Algebra of derivations of Lie algebras☆

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Abstract

We show a method to determine the space of derivations of any Lie algebra, and in particular we apply this method to a special class of Lie algebras, those nilpotent with low nilindex. Most calculations have been supported by the software *Mathematica* 3.0.

1. Introduction

Lots of geometric and algebraic properties of Lie algebras, such as the computation of the dimension of their orbits or the determination of their first spaces of cohomology [8] can be studied based on the knowledge of the corresponding space of derivations, $\mathscr{D}\mathrm{er}(\mathfrak{g})$. But $\mathscr{D}\mathrm{er}(\mathfrak{g})$ has also applications to other problems in mathematics, like the classification of solvable Lie algebras from the nilpotent or Gelfand–Kirillov's conjecture about the field of fractions of enveloping algebras of Lie algebras [1,2], and also in Physics as in the study of the interaction of particles [11], etc.

Up to now, in order to determine the space of derivations of Lie algebras was needed a suitable gradation for the algebra containing homogeneous subspaces of

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dimension as low as possible [3,7], and this fact is specially difficult to obtain when the nilindex is low.

We are now going to describe a process that allows us to determine the space of derivations $\mathscr{D}\mathrm{er}(\mathfrak{g})$ for a given Lie algebra $\mathfrak{g}$ and with that method we will not need to have a special gradation of the algebra and particularly, this method will be more precise with algebras of low nilindex, but it is also valid to determine the algebra of derivations of Lie algebras of high nilindex or, even, for non-nilpotent Lie algebras.

The *descending central sequence* of a Lie algebra $\mathfrak{g}$ of dimension n is defined by $\mathscr{C}^0(\mathfrak{g}) = \mathfrak{g}$, $\mathscr{C}^i(\mathfrak{g}) = [\mathfrak{g}, \mathscr{C}^{i-1}(\mathfrak{g})]$. If $\mathscr{C}^k(\mathfrak{g}) = 0$ for some k, the corresponding Lie algebra is called *nilpotent*. The smallest integer k such that the equality $\mathscr{C}^k(\mathfrak{g}) = 0$ holds is called the *nilindex* of $\mathfrak{g}$. In general, the nilindex ranges from 1 (*abelian*) to $n - 1$ (*filiform Lie algebra*).

In this way, the *derived sequence* $(\mathscr{D}^n(\mathfrak{g}))$ of ideals in $\mathfrak{g}$ is defined inductively by $\mathscr{D}^1(\mathfrak{g}) = \mathfrak{g}$, and $\mathscr{D}^n(\mathfrak{g}) = [\mathscr{D}^{n-1}(\mathfrak{g}), \mathscr{D}^{n-1}(\mathfrak{g})]$ for $n > 1$. If there exists an integer n such that $\mathscr{D}^n(\mathfrak{g}) = \{0\}$, then $\mathfrak{g}$ is said to be a *solvable Lie algebra*, for more details see [12].

Let $\mathfrak{g}$ be a Lie algebra over $\mathbf{K}$. A linear endomorphism d of $\mathfrak{g}$ is called a derivation of $\mathfrak{g}$ if it satisfies

$$d([X, Y]) = [d(X), Y] + [X, d(Y)] \quad \forall X, Y \in \mathfrak{g}.$$

It is easy to see that the set $\mathscr{D}\mathrm{er}(\mathfrak{g})$ of all derivations of $\mathfrak{g}$ is a vector subspace of $\mathrm{End}\,\mathfrak{g}$, furthermore $\mathscr{D}\mathrm{er}(\mathfrak{g})$ is a Lie algebra over $\mathbf{K}$ for the bracket $[d, d'] = d \circ d' - d' \circ d \; \forall d, d' \in \mathscr{D}\mathrm{er}(\mathfrak{g})$.

A $\mathbf{Z}$-gradation for a Lie algebra $\mathfrak{g}$ consists in a decomposition as direct sum into vectorial spaces $\mathfrak{g} = \bigoplus_{i \in \mathbf{Z}} \mathfrak{g}_i$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$.

Let $\mathfrak{g}$ be a Lie algebra on a field $\mathbf{K}$. It is often convenient to use the language of modules along with the (equivalent) language of representations. As in other algebraic theories, there is a natural definition (see [9]). A vector space V over $\mathbf{K}$, endowed with a bilinear operation $\mathfrak{g} \times V \to V$ (denoted $(x, v) \mapsto x \cdot v$ or just xv) is called a $\mathfrak{g}$-*module* if the following condition is satisfied:

$$[x, y] \cdot v = x \cdot y \cdot v - y \cdot x \cdot v \quad \forall x, y \in \mathfrak{g}, \quad v \in V.$$

For example, if $\phi : \mathfrak{g} \to \mathrm{End}(V)$ is a representation of $\mathfrak{g}$ (i.e., a homomorphism of Lie algebras), then V may be viewed as a $\mathfrak{g}$-module via the action $x \cdot v = \phi(x)(v)$. Conversely, given a $\mathfrak{g}$-module V, this equation defines a representation $\phi : \mathfrak{g} \to \mathrm{End}(V)$.

Throughout the following sections, $\mathbf{K}$ is a field of characteristic 0, and all algebras and modules are finite-dimensional over $\mathbf{K}$.

1.1. Semisimple Lie algebras

Let $\mathfrak{g}$ be a Lie algebra. Then always there exists a solvable ideal in $\mathfrak{g}$ which contains all other solvable ideals. This largest solvable ideal is called the *radical* of $\mathfrak{g}$.

One says that $\mathfrak{g}$ is *semisimple* if its radical 0 or an equivalent condition is that $\mathfrak{g}$ *contains no non-zero abelian ideal.*

Now, let V be a $\mathfrak{g}$-module, and $\phi : \mathfrak{g} \to \mathrm{End}(V)$ the corresponding representation.

Definition 1.1. V (or ϕ) is called *simple* (or *irreducible*) if $V \neq (0)$ and V has no submodules other than (0) and V.

V (or ϕ) is called *semisimple* (or *completely reducible*) if V is the direct sum of simple submodules.

Note that $\mathfrak{g}$ may be semisimple as a $\mathfrak{g}$-module without being a semisimple Lie algebra; for example, $\mathfrak{g} = \mathbf{K}$.

Theorem 1.1 (H. Weyl [11]). *If* $\mathfrak{g}$ *is semisimple, all* $\mathfrak{g}$*-modules* (*of finite dimension*) *are semisimple.*

1.2. Reductive Lie algebras

It is possible to use the foregoing results to obtain some general results concerning semisimple representations of arbitrary Lie algebras and the structure of reductive Lie algebras.

A Lie algebra $\mathfrak{g}$ is called *reductive* if its radical (rad $\mathfrak{g}$) coincides with its center (cf. [10]), and in this case, $\mathfrak{g}$ is the direct sum of its center (that is an abelian algebra) and $\mathscr{D}(\mathfrak{g})$ (that is semisimple) [13].

We now show the following decisive criterion for the semisimplicity of a representation of an arbitrary Lie algebra.

Theorem 1.2 [13]. *Let* $\mathfrak{g}$ *be a reductive Lie algebra over* $\mathbf{K}$, *and let* ϕ *be a finite-dimensional representation of* $\mathfrak{g}$. *Then* ϕ *is completely reducible if and only if for every element X in the radical of* $\mathfrak{g}$, $\phi(X)$ *is a semisimple endomorphism* (*i.e.*, *diagonalizable over the algebraic closure of* $\mathbf{K}$).

Furthermore, let V be the space on which ϕ acts, so if the hypothesis of the theorem is satisfied we can then find distinct elements $\lambda_1, \ldots, \lambda_p$ on the dual space $(\mathrm{rad}\ \mathfrak{g})^*$ and subspaces $V_1, \ldots, V_p$ of V such that V is the direct sum of the V_i and $V_i = \{v\colon\ v \in V,\ Xv = \lambda_i(X)v\ \forall X \in \mathrm{rad}\ \mathfrak{g}\}$, obtaining with that direct sum the decomposition into simple $\mathfrak{g}$-submodules of the semisimple $\mathfrak{g}$-module V.

The following result holds if we restrict to the algebras $\mathfrak{g} = \mathfrak{a}$ with $\mathfrak{a}$ abelian instead of considering general reductive Lie algebras $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{a}$ with $\mathfrak{g}_1$ semisimple and $\mathfrak{a}$ abelian Lie algebras. By applying the preceding theorem to a particular abelian Lie algebra (which always coincides with its radical) we are going to obtain a new method that simplifies the computations involved in the determination of the algebra of derivations of any Lie algebra.

1.3. Toral-method

Let $\mathfrak{g}$ be an arbitrary Lie algebra, and $\mathscr{D}\mathrm{er}(\mathfrak{g})$ the corresponding algebra of derivations that is also a Lie algebra. We now consider a special subalgebra of $\mathscr{D}\mathrm{er}(\mathfrak{g})$ called T that it is constituted by the set, the largest that can be found, of derivations that are simultaneously diagonalizable. This algebra is abelian, coincides with its radical and it is formed by diagonal endomorphisms X, so any endomorphism of the form $\phi(X)$ there will be a semisimple endomorphism. Thus, by applying the precedent theorem every T-module will be semisimple via the correspondence between representations completely reducible and semisimple modules.

An easy calculation shows that $\mathfrak{g}$ and $\mathscr{D}\mathrm{er}(\mathfrak{g})$ are both structure of T-modules by considering the operations or 'products' $t \cdot g = t(g)$, $\forall t \in T$, $g \in \mathfrak{g}$ and $t \cdot d = [t, d]$, $\forall t \in T, d \in \mathscr{D}\mathrm{er}(\mathfrak{g})$, respectively. So $\mathfrak{g}$ and $\mathscr{D}\mathrm{er}(\mathfrak{g})$ are semisimple T-modules, obtaining from that fact the following decompositions into direct sums, in the sense of vectorial spaces, of simple T-modules:

- $\mathfrak{g} = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \cdots$
  $\mathfrak{g}_{\alpha_i} = \{g/t \cdot g = \alpha_i(t)g \ \forall t \in T\}$
  with $\alpha_i \in T^*$, such that $\alpha_i(T_j) = \delta_{i,j}$ with $\{T_i\}$ a basis of T, and the subspaces $\mathfrak{g}_{\alpha_i}$ verifies that $[\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{\alpha_j}] \subset \mathfrak{g}_{\alpha_i+\alpha_j}$, i.e., if there exist $\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{\alpha_j}$ with $[\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{\alpha_j}] \neq 0$, then the space $\mathfrak{g}_{\alpha_i+\alpha_j}$ also exists.
- $\mathscr{D}\mathrm{er}(\mathfrak{g}) = \mathscr{D}_{\lambda_1} \oplus \mathscr{D}_{\lambda_2} \oplus \cdots$
  $\mathscr{D}_{\lambda_i} = \{d \in \mathscr{D}\mathrm{er}(\mathfrak{g})/t \cdot d = \lambda_i(t) \cdot d \ \forall t \in T\}$
  $\lambda_i \in T^*$, and the subspaces $\mathscr{D}_{\lambda_i}$ verify similar conditions to the precedents $\mathfrak{g}_{\alpha_i}$.

It remains to determine which is the form of each space $\mathscr{D}_{\lambda_i}$ belonging to the gradation obtained for $\mathscr{D}\mathrm{er}(\mathfrak{g})$. On the other hand, if we use the previous gradation for $\mathfrak{g}$, any derivation $d \in \mathscr{D}_{\lambda_i}$ verifies $d(\mathfrak{g}_{\alpha_i}) \subset \mathfrak{g}_{\alpha_j}$, from that, with an easy calculation we lead to the following form to λ_i, that is, $\lambda_i(t) = \alpha_j(t) - \alpha_i(t)$. This last fact determines the type of derivations that form part of every $\mathscr{D}_{\alpha_j-\alpha_i}$.

The steps that can be followed to calculate the algebra of derivations of a Lie algebra are the following:

Step 1: The first step consists in determining into $\mathscr{D}\mathrm{er}(\mathfrak{g})$ a particular set, the biggest possible, constituted by diagonal derivations. For that, we choose a gradation for $\mathfrak{g}$ ($\mathfrak{g} = \bigoplus_{i\in\mathbf{Z}} \mathfrak{g}_i$) in the sense that this gradation must contain as many homogeneous subspaces of dimension 1 as possible.

An easy way to find diagonal derivations is to determine the subspace d_0 of $\mathscr{D}\mathrm{er}(\mathfrak{g})$ such that

$$d_0(\mathfrak{g}_i) \subset \mathfrak{g}_i, \quad i \in \mathbf{Z},$$

continuously, we selected those elements of a certain basis in which they are diagonals.

Step 2: Let T be a subalgebra form by the diagonal derivations. Thus,

$$t \in T \implies t = \sum_{i \in \mathbf{Z}} \alpha_i(t) d_0^i,$$

where the α_i's are the dual basis of the d_0^i's, these last ones are the diagonal derivations.

Step 3: In this step, we are going to determine the differences of the 'weights' [1] of the subspaces. Then imposing to a generic derivation in each subspace the conditions that actual derivation must hold, we determine all possible derivations. In this step, we have used as an invaluable tool the software *Mathematica* given the huge amount of data involved in these computations.

Remark 1.3. The method essentially reduces the computations comparing with the methods that have been used up to now, being specially efficient for that algebras of low nilindex whose treatment results very complicated by the precedent methods.

Remark 1.4. If the subalgebra T is one-dimensional, our method is reduced to the one used in [3], so the new method (called *Toral-method*) generalizes the precedent.

Remark 1.5. The Toral-method can be used to determine $\mathscr{D}\mathrm{er}(\mathfrak{g})$ even if $\mathfrak{g}$ is not nilpotent.

2. Applying the Toral-method to $(n-3)$-filiform Lie algebras

We now apply the method described in the precedent section to a wide family $\mathfrak{g}$ of Lie algebras with nilindex 3 and from which we already know $\mathscr{D}\mathrm{er}(\mathfrak{g})$ what allows us to contrast the methods.

In [4,5], $(n-3)$-filiform Lie algebras are classified (by different methods), obtaining the algebras $\mathfrak{g}_n^i$'s with i ranging from 1 to $n-2$, and their laws can be expressed in a certain adapted basis $\{X_0, X_1, X_2, X_3, Y_1, Y_2, \ldots, Y_{n-5}, Y_{n-4}\}$ by

$$\mathfrak{g}_n^{2q-1}\colon \begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant 2, \\ [Y_{2k-1}, Y_{2k}] = X_3, & 1 \leqslant k \leqslant q-1, \end{cases} \qquad 1 \leqslant q \leqslant E\left(\frac{n-2}{2}\right);$$

$$\mathfrak{g}_n^{2s}\colon \begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant 2, \\ [X_1, Y_{n-4}] = X_3, & \\ [Y_{2k-1}, Y_{2k}] = X_3, & 1 \leqslant k \leqslant s-1, \end{cases} \qquad 1 \leqslant s \leqslant E\left(\frac{n-3}{2}\right);$$

$$\mathfrak{g}_n^{n-2}\colon \begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant 2, \\ [X_1, X_2] = Y_{n-4}. & \end{cases}$$

[1] The notion of weight is usually used in the theory of semisimple algebras though we used it for arbitrary algebras using an analogous meaning with the semisimple case.

As an application of the Toral-method, continuously we are going to calculate the space of derivations of the family of algebras $\mathfrak{g}_n^{2q-1}$.

Lemma 2.1. *The linear mappings* $ad(X_0)$, $ad(X_1)$, $ad(X_2)$, $ad(Y_{2i})$, $ad(Y_{2i-1})$, t_0, t_1, d_0^i, h_0^r, $d_{\beta-\alpha}$, $d_{\alpha+\beta-\lambda_i}$, $d_{-\alpha+\lambda_i}$, $d_{2\alpha}$, $d_{2\alpha+\beta-\delta_r}$, $d_{\lambda_i-\lambda_k}$, $d_{-2\alpha-\beta+\lambda_i+\lambda_j}$, $d_{-2\alpha-\beta+2\lambda_i}$, $d_{2\alpha+\beta-\lambda_i-\lambda_j}$, $d_{2\alpha+\beta-2\lambda_i}$, $d_{\delta_r-\alpha}$, $d_{\delta_r-\beta}$, $d_{\delta_r-\lambda_i}$, $d_{-2\alpha-\beta+\delta_r+\lambda_i}$, $d_{\delta_r-\delta_s}$, $1 \leqslant i, j, k \leqslant q-1$, $i \neq k$, $i < j$ *and* $2q-1 \leqslant r, s \leqslant n-4$ *and* $r \neq s$ *of the space* $\mathfrak{g}_n^{2q-1}$, *with*

$$\begin{cases} t_0(X_0) = X_0, \\ t_0(X_2) = X_2, \\ t_0(X_3) = 2X_3, \\ t_0(Y_{2j}) = 2Y_{2j}, \end{cases} \quad \begin{cases} t_1(X_1) = X_1, \\ t_1(X_2) = X_2, \\ t_1(X_3) = X_3, \\ t_1(Y_{2j}) = Y_{2j}, \end{cases} \quad \begin{cases} d_0^i(Y_{2i}) = -Y_{2i}, \\ d_0^i(Y_{2i-1}) = Y_{2i-1}, \end{cases}$$

$$\begin{cases} h_0^r(Y_r) = Y_r, \end{cases} \quad \begin{cases} d_{\beta-\alpha}(X_0) = X_1, \end{cases} \quad \begin{cases} d_{\alpha+\beta-\lambda_i}(X_0) = Y_{2i}, \\ d_{\alpha+\beta-\lambda_i}(Y_{2i-1}) = X_2, \end{cases}$$

$$\begin{cases} d_{-\alpha+\lambda_i}(X_0) = Y_{2i-1}, \\ d_{-\alpha+\lambda_i}(Y_{2i}) = -X_2, \end{cases} \quad \{d_{2\alpha}(X_1) = X_3, \quad \begin{cases} d_{2\alpha+\beta-\delta_r}(Y_r) = X_3, \end{cases}$$

$$\begin{cases} d_{-2\alpha-\beta+\lambda_i+\lambda_j}(Y_{2j}) \\ \quad = Y_{2i-1}, \\ d_{-2\alpha-\beta+\lambda_i+\lambda_j}(Y_{2i}) \\ \quad = Y_{2j-1}, \end{cases} \quad \begin{cases} d_{-2\alpha-\beta+2\lambda_i}(Y_{2i}) \\ \quad = Y_{2i-1}, \end{cases} \quad \begin{cases} d_{2\alpha+\beta-\lambda_i-\lambda_j}(Y_{2j-1}) = Y_{2i}, \\ d_{2\alpha+\beta-\lambda_i-\lambda_j}(Y_{2i-1}) = Y_{2j}, \end{cases}$$

$$\begin{cases} d_{2\alpha+\beta-2\lambda_i}(Y_{2i-1}) = Y_{2i}, \end{cases} \quad \begin{cases} d_{\lambda_i-\lambda_k}(Y_{2k-1}) = -Y_{2i-1}, \\ d_{\lambda_i-\lambda_k}(Y_{2i}) = Y_{2k}, \end{cases} \quad \begin{cases} d_{\delta_r-\alpha}(X_0) = Y_r, \end{cases}$$

$$\begin{cases} d_{\delta_r-\beta}(X_1) = Y_r, \end{cases} \quad \begin{cases} d_{\delta_r-\lambda_i}(Y_{2i-1}) = Y_r \end{cases} \quad \begin{cases} d_{-2\alpha-\beta+\delta_r+\lambda_i}(Y_{2i}) = Y_r, \end{cases}$$

$$\begin{cases} d_{\delta_r-\delta_s}(Y_s) = Y_r \end{cases}$$

are derivations over $\mathfrak{g}_n^{2q-1}$.

Theorem 2.2. *The linear mappings described above constitute a basis for* $\mathscr{D}\mathrm{er}(\mathfrak{g}_n^{2q-1})$.

Proof. Following the method description it is necessary to calculate the diagonal derivations and in order to do it we first need a gradation for the algebra. So let

$$\mathfrak{g}_i = \begin{cases} \langle Y_{-2i} \rangle, & 1-q \leqslant i \leqslant -1, \\ \langle 0 \rangle, & i = 0, \\ \langle X_{i-1} \rangle, & 1 \leqslant i \leqslant 4, \\ \langle Y_{2i-9} \rangle, & 5 \leqslant i \leqslant q+3, \\ \langle Y_{i+q-5} \rangle & q+4 \leqslant i \leqslant n-(q-1), \end{cases}$$

$$\begin{aligned} &\langle Y_{2q-2} \rangle \oplus \cdots \oplus \langle Y_6 \rangle \oplus \langle Y_4 \rangle \oplus \langle Y_2 \rangle \oplus \langle 0 \rangle \\ &\quad \oplus \langle X_0 \rangle \oplus \langle X_1 \rangle \oplus \langle X_2 \rangle \oplus \langle X_3 \rangle \\ &\quad \oplus \langle Y_1 \rangle \oplus \langle Y_3 \rangle \oplus \cdots \oplus \langle Y_{2q-3} \rangle \oplus \langle Y_{2q-1} \rangle \oplus \cdots \oplus \langle Y_{n-4} \rangle, \end{aligned}$$

a gradation of the mentioned family. Using this gradation we can determine the space of derivations d_0, such that $d_0(\mathfrak{g}_i) \subset \mathfrak{g}_i$; and from this space we form the subalgebra T by choosing those of them that be diagonal.

Computation to d_0.

$$\begin{aligned} &d_0(X_i) = \alpha_i X_i, \quad 0 \leqslant i \leqslant 3, \\ &d_0(Y_j) = \beta_j Y_j, \quad 1 \leqslant j \leqslant n-4, \end{aligned}$$

by imposing the derivation condition for each pair of vectors belonging to the basis we lead to the following restrictions for the parameters:

$$\begin{aligned} &\alpha_2 = \alpha_0 + \alpha_1, \\ &\alpha_3 = 2\alpha_0 + \alpha_1, \\ &\beta_{2k} = 2\alpha_0 + \alpha_1 - \beta_{2k-1}, \quad 1 \leqslant k \leqslant q-1. \end{aligned}$$

Obtaining that a possible basis for d_0 can be formed by

$$\{t_0, t_1\} \cup \left\{d_0^i,\ 1 \leqslant i \leqslant q-1\right\} \cup \left\{h_0^j,\ 2q-1 \leqslant j \leqslant n-4\right\}$$

with $1 \leqslant q \leqslant E\left(\frac{n-2}{2}\right)$ and $\dim(d_0) = n-q-1$.

To construct a *torus of derivations*, that is, a subalgebra T formed by diagonal derivations, we have to take those derivations in d_0 that have the desired form. In the particular case in which we have been working now $T = d_0$, so if $d \in T$, then

$$\begin{cases} d(X_0) = \alpha X_0, \\ d(X_1) = \beta X_1, \\ d(X_2) = (\alpha+\beta) X_2, \\ d(X_3) = (2\alpha+\beta) X_3, \\ d(Y_{2i-1}) = \lambda_i Y_{2i-1}, & 1 \leqslant i \leqslant q-1, \\ d(Y_{2i}) = (2\alpha+\beta-\lambda_i) Y_{2i}, & 1 \leqslant i \leqslant q-1, \\ d(Y_j) = \delta_j Y_j, & 2q-1 \leqslant j \leqslant n-4, \end{cases}$$

and we have $T = \alpha t_0 + \beta t_1 + \sum_{i \in \{1,q-1\}} \lambda_i d_0^i + \sum_{j \in \{2q-1,n-4\}} \delta_j h_0^j$, leading to the following gradation for the algebra:

$$\begin{aligned}\mathfrak{g}_n^{2q-1} &= \mathfrak{g}_{2\alpha+\beta-\lambda_{q-1}} \oplus \cdots \oplus \mathfrak{g}_{2\alpha+\beta-\lambda_1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{\alpha+\beta} \\ &\quad \oplus \mathfrak{g}_{2\alpha+\beta} \oplus \mathfrak{g}_{\lambda_1} \oplus \cdots \oplus \mathfrak{g}_{\lambda_{q-1}} \oplus \mathfrak{g}_{\delta_{2q-1}} \oplus \cdots \oplus \mathfrak{g}_{\delta_{n-4}}.\end{aligned}$$

Once, we have found the weights, we calculate all possible differences between them determining the corresponding subspaces of the space of derivations.

Calculate the differences between weights:

Case 1. $\alpha - \beta$; in this case, $d(X_1) = \alpha_1 X_0$ and the remaining ones vanish. Imposing the fact that d is a derivation we arrive at $\alpha_1 = 0$. Thus, $\mathscr{D}_{\alpha-\beta} = \{0\}$.

Case 2. $\beta - \alpha$; in this case, $d(X_0) = \alpha_0 X_1$ and the rest vanish. Imposing again that d is a derivation, we obtain that the parameter α_0 can take any value, leading to a derivation that will be represented by $d_{\beta-\alpha}(X_0) = X_1$. So $\mathscr{D}_{\beta-\alpha} = \langle d_{\beta-\alpha}\rangle$.

Case 3. $-2\alpha - \beta + \lambda_i + \lambda_j$, $1 \leqslant i < j \leqslant q-1$; if $i \neq j$, we have

$$d(Y_{2j}) = \beta_{2j} Y_{2i-1}, \quad 1 \leqslant i, j \leqslant q-1,$$

$$d(Y_{2i}) = \beta_{2i} Y_{2j-1}, \quad 1 \leqslant i, j \leqslant q-1.$$

Imposing once again the derivation condition, $(q-1)(q-2)/2$ derivations are obtained, $d_{-2\alpha-\beta+\lambda_i+\lambda_j}$.

If $i = j$, we obtain $q-1$ derivations, $d_{-2\alpha-\beta+2\lambda_i}$. Thus,

$$\mathscr{D}_{-2\alpha-\beta+2\lambda_i} = \langle d_{-2\alpha-\beta+2\lambda_i}\rangle, \quad 1 \leqslant i \leqslant q-1,$$

$$\mathscr{D}_{-2\alpha-\beta+\lambda_i+\lambda_j} = \langle d_{-2\alpha-\beta+\lambda_i+\lambda_j}\rangle, \quad 1 \leqslant i < j \leqslant q-1.$$

The remaining differences are in correspondence with analogous situations that we have yet to treat.

Concluding, in Table 1 we show all the differences and all linear mappings from each one, together to the dimension of the corresponding subspace of derivations.

The mappings in Table 1 together to *the torus of derivations*, constitute a basis for the space of derivations. Then

$$\dim\Big(\mathscr{D}\mathrm{er}\big(\mathfrak{g}_n^{2q-1}\big)\Big) = n^2 - (3+2q)n + 2q^2 + 3q + 6. \qquad \square$$

Most calculations have been supported by the package of symbolic calculus, *Mathematica* 3.0. The program that we explicit allows us to obtain the space of derivations of any Lie algebras for concrete dimensions. It is very useful in order to guess the

Table 1

Differences	Derivations	Dimension
$\beta-\alpha$	$d_{\beta-\alpha}$	$\dim(\mathscr{D}_{\beta-\alpha})=1$
β	$ad(X_1)$	$\dim(\mathscr{D}_{\beta})=1$
α	$ad(X_0)$	$\dim(\mathscr{D}_{\alpha})=1$
$\alpha+\beta-\lambda_i$	$d_{\alpha+\beta-\lambda_i}$	$\dim(\oplus\mathscr{D}_{\alpha+\beta-\lambda_i})=q-1$
$\lambda_i-\alpha$	$d_{\lambda_i-\alpha}$	$\dim(\oplus\mathscr{D}_{\lambda_i-\alpha})=q-1$
$\alpha+\beta$	$ad(X_2)$	$\dim(\mathscr{D}_{\alpha+\beta})=1$
2α	$d_{2\alpha}$	$\dim(\mathscr{D}_{2\alpha})=1$
$2\alpha+\beta-\lambda_i$	$ad(Y_{2i})$	$\dim(\oplus\mathscr{D}_{2\alpha+\beta-\lambda_i})=q-1$
λ_i	$ad(Y_{2i-1})$	$\dim(\oplus\mathscr{D}_{\lambda_i})=q-1$
$2\alpha+\beta-\delta_h$	$d_{2\alpha+\beta-\delta_h}$	$\dim(\oplus\mathscr{D}_{2\alpha+\beta-\delta_h})=n-2q-2$
$\lambda_i-\lambda_k$	$d_{\lambda_i-\lambda_k}$	$\dim(\oplus\mathscr{D}_{\lambda_i-\lambda_k})=q^2-3q+2$
$-2\alpha-\beta+\lambda_i+\lambda_j$	$d_{-2\alpha-\beta+\lambda_i+\lambda_j}$	$\dim(\oplus\mathscr{D}_{-2\alpha-\beta+\lambda_i+\lambda_j})=\frac{(q-1)(q-2)}{2}$
$-2\alpha-\beta+2\lambda_i$	$d_{-2\alpha-\beta+2\lambda_i}$	$\dim(\oplus\mathscr{D}_{-2\alpha-\beta+2\lambda_i})=q-1$
$2\alpha+\beta-\lambda_i-\lambda_j$	$d_{2\alpha+\beta-\lambda_i-\lambda_j}$	$\dim(\oplus\mathscr{D}_{2\alpha+\beta-\lambda_i-\lambda_j})=\frac{(q-1)(q-2)}{2}$
$2\alpha+\beta-2\lambda_i$	$d_{2\alpha+\beta-2\lambda_i}$	$\dim(\oplus\mathscr{D}_{2\alpha+\beta-2\lambda_i})=q-1$
$\delta_h-\alpha$	$d_{\delta_h-\alpha}$	$\dim(\oplus\mathscr{D}_{\delta_h-\alpha})=n-2q-2$
$\delta_h-\beta$	$d_{\delta_h-\beta}$	$\dim(\oplus\mathscr{D}_{\delta_h-\beta})=n-2q-2$
$\delta_h-\lambda_i$	$d_{\delta_h-\lambda_i}$	$\dim(\oplus\mathscr{D}_{\delta_h-\lambda_i})=(n-2q-2)(q-1)$
$-2\alpha-\beta+\delta_h+\lambda_i$	$d_{-2\alpha-\beta+\delta_h+\lambda_i}$	$\dim(\oplus\mathscr{D}_{-2\alpha-\beta+\delta_h+\lambda_i})=(n-2q-2)(q-1)$
$\delta_h-\delta_s$	$d_{\delta_h-\delta_s}$	$\dim(\oplus\mathscr{D}_{\delta_h-\delta_s})=(n-2q-2)(n-2q-3)$

dimension and a basis of the aforesaid space for generic dimension of the Lie algebra (see for instance [6]). The program can be separated into four parts.

1. Using T (the algebra form by diagonal derivations) we calculate all of the possible differences between weights of subspaces.
2. We now create the generic space of derivations associated to each weight.
3. We impose that the bellow space is effectively the space of derivations of the algebra.
4. Finally, we arrive at the dimension and a basis of the algebra of derivations as sum and union, respectively, of the dimension and particular basis of each space.

Note: The program allows us to obtain the space of derivations of any Lie algebra for concrete dimensions. By induction we can compute the space of derivations of Lie algebras in arbitrary dimension.

References

[1] J. Alev, F. Dumas, Sur le corps des fractions de certaines algèbres quantiques, J. Algebra 170 (1994) 229–265.

[2] J. Alev, A. Ooms, M. Van der Bergh, A class of counter examples to the Gel'fand–Kirillov conjeture, Trans. Am. Math. Soc. 348 (1996) 1709–1715.

[3] J.M. Cabezas, Una generalización de las álgebras de Lie filiformes, Ph.D. thesis, Universidad de Sevilla, 1996.

[4] J.M. Cabezas, J.R. Gómez, Las álgebras de Lie $(n-3)$-filiformes como extensiones por derivaciones, Extracta Mathematicae 13 (3) (1998) 383–391.

[5] J.M. Cabezas, J.R. Gómez, A. Jiménez-Merchán, Family of p-filiform Lie algebras, in: Algebra and Operator Theory, Kluwer Academic Publishers, Dordrecht, 1998, pp. 93–102.

[6] L.M. Camacho, J.R. Gómez, R.M. Navarro, I. Rodríguez, Effective computation of algebra of derivations of Lie algebras, in: Computer Algebra in Scientific Computing, CASC 2000, Springer, Berlin, 2000, pp. 101–113.

[7] J.R. Gómez, R.M. Navarro, Espacios de derivaciones de álgebras de Lie con Mathematica, in: Proceedings of the IV Encuentro de Álgebra Computacional y Aplicaciones (EACA'98), Sigüenza, 1998.

[8] M. Goze, Y. Khakimdjanov, Nilpotent Lie Algebras, Kluwer Academic Publishers, Dordrecht, 1996.

[9] J.E. Humphreys, Introduction to Lie Algebras and Representation Theory, Graduate Texts in Mathematics, vol. 9, Springer, Berlin, 1987.

[10] J.L. Koszul, Homologie et cohomologie des algèbres de Lie, Bull. Soc. Math. France 78 (1950) 65–127.

[11] D.H. Sattinger, O.L. Weaver, Lie Groups and Algebras with Applications to Physics, Geometry, and Mechanics, Springer, New York, 1986.

[12] J-P. Serre, Lie Algebras and Lie Groups, Lectures Notes in Mathematics, Springer, Berlin, 1992.

[13] V.S. Varadarajan, Lie Groups, Lie Algebras, and Their Representations, Graduate Texts in Mathematics, vol. 102, Springer, Berlin, 1984.