}^{m} a_{1k}^i x_k - \sum_{k=0}^{m} a_{1k}^i (m-2k)x_k = \sum_{k=0}^{m} a_{1k}^i (m-2i-(m-2k))x_k$$
$$= \sum_{k=0}^{m} 2a_{1k}^i (k-i)x_k,$$

from which we have $a_{1k}^i = 0$ with $0 \le i \le m$ and $i \ne k$. Thus,

$$[x_i, y_1] = a_{1i}^i x_i = a_{1i} x_i \qquad \text{with } 0 \le i \le m.$$

Similarly,

$$0 = [x_i, [h, y_2]] = [[x_i, h], y_2] - [[x_i, y_2], h] = (m-2i)[x_i, y_2] - \sum_{k=0}^{m} a_{2k}^i [x_i, h]$$
$$= (m-2i)\sum_{k=0}^{m} a_{2k}^i x_i - \sum_{k=0}^{m} a_{2k}^i (m-2k)x_k = \sum_{k=0}^{m} a_{2k}^i (m-2i-(m-2k))x_k$$
$$= \sum_{k=0}^{m} 2a_{2k}^i (k-i)x_k,$$

we get $[x_i, y_2] = a_{2i}^i x_i$ with $0 \le i \le m$.

From the identity $[x_i, [y_1, y_2]] = [[x_i, y_1], y_2] - [[x_i, y_2], y_1]$, we deduce

$$\left[x_i, y_1 + \sum_{k=0}^{m} b_{12}^k x_k\right] = a_{1i}^i [x_i, y_2] - a_{2i}^i [x_i, y_1]$$
$$\Rightarrow [x_i, y_1] = a_{1i}^i a_{2i}^i x_i - a_{2i}^i a_{1i}^i x_i = 0,$$

and we have $[x_i, y_1] = 0$ with $0 \le i \le m$, that is, $[I, y_1] = 0$.

We consider the Leibniz identity

$$[x_i, [y_2, e]] = [[x_i, y_2], e] - [[x_i, e], y_2], \qquad \text{for } 0 \le i \le m.$$

Then,

$$0 = a_{2i}^i [x_i, e] - (-mi + i(i-1))[x_{i-1}, y_2]$$
$$= a_{2i}^i (-mi + i(i-1))x_{i-1} - a_{2,i-1}^i (-mi + i(i-1))x_{i-1}$$
$$= -(-mi + i(i-1))(a_{2i}^i - a_{2,i-1}^i)x_{i-1},$$

which leads to $a_{2i}^i = a_{2i-1}^i$, that is, we get $[x_i, y_2] = ax_i$ with $0 \le i \le m$ and some $a \in \mathbb{C}$.

Thus, we have obtained the product $[I, R^{-1}]$.

Verifying that

$$0 = \left[y_1, \sum_{j=0}^{m} a_{1f}^{j} x_j \right] = [y_1, [y_1, f]] = [[y_1, y_1], f] - [[y_1, f], y_1] = [[y_1, y_1], f]$$

$$= \sum_{j=0}^{m} b_1^j [x_j, f] = \sum_{j=0}^{m-1} b_1^j x_{j+1},$$

we obtain $b_1^j = 0$ for $0 \le j \le m-1$, i.e., $[y_1, y_1] = b_1{}^m x_m$.

The equalities

$$0 = [y_1, [y_1, h]] = [[y_1, y_1], h] - [[y_1, h], y_1] = [[y_1, y_1], h] = a_1^m [x_m, h] = -m a_1^m x_m,$$

deduce $b_1^m = 0$, hence $[y_1, y_1] = 0$.

From the identities

$$0 = [y_2, [y_1, h]] = [[y_2, y_1], h] - [[y_2, h], y_1] = -[y_1, h],$$

$$0 = [y_2, [y_1, f]] = [[y_2, y_1], f] - [[y_2, f], y_1] = -[y_1, f],$$

$$0 = [y_2, [y_1, e]] = [[y_2, y_1], e] - [[y_2, e], y_1] = -[y_1, e],$$

we obtain $[y_1, h] = [y_1, f] = [y_1, e] = 0$.

Using the above obtained products and

$$0 = [y_1, [y_2, f]] = [[y_1, y_2], f] - [[y_1, f], y_2] = [[y_1, y_2], f]$$

$$= \left[y_1 + \sum_{k=0}^{m} b_{12}^k x_k, f \right] = \sum_{k=0}^{m} b_{12}^k [x_k, f] = \sum_{k=0}^{m-1} b_{12}^k x_{k+1},$$

we get $b_{12}^i = 0$ with $0 \le i \le m-1$.

Now from

$$0 = [y_1, [y_2, h]] = [[y_1, y_2], h] - [[y_1, h], y_2] = [[y_1, y_2], h]$$

$$= [y_1 + b_{12}^m x_m, h] = -m b_{12}^m x_m,$$

we get $b_{12}^m = 0$. Consequently, $b_{12}^m = 0$ for all $0 \le i \le m$, i.e., $[y_1, y_2] = y_1$.

Thus, we obtain the following table of multiplication:

$$
\begin{array}{lll}
[e,h]=2e, & [h,f]=2f, & [e,f]=h,\\
[h,e]=-2e, & [f,h]=-2f, & [f,e]=-h,\\
[x_k,h]=(m-2k)x_k, & 0\le k\le m, & \\
[x_k,f]=x_{k+1}, & 0\le k\le m-1, & \\
[x_k,e]=-k(m+1-k)x_{k-1}, & 1\le k\le m, & \\
[y_2,e]=\sum_{j=0}^{m}a_{2e}^{j}x_j, & [y_2,f]=\sum_{j=0}^{m}a_{2f}^{j}x_j, & [y_2,h]=\sum_{j=0}^{m}a_{2h}^{j}x_j,\\
[y_1,y_2]=y_1, & [y_2,y_1]=-y_1, & [y_2,y_2]=\sum_{j=0}^{m}b_2^{j}x_j,\\
[x_k,y_2]=ax_k, & 0\le k\le m. &
\end{array}
$$

In order to complete the proof of the theorem, we need to prove that $[y_2, y_2] = 0$, and $[R^{-1}, sl_2^{-1}] = 0$.

Consider two cases.

Case 1. Let $a \neq 0$ be. Then taking the change of the basic element

$$y_2{}' = y_2 - \sum_{j=0}^{m} \frac{b_2^j}{a} x_j,$$

we get

$$
\begin{aligned}
[y_2{}', y_2{}'] &= \left[y_2 - \sum_{j=0}^{m} \frac{b_2^j}{a} x_j, y_2 - \sum_{j=0}^{m} \frac{b_2^j}{a} x_j \right] \\
&= [y_2, y_2] - \left[\sum_{j=0}^{m} \frac{b_2^j}{a} x_j, y_2 \right] = \sum_{j=0}^{m} b_2^j x_j - \sum_{j=0}^{m} b_2^j x_j = 0,
\end{aligned}
$$

which leads to $[y_2, y_2] = 0$.

Consider

$$
\begin{aligned}
0 &= [y_2, [y_2, h]] = [[y_2, y_2], h] - [[y_2, h], y_2] = -[[y_2, h], y_2] \\
&= -\sum_{j=0}^{m} a_{2h}^j [x_j, y_2] = -\sum_{j=0}^{m} a_{2h}^j a x_j,
\end{aligned}
$$

which gives $a_{2h}^j = 0$ for $0 \le j \le m$.

Similarly, from the equalities

$$0 = [y_2, [y_2, f]] = [[y_2, y_2], f] - [[y_2, f], y_2] = -\sum_{j=0}^{m} a_{2f}^j [x_j, y_2] = -\sum_{j=0}^{m} a_{2f}^j a x_j,$$

$$0 = [y_2, [y_2, e]] = [[y_2, y_2], e] - [[y_2, e], y_2] = -\sum_{j=0}^{m} a_{2e}^j [x_j, y_2] = -\sum_{j=0}^{m} a_{2e}^j a x_j,$$

we get $a_{2f}^j = a_{2e}^j = 0$ for $0 \le j \le m$. Hence, $[R^{-1}, sl_2^{-1}] = 0$.

Thus, we proved the theorem for the case of $a \neq 0$.

Case 2. Let $a = 0$ be. Then, we consider the identity

$$[y_2, [y_2, f]] = [[y_2, y_2], f] - [[y_2, f], y_2],$$

and we derive

$$\begin{aligned} 0 &= \sum_{j=0}^{m} b_2^i [x_i, f] = \sum_{j=0}^{m-1} b_2^i x_{i+1} \\ &\Rightarrow b_2^i = 0, \ 0 \leq i \leq m-1, \qquad i.e., \\ &[y_2, y_2] = b_2^m x_m. \end{aligned}$$

From the chain of the equalities

$$0 = [y_2, [y_2, h]] = [[y_2, y_2], h] - [[y_2, h], y_2] = b_2^m [x_m, h] = -m b_2^m x_m,$$

we obtain $b_2^m = 0$, that is, $[y_2, y_2] = 0$.

Let us take the change of the basic element in the form

$$y_2{}' = y_2 - \sum_{j=1}^{m} \frac{a_{2e}^{j-1}}{-mj + j(j-1)} x_j.$$

Then

$$\begin{aligned} [y_2{}', e] &= [y_2, e] - \sum_{j=1}^{m} \frac{a_{2e}^{j-1}}{-mj + j(j-1)} [x_j, e] \\ &= [y_2, e] - \sum_{j=1}^{m} \frac{a_{2e}^{j-1}}{-mj + j(j-1)} (-mj + j(j-1)) x_{j-1} \\ &= \sum_{j=0}^{m} a_{2e}^j x_j - \sum_{j=1}^{m} a_{2e}^{j-1} x_{j-1} \\ &= \sum_{j=0}^{m} a_{2e}^j x_j - \sum_{j=0}^{m-1} a_{2e}^j x_j = a_{2e}^m x_m. \end{aligned}$$

Thus, we can assume that

$$[y_2, e] = a_{2e}^m x_m, \qquad [y_2, h] = \sum_{j=0}^{m} a_{2h}^j x_j, \qquad [y_2, f] = \sum_{j=0}^{m} a_{2f}^j x_j.$$

We have

$$\begin{aligned} [y_2, [e, h]] &= [[y_2, e], h] - [[y_2, h], e] = a_{2e}^m [x_m, h] - \sum_{j=0}^{m} a_{2h}^j [x_j, e] \\ &= -m a_{2e}^m x_m - \sum_{j=0}^{m} a_{2h}^j (-mj + j(j-1)) x_{j-1}. \end{aligned}$$

On the other hand, $[y_2, [e, h]] = 2[y_2, e] = 2a_{2e}^m x_m$.

Comparing the coefficients at the basic elements, we get $a_{2e}^m = 0$ and $a_{2h}^j = 0$, where $1 \leq j \leq m$. Hence,

$$[y_2, e] = 0, \qquad [y_2, f] = \sum_{j=0}^{m} a_{2f}^j x_j, \qquad [y_2, h] = a_{2h}^0 x_0.$$

Consider

$$\begin{aligned}[y_2, [e, f]] = [[y_2, e], f] - [[y_2, f], e] &= -\sum_{j=0}^{m} a_{2f}^j [x_j, e] \\ &= -\sum_{j=0}^{m} a_{2f}^j (mj + j(j-1)) x_{j-1} \\ &= m a_{2f}^1 x_0 - \sum_{j=2}^{m} a_{2f}^j (mj + j(j-1)) x_{j-1}.\end{aligned}$$

On the other hand,

$$[y_2, [e, f]] = [y_2, h] = a_{2h}^0 x_0.$$

Comparing the coefficients, we obtain $a_{2h}^0 = m a_{2f}^1$ and $a_{2f}^j = 0$ for $2 \leq j \leq m$. Then, we have the product $[y_2, f] = a_{2f}^0 x_0 + a_{2f}^1 x_1$.

Now, we consider the equalities

$$-2[y_2, f] = [y_2, [f, h]] = [[y_2, f], h] - [[y_2, h], f] = [a_{2f}^0 x_0 + a_{2f}^1 x_1, h] - m a_{2f}^1 [x_0, f],$$

and we have

$$-2a_{2f}^0 x_0 - 2a_{2f}^1 x_1 = m a_{2f}^0 x_0 + a_{2f}^1 (m-2) x_1 - m a_{2f}^1 x_1 \Rightarrow a_{2f}^0 = 0.$$

Therefore, $[y_2, f] = a_{2f}^1 x_1$ and $[y_2, h] = m a_{2f}^1 x_0$.

Taking the change $y_2{}' = y_2 - a_{2f}^1 x_0$, we obtain

$$[y_2{}', f] = [y_2{}', h] = 0.$$

Thus, we have $[R^{-1}, sl_2^{-1}] = 0$, which completes the proof of the theorem. □

Remark 3.2. From the description of Theorem 3.1, we conclude that $L = (sl_2 \oplus R) \dot{+} I$. Moreover, if $L/I \cong sl_2 \oplus R$ with $dim I \neq 3$ and there exist two element $\bar{x}, \bar{y}$ of the finite-dimensional solvable radical R such that $[\bar{x}, \bar{y}] = -[\bar{y}, \bar{x}] = \bar{x}$, then in a similar way to the proof of Theorem 3.1, one can prove that the restriction of the operator of right multiplication on an element y to ideal I is $\alpha \cdot id$ and the restriction of the operator of right multiplication on an element x is trivial.

When the dimension of the ideal I is equal to three, we have the family of Leibniz algebras with the following table of multiplication:

$$\begin{array}{lll}
[e,h]=2e, & [h,f]=2f, & [e,f]=h, \\
[h,e]=-2e, & [f,h]=-2f, & [f,e]=-h, \\
[x_1,e]=-2x_0, & [x_2,e]=-2x_1, & [x_0,f]=x_1, \\
[x_1,f]=x_2, & [x_0,h]=2x_0, & [x_2,h]=-2x_2, \\
[e,y_1]=\lambda x_0, & [f,y_1]=\frac{1}{2}\lambda x_2, & [h,y_1]=\lambda x_1, \\
[e,y_2]=\mu x_0, & [f,y_2]=\frac{1}{2}\mu x_2, & [h,y_2]=\mu x_1, \\
[y_1,y_2]=y_1, & [y_2,y_1]=-y_1, & [y_2,y_2]=-\frac{ab}{2}x_2, \\
[x_0,y_2]=ax_0, & [x_1,y_2]=ax_1, & [x_2,y_2]=ax_2, \\
[y_2,e]=bx_1, & [y_2,h]=bx_2. &
\end{array}$$

Verifying the Leibniz identity of the above family of algebras and using the program in software *Mathematica* (see in [3]), we derive the condition $\lambda(1-a)=0$.

Taking the change of the basic element in the form $y_2' = y_2 + \frac{b}{2}x_2$, we obtain

$$[y_2', e] = [y_2', h] = [y_2', y_2'] = 0.$$

Thus, we have the family of algebras $L(\lambda, \mu, a)$:

$$\begin{array}{lll}
[e,h]=2e, & [h,f]=2f, & [e,f]=h, \\
[h,e]=-2e, & [f,h]=-2f, & [f,e]=-h, \\
[x_1,e]=-2x_0, & [x_2,e]=-2x_1, & [x_0,f]=x_1, \\
[x_1,f]=x_2, & [x_0,h]=2x_0, & [x_2,h]=-2x_2, \\
[e,y_1]=\lambda x_0, & [f,y_1]=\frac{1}{2}\lambda x_2, & [h,y_1]=\lambda x_1, \\
[e,y_2]=\mu x_0, & [f,y_2]=\frac{1}{2}\mu x_2, & [h,y_2]=\mu x_1, \\
[x_0,y_2]=ax_0, & [x_1,y_2]=ax_1, & [x_2,y_2]=ax_2, \\
[y_1,y_2]=y_1, & [y_2,y_1]=-y_1, &
\end{array}$$

with the condition $\lambda(1-a)=0$.

Theorem 3.3. *Let L be a Leibniz algebra such that $L/I \cong sl_2 \oplus R$, where R is a two-dimensional solvable ideal and I is a three-dimensional right irreducible module over sl_2. Then, L is isomorphic to one of the following pairwise non isomorphic algebras*:

$$L(1,0,1); \qquad L(0,1,a); \qquad L(0,0,a), \qquad \text{with } a \in F.$$

Proof. We shall consider the equality $\lambda(1-a)=0$.

Let $\lambda \neq 0$. Then $a=1$. Making the basis transformation

$$y_1' = \frac{1}{\lambda}y_1, \quad y_2' = -\frac{\mu}{\lambda}y_1 + y_2,$$

we deduce that

$$[e, y_1'] = x_0, \qquad [f, y_1'] = \frac{1}{2}x_2, \qquad [h, y_1'] = x_1, \qquad [e, y_2'] = [f, y_2'] = [h, y_2'] = 0$$

and the rest products of the family of algebra $L(\lambda, \mu, a)$ are not changed.

Thus, we can assume that $\lambda = 1$ and $\mu = 0$. Hence, we get the algebra $L(1, 0, 1)$.

If $\lambda = 0$, then for $\mu \neq 0$ by a suitable scaling of the basic elements of I, we can suppose that $\mu = 1$. Thus, we obtain the algebra $L(0, 1, a)$.

If $\lambda = 0$, then for $\mu = 0$ we get the algebra $L(0, 0, a)$.

Using a program in software *Mathematica*, we conclude that these algebras are nonisomorphic. This program establishes when two algebras are isomorphic; moreover, we have added some subroutines to know if two algebras are isomorphic or not, when one of them is an uniparameter family. It returns the value of the parameter for which it would be isomorphic. The implementation of this program is presented for low and fixed dimension. Then we will formulate the generalizations, proving by induction the results for arbitrary fixed dimension. Finally, to point out that the algorithmic method of these programs is presented with a step-by-step explanation in the following web site: http://personal.us.es/jrgomez.

Thus the theorem is proved. □

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