A class of nilpotent lie algebras

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Abstract

A p-filiform Lie algebra $\mathfrak{g}$ is a nilpotent Lie algebra for which Goze's invariant is $(n - p, 1, \ldots, 1)$. These Lie algebras are well known for $p \geq n - 4$, $n = dim(\mathfrak{g})$. In this paper we describe the p-filiform Lie algebras, for $p = n - 5$ and we give their classification when the derived subalgebra is maximal.

1 Introduction

Nilpotent Lie algebras are known up to dimension 7 [1]. But in higher dimensions their classification is a problem that is far from being completely solved [10]. Filiform Lie algebras have been classified up to dimension 11 (cf. [2], [7], [9]). In 1997, an explicit classification of 2-abelian filiform Lie algebras in any dimension was given in [8].

Cabezas, Gómez and Jiménez-Merchán study the p-filiform Lie algebras, that is, a wider class including filiform algebras and defined by the characteristic sequence $(n - p, 1, \ldots, 1)$ with $n = dim(\mathfrak{g})$, giving their classification for high values of p close to n ($n - 4 \leq p \leq n - 2$) [3, 5, 6].

In this paper, we obtain an expression for $(n - 5)$-filiform family of laws in arbitrary dimension, classifying them when the dimension of the derived algebra is maximal, that is, 6. Using that classification, we determine which of those algebras are contact or naturally graded algebras. In what follows, Lie algebras will be considered over the complex field and will have finite dimension.

The *descending central sequence* of a Lie algebra $\mathfrak{g}$ is defined by $(\mathcal{C}^k(\mathfrak{g}))$, $k \in \mathbb{N} \cup \{0\}$, where

$$\mathcal{C}^0(\mathfrak{g}) = \mathfrak{g} \quad \text{and} \quad \mathcal{C}^i(\mathfrak{g}) = [\mathfrak{g}, \mathcal{C}^{i-1}(\mathfrak{g})], \quad i \in \mathbb{N}$$

If there exists a $k \in \mathbb{N}$ such that $\mathcal{C}^k(\mathfrak{g}) = 0$, the Lie algebra $\mathfrak{g}$ is said to be *nilpotent* and the smallest integer verifying this equation is called the *nilindex* of $\mathfrak{g}$.

A Lie algebra $\mathfrak{g}$, with $dim(\mathfrak{g}) = n$, is called *filiform* if it verifies $dim(\mathcal{C}^i(\mathfrak{g})) = n - i - 1$ for $1 \leq i \leq n - 1$. These algebras have maximal nilindex, equal to $n - 1$. The Lie algebras with nilindex $n - 2$ are called *quasifiliform* and those whose nilindex is 1 are called *abelian*.

Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension n. For all $X \in \mathfrak{g} - [\mathfrak{g}, \mathfrak{g}]$, $c(X) = (c_1(X), c_2(X), \ldots, 1)$ is the sequence, in decreasing order, of the dimensions of the characteristic subspaces of the *nilpotent operator* $ad(X)$, where the adjoint operator of an element $X \in \mathfrak{g}$, $ad(X)$, is defined by

$$ad: \quad \begin{array}{ccc} \mathfrak{g} & \longrightarrow & \mathfrak{g} \\ Y & \longrightarrow & [X, Y] \end{array}$$

$c(\mathfrak{g}) = sup\{c(X) : X \in \mathfrak{g} - [\mathfrak{g}, \mathfrak{g}]\}$ is called the *characteristic sequence* or *Goze's invariant* of the nilpotent Lie algebra $\mathfrak{g}$. The filiform, quasifiliform and abelian Lie algebras of dimension n have as their Goze invariant $(n - 1, 1)$, $(n - 2, 1, 1)$ and $(1, 1, \ldots, 1)$, respectively.

Definition 1.1. *[3] A nilpotent Lie algebra $\mathfrak{g}$, of dimension n, is called p-filiform if its Goze invariant is $(n - p, 1, \ldots, 1)$.*

Thus *p*-filiform Lie algebras are a large family of Lie algebras of which the last ones are only special cases.

Note that a complex Lie algebra $\mathfrak{g}$ is naturally filtered by the central descending sequence. This result leads to associate to any Lie algebra $\mathfrak{g}$ a graded Lie algebra, $gr\mathfrak{g}$ with equal nilindex:

$$gr\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i, \qquad \mathfrak{g}_i = \mathcal{C}^{i-1}(\mathfrak{g})/\mathcal{C}^i(\mathfrak{g})$$

By nilpotency, the above gradation is finite, that is

$$gr\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k$$

with $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, for $i + j \leq k$, verifying that $2 \leq dim(\mathfrak{g}_1) \leq 4$ and $1 \leq dim(\mathfrak{g}_i) \leq 3$, for $2 \leq i \leq k$. A Lie algebra $\mathfrak{g}$ is said to be naturally graded if $gr\mathfrak{g}$ is isomorphic to $\mathfrak{g}$, what will be denoted henceforth by $gr\mathfrak{g}=\mathfrak{g}$ [11].

2 Family of laws of $(n - 5)$-filiform Lie algebras

In this section we describe the generic family of *p*-filiform Lie algebras for $p = n - 5$. In order to get our goal, we use theorems on nilpotency, Goze's invariant and *p*-filiformity.

Theorem 2.1. *Any complex $(n - 5)$-filiform Lie algebra of dimension n, $n \geq 6$, is isomorphic to one which law can be expressed, in an adapted basis $\{X_0, X_1, X_2, X_3, X_4, X_5, Y_1, Y_2, \ldots, Y_{n-6}\}$, by*

$$\begin{array}{rcll} [X_0, X_i] & = & X_{i+1} & 1 \leq i \leq 4 \\[2mm] [X_1, X_2] & = & cX_4 + dX_5 + \displaystyle\sum_{k=1}^{n-6} \alpha_k Y_k & \\[4mm] [X_1, X_3] & = & cX_5 & \\[2mm] [X_1, X_4] & = & eX_5 + \displaystyle\sum_{k=1}^{n-6} \beta_k Y_k & \end{array}$$

$$
\begin{aligned}
[X_2, X_3] &= -eX_5 - \sum_{k=1}^{n-6} \beta_k Y_k \\
[X_1, Y_i] &= a_{3i}X_3 + a_{2i}X_4 + a_{1i}X_5 & 1 \le i \le n-6 \\
[X_2, Y_i] &= a_{3i}X_4 + a_{2i}X_5 & 1 \le i \le n-6 \\
[X_3, Y_i] &= a_{3i}X_5 & 1 \le i \le n-6 \\
[Y_i, Y_j] &= b_{ij}X_5 & 1 \le i < j \le n-6,
\end{aligned}
$$

where the parameters that appear verify the following restrictions

$$
\begin{aligned}
a_{3i}\beta_k &= 0 & 1 \le i, k \le n-6 \\
a_{3i}\alpha_k &= 0 & 1 \le i, k \le n-6 \\
a_{1i}\beta_k &= 0 & 1 \le i, k \le n-6 \\
a_{2i}c\beta_k &= 0 & 1 \le i, k \le n-6 \\
a_{3i}e &= 0 & 1 \le i \le n-6 \\
\sum_{k=1}^{n-6} \beta_k a_{2k} &= 0 \\
\sum_{k=2}^{n-6} \alpha_k b_{1k} &= 0 \\
\sum_{k=i+1}^{n-6} \alpha_k b_{ik} &= \sum_{r=1}^{i-1} \alpha_r b_{ri} & 2 \le i \le n-7 \\
\sum_{k=1}^{n-7} \alpha_k b_{k,n-6} &= 0 \\
\sum_{k=2}^{n-6} \beta_k b_{1k} &= 0 \\
\sum_{k=i+1}^{n-6} \beta_k b_{ik} &= \sum_{r=1}^{i-1} \beta_r b_{ri} & 2 \le i \le n-7 \\
\sum_{k=1}^{n-7} \beta_k b_{k,n-6} &= 0.
\end{aligned}
$$

Proof: Let $\mathfrak{g}$ be a complex nilpotent Lie algebra of dimension n and with Goze's invariant $(5, 1, 1, \ldots, 1)$.

If $X_0 \notin [\mathfrak{g}, \mathfrak{g}]$ is a characteristic vector and $\{X_0, X_1, X_2, X_3, X_4, X_5, Y_1, Y_2, \ldots, Y_{n-6}\}$ is an adapted basis of $\mathfrak{g}$, then it follows that

$$
\begin{aligned}
[X_0, X_i] &= X_{i+1} & 1 \le i \le 4 \\
[X_0, X_5] &= 0 \\
[X_0, Y_j] &= 0 & 1 \le j \le n-6.
\end{aligned}
$$

Since $Y_1, Y_2, \ldots, Y_{n-6} \notin Im(ad(X_0))$ we can get, by some appropriate Jacobi identities, that the nilpotent Lie algebra $\mathfrak{g}$ belongs to the family

$$
\begin{aligned}
[X_0, X_i] &= X_{i+1} & 1 \le i \le 4 \\[6pt]
[X_1, X_2] &= 2\gamma_{24}X_2 + (\gamma_{23} + \gamma_{14})X_3 + \gamma_{13}X_4 + \gamma_{12}X_5 + \sum_{k=1}^{n-6} \alpha_{12}^k Y_k \\[6pt]
[X_1, X_3] &= 2\gamma_{24}X_3 + (\gamma_{23} + \gamma_{14})X_4 + \gamma_{13}X_5 \\[6pt]
[X_1, X_4] &= \gamma_{24}X_4 + \gamma_{14}X_5 + \sum_{k=1}^{n-6} \alpha_{14}^k Y_k
\end{aligned}
$$

$$[X_2, X_3] = \gamma_{24}X_4 + \gamma_{23}X_5 - \sum_{k=1}^{n-6} \alpha_{14}^k Y_k$$

$$[X_2, X_4] = \gamma_{24}X_5$$

$$[X_1, Y_i] = a_{4i}X_2 + a_{3i}X_3 + a_{2i}X_4 + a_{1i}X_5 + \sum_{k=1}^{n-6} d_i^k Y_k \quad 1 \le i \le n-6$$

$$[X_2, Y_i] = a_{4i}X_3 + a_{3i}X_4 + a_{2i}X_5 \qquad\qquad 1 \le i \le n-6$$

$$[X_3, Y_i] = a_{4i}X_4 + a_{3i}X_5 \qquad\qquad\qquad 1 \le i \le n-6$$

$$[X_4, Y_i] = a_{4i}X_5 \qquad\qquad\qquad\qquad 1 \le i \le n-6$$

$$[Y_i, Y_j] = b_{ij}X_5 + \sum_{k=1}^{n-6} c_{ij}^k Y_k \qquad\qquad 1 \le i < j \le n-6$$

For all $A \in \mathbf{C} - \{0\}$ it follows that $X_0 + AY_i \notin [\mathfrak{g}, \mathfrak{g}]$, $1 \le i \le n-7$ and the adjoint matrix of the vector $X_0 + AY_1$ will be given by

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\
0 & 1-Aa_{41} & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\
0 & -Aa_{31} & 1-Aa_{41} & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\
0 & -Aa_{21} & -Aa_{31} & 1-Aa_{41} & 0 & 0 & 0 & 0 & \ldots & 0 \\
0 & -Aa_{11} & -Aa_{21} & -Aa_{31} & 1-Aa_{41} & 0 & 0 & Ab_{12} & \ldots & Ab_{1,n-6} \\
0 & -Ad_1^1 & 0 & 0 & 0 & 0 & 0 & Ac_{12}^1 & \ldots & Ac_{1,n-6}^1 \\
0 & -Ad_1^2 & 0 & 0 & 0 & 0 & 0 & Ac_{12}^2 & \ldots & Ac_{1,n-6}^2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & -Ad_1^{n-6} & 0 & 0 & 0 & 0 & 0 & Ac_{12}^{n-6} & \ldots & Ac_{1,n-6}^{n-6}
\end{pmatrix}
$$

Since Goze's invariant is $(5,1,1,\ldots,1)$, it cannot exist a non-zero minor of order 5. Then if $1 - Aa_{41} \ne 0$ it follows that

$$
\begin{vmatrix}
1-Aa_{41} & 0 & 0 & 0 & 0 \\
-Aa_{31} & 1-Aa_{41} & 0 & 0 & 0 \\
-Aa_{21} & -Aa_{31} & 1-Aa_{41} & 0 & 0 \\
-Aa_{11} & -Aa_{21} & -Aa_{31} & 1-Aa_{41} & Ab_{1j} \\
-Ad_1^k & 0 & 0 & 0 & Ac_{1j}^k
\end{vmatrix} = (1-Aa_{41})^4 Ac_{1j}^k = 0 \Rightarrow
$$

$$\Rightarrow c_{1j}^k = 0, \quad 1 \le k \le n-6 \quad 2 \le j \le n-6.$$

and a simple process of finite induction leads to

$$c_{ij}^k = 0, \quad 1 \le k \le n-6, \quad i+1 \le j \le n-6, \quad 1 \le i \le n-7.$$

¿From considerations of adjoint nilpotency on the vector X_2, we obtain that

$$|\lambda I_n - ad(X_2)| = \lambda^n \quad \Leftrightarrow \quad \sum_{k=1}^{n-6} \alpha_{14}^k a_{4k} = 0.$$

It now can be easily assumed that $\gamma_{23} + \gamma_{14} = 0$ implementing an easy change of basis defined by

$$
\begin{cases}
X_t^* &= X_t \qquad\qquad\qquad 0 \le t \le 4 \quad (t \ne 1) \\
X_1^* &= X_1 - (\gamma_{23} + \gamma_{14})X_0 \\
Y_k^* &= Y_k \qquad\qquad\qquad 1 \le k \le n-6
\end{cases}
$$

and, therefore, considering nilpotency, γ_{24} vanishes, leading to the following family of laws

$$
\begin{aligned}
{[X_0, X_i]} &= X_{i+1} & & 1 \le i \le 4 \\
{[X_1, X_2]} &= cX_4 + dX_5 + \sum_{k=1}^{n-6} \alpha_k Y_k & & (d = \gamma_{12} \quad \alpha_k = \alpha_{12}^k) \\
{[X_1, X_3]} &= cX_5 & & (c = \gamma_{13}) \\
{[X_1, X_4]} &= eX_5 + \sum_{k=1}^{n-6} \beta_k Y_k & & (e = \gamma_{14} \quad \beta_k := \alpha_{14}^k) \\
{[X_2, X_3]} &= -eX_5 - \sum_{k=1}^{n-6} \beta_k Y_k & & \\
{[X_1, Y_i]} &= a_{4i}X_2 + a_{3i}X_3 + a_{2i}X_4 + a_{1i}X_5 + \sum_{k=1}^{n-6} d_i^k Y_k & & 1 \le i \le n-6 \\
{[X_2, Y_i]} &= a_{4i}X_3 + a_{3i}X_4 + a_{2i}X_5 & & 1 \le i \le n-6 \\
{[X_3, Y_i]} &= a_{4i}X_4 + a_{3i}X_5 & & 1 \le i \le n-6 \\
{[X_4, Y_i]} &= a_{4i}X_5 & & 1 \le i \le n-6 \\
{[Y_i, Y_j]} &= b_{ij}X_5 & & 1 \le i < j \le n-6,
\end{aligned}
$$

with $\displaystyle\sum_{k=1}^{n-6} \beta_k a_{4k} = 0$.

By conditions of $(n-5)$-filiformity on the adjoint matrices of the vectors $AX_0 + X_i \notin [\mathfrak{g}, \mathfrak{g}]$, $1 \le i \le 2, \forall A \in \mathbf{C}$, we conclude that

$$
\begin{aligned}
d_i^k &= 0 & & 1 \le i, k \le n-6 \\
a_{3i}\beta_k &= 0 & & 1 \le i, k \le n-6 \\
a_{3i}\alpha_k &= 0 & & 1 \le i, k \le n-6 \\
a_{1i}\beta_k &= 0 & & 1 \le i, k \le n-6 \\
a_{2i}c\beta_k &= 0 & & 1 \le i, k \le n-6.
\end{aligned}
$$

¿From the remaining Jacobi identities and the adjoint nilpotency it follows that

$$
\begin{aligned}
\sum_{k=1}^{n-6} \alpha_k a_{4k} &= \sum_{k=1}^{n-6} \beta_k a_{2k} \\
2a_{31}e &= -\sum_{k=2}^{n-6} \alpha_k b_{1k} \\
2a_{3i}e &= \sum_{r=1}^{i-1} \alpha_r b_{ri} - \sum_{k=i+1}^{n-6} \alpha_k b_{ik} & & 2 \le i \le n-7 \\
2a_{3,n-6}e &= \sum_{k=1}^{n-7} \alpha_k b_{k,n-6} \\
\sum_{k=2}^{n-6} \beta_k b_{1k} &= 0 \\
\sum_{k=i+1}^{n-6} \beta_k b_{ik} &= \sum_{r=1}^{i-1} \beta_r b_{ri} & & 2 \le i \le n-7 \\
\sum_{k=1}^{n-7} \beta_k b_{k,n-6} &= 0 \\
\sum_{k=1}^{n-6} \alpha_k a_{4k} &= 0
\end{aligned}
$$

$$\sum_{k=1}^{n-6} \beta_k a_{2k} \quad = \quad 0$$

Since $a_{3i}\alpha_k = 0$, $1 \le i, k \le n-6$, the expression of the above mentioned conditions is dramatically simplified. This result together with the change of basis defined by

$$\begin{cases} X_t^* &= X_t & 1 \le t \le 5 \\ Y_i^* &= Y_i - a_{4i}X_0 & 1 \le i \le n-6 \end{cases}$$

leads to $a_{4i} = 0$, $1 \le i \le n-6$. Hence, some of above mentioned restrictions vanish because they are trivial.   □

3   Classification of $(n-5)$-filiform algebras with maximal derived algebra

We determine all the $(n-5)$-filiform algebras in arbitrary dimension for the case in which the derived algebra is maximal, that is, 6.

Proposition 3.1. *If $\mathfrak{g}$ is an $(n-5)$-filiform Lie algebra of dimension n with $dim[\mathfrak{g},\mathfrak{g}] = 6$ and $n \ge 8$, then its law can be expressed in an adapted basis $\{X_0, X_1, X_2, X_3, X_4, X_5, Y_1, Y_2, \ldots, Y_{n-6}\}$, by*

$$\begin{cases} [X_0, X_i] = X_{i+1} & 1 \le i \le 4 \\ [X_1, X_2] = cX_4 + Y_{n-7} \\ [X_1, X_3] = cX_5 \\ [X_1, X_4] = Y_{n-6} \\ [X_2, X_3] = -Y_{n-6} \\ [X_1, Y_i] = a_{2i}X_4 \quad\cdot & 1 \le i \le n-7 \\ [X_2, Y_i] = a_{2i}X_5 & 1 \le i \le n-7 \\ [Y_i, Y_j] = b_{ij}X_5 & 1 \le i < j \le n-8 \end{cases}$$

with $a_{2i}c = 0$, $1 \le i \le n-7$.

Proof: Implementing an easy change of basis in the family of laws of the above section, that is

$$\begin{cases} Y'_{n-7} = dX_5 + \displaystyle\sum_{k=1}^{n-6} \alpha_k Y_k \\ Y'_{n-6} = eX_5 + \displaystyle\sum_{k=1}^{n-6} \beta_k Y_k \end{cases}$$

it can be assumed that $\alpha_{n-7} = \beta_{n-6} = 1$ and the other ones of $\alpha'_k s$, $\beta'_k s$ are equal to zero. This result reduced the restrictions to $a_{2i}c = 0$, $1 \le i \le n-7$.   □

Proposition 3.2. *If $\mathfrak{g}$ is an $(n-5)$-filiform Lie algebra of dimension n with $dim[\mathfrak{g},\mathfrak{g}] = 6$ and $n \ge 9$, then it is isomorphic to one of the following families of laws, which are pairwise non-isomorphic.*

$$\mu_{(5,1,\ldots,1)}^{1,r} : \begin{cases} [X_0, X_i] = X_{i+1} & 1 \leq i \leq 4 \\ [X_1, X_2] = Y_{n-7} \\ [X_1, X_4] = Y_{n-6} \\ [X_2, X_3] = -Y_{n-6} \\ [Y_{2i-1}, Y_{2i}] = X_5 & 1 \leq i \leq r-1 \end{cases} \qquad 1 \leq r \leq E\left(\frac{n-6}{2}\right)$$

$$\mu_{(5,1,\ldots,1)}^{2,r} : \begin{cases} [X_0, X_i] = X_{i+1} & 1 \leq i \leq 4 \\ [X_1, X_2] = Y_{n-7} \\ [X_1, X_4] = Y_{n-6} \\ [X_2, X_3] = -Y_{n-6} \\ [X_1, Y_{n-8}] = X_4 \\ [X_2, Y_{n-8}] = X_5 \\ [Y_{2i-1}, Y_{2i}] = X_5 & 1 \leq i \leq r-1 \end{cases} \qquad 1 \leq r \leq E\left(\frac{n-7}{2}\right)$$

$$\mu_{(5,1,\ldots,1)}^{3,r} : \begin{cases} [X_0, X_i] = X_{i+1} & 1 \leq i \leq 4 \\ [X_1, X_2] = Y_{n-7} \\ [X_1, X_4] = Y_{n-6} \\ [X_2, X_3] = -Y_{n-6} \\ [X_1, Y_{n-7}] = X_4 \\ [X_2, Y_{n-7}] = X_5 \\ [Y_{2i-1}, Y_{2i}] = X_5 & 1 \leq i \leq r-1 \end{cases} \qquad 1 \leq r \leq E\left(\frac{n-6}{2}\right)$$

$$\mu_{(5,1,\ldots,1)}^{4,r} : \begin{cases} [X_0, X_i] = X_{i+1} & 1 \leq i \leq 4 \\ [X_1, X_2] = X_4 + Y_{n-7} \\ [X_1, X_3] = X_5 \\ [X_1, X_4] = Y_{n-6} \\ [X_2, X_3] = -Y_{n-6} \\ [Y_{2i-1}, Y_{2i}] = X_5 & 1 \leq i \leq r-1 \end{cases} \qquad 1 \leq r \leq E\left(\frac{n-6}{2}\right)$$

Remark 3.3. In dimension 8, we only obtain three $(n-5)$-filiform Lie algebras with $dim[\mathfrak{g}, \mathfrak{g}] = 6$: $\mu_{(5,1,1,1)}^{1,1}$, $\mu_{(5,1,1,1)}^{3,1}$ and $\mu_{(5,1,1,1)}^{4,1}$

Proof: The structure constant $c_{1,3}^5$ is always a multiple of c. In fact:

1). Jordan's change of basis.

$$X_0' = X_0$$

$$X_1' = A_0 X_0 + A_1 X_1 + A_2 X_2 + A_3 X_3 + A_4 X_4 + A_5 X_5 + \sum_{k=1}^{n-6} B_k Y_k$$

Two conditions must be satisfied by the constants above. Firstly, by conditions of change of basis, we get that $A_1 \neq 0$. Secondly, as we want the law to be in the family, A_0 must be zero. The resulting parameter is $c' = A_1 c$ that is multiple of c.

2). Change of characteristic vector.

$$X_0' = E_0 X_0 + E_1 X_1 + E_2 X_2 + E_3 X_3 + E_4 X_4 + E_5 X_5 + \sum_{k=1}^{n-6} D_k Y_k$$

$$X_1' = X_1$$

with $E_0 \neq 0$ and $E_1 a_{2,n-7} = 0$. In this case the resulting parameter is $c" = \dfrac{c}{E_0^2}$.

Thus, the generic family obtained in the above proposition splits up in two families of non-isomorphic laws.

Case 1: $c = 0$

The law of the algebra can be expressed, up to antisymmetry, by

$$
\begin{aligned}
[X_0, X_i] &= X_{i+1} & 1 \leq i \leq 4 \\
[X_1, X_2] &= Y_{n-7} \\
[X_1, X_4] &= Y_{n-6} \\
[X_2, X_3] &= -Y_{n-6} \\
[X_1, Y_i] &= a_{2i} X_4 & 1 \leq i \leq n-7 \\
[X_2, Y_i] &= a_{2i} X_5 & 1 \leq i \leq n-7 \\
[Y_i, Y_j] &= b_{ij} X_5 & 1 \leq i < j \leq n-8
\end{aligned}
$$

Considering the dimension of the subalgebra $[\mathcal{C}^1(\mathfrak{g}), \mathcal{C}^2(\mathfrak{g})]$, we obtain that

$$
dim([\mathcal{C}^1(\mathfrak{g}), \mathcal{C}^2(\mathfrak{g})]) = \begin{cases} 1 & \text{if } a_{2,n-7} = 0 \\ \\ 2 & \text{if } a_{2,n-7} \neq 0 \end{cases}
$$

leading to two families of non-isomorphic laws.

(1.1) $a_{2,n-7} = 0$

Considering the centralizer of $\mathfrak{g}$ for this case, we obtain that

$$
dim(\mathcal{C}ent(\mathcal{C}^1(\mathfrak{g}))) = \begin{cases} n-4 & \text{if } a_{2i} = 0 \ \forall i : \ 1 \leq i \leq n-8 \\ \\ n-5 & \text{if } \exists i : \ a_{2i} \neq 0 \end{cases}
$$

Subcase 1.1.1. $a_{2i} = 0 \ \forall i : \ 1 \leq i \leq n-8$.

The law of the algebra can be expressed, up to antisymmetry, by

$$
\begin{aligned}
[X_0, X_i] &= X_{i+1} & 1 \leq i \leq 4 \\
[X_1, X_2] &= Y_{n-7} \\
[X_1, X_4] &= Y_{n-6} \\
[X_2, X_3] &= -Y_{n-6} \\
[Y_i, Y_j] &= b_{ij} X_5 & 1 \leq i < j \leq n-8
\end{aligned}
$$

· If b_{ij} vanishes $\forall i, j$ then $\mu_{(5,1,\ldots,1)}^{1,1}$ is obtained.

· If there exists $b_{ij} \neq 0$, it can be easily assumed that $b_{12} \neq 0$ and implementing an easy change of basis $(Y_1' = \dfrac{1}{b_{12}} Y_1, \ Y_2' = Y_2, \ Y_i' = Y_i + \dfrac{b_{2i}}{b_{12}} Y_1 - \dfrac{b_{1i}}{b_{12}} Y_2, \ 3 \leq i \leq n-8)$ we obtain that $b_{12} = 1, \ b_{1i} = b_{2i} = 0, \ 3 \leq i \leq n-8$.

By continuing this process, the family of algebras $\mu_{(5,1,\ldots,1)}^{1,r}$ is obtained. Moreover, distinct values of r lead to non-isomorphic algebras; this follows from considering the dimension of the center

$$
dim(\mathcal{Z}(\mathfrak{g})) = n - 2r - 3, \quad 1 \leq r \leq E\left(\frac{n-6}{2}\right)
$$

Subcase 1.1.2. $\exists i : \ a_{2i} \neq 0$.

We can always suppose that $a_{21} \neq 0$ and with no further operation than the change of

basis defined by $(Y_1' = \frac{1}{a_{21}}Y_1, Y_i' = a_{21}Y_i - a_{2i}Y_1,\ 2 \le i \le n-7)$ it can be assumed that $a_{21} = 1$ and $a_{2i} = 0,\ i \ne 1$. Then $\mathfrak{g}$ is isomorphic to an algebra of law

$$[X_0, X_i] = X_{i+1} \qquad 1 \le i \le 4$$
$$[X_1, X_2] = Y_{n-7}$$
$$[X_1, X_4] = Y_{n-6}$$
$$[X_2, X_3] = -Y_{n-6}$$
$$[X_1, Y_1] = X_4$$
$$[X_2, Y_1] = X_5$$
$$[Y_i, Y_j] = b_{ij}X_5 \qquad 1 \le i < j \le n-8$$

Changes of basis similar to some of those already seen (subcase 1.1.1) lead to $\mu^{2,r}_{(5,1,\dots,1)}$ providing pairwise non-isomorphic algebras. This result follows from

$$dim(\mathcal{Z}(\mathfrak{g})) = n - 2r - 4, \qquad 1 \le r \le E\left(\frac{n-7}{2}\right)$$

(1.2) $a_{2,n-7} \ne 0$

The change of basis defined by $Y_i' = a_{2,n-7}Y_i - a_{2i}Y_{n-7},\ 1 \le i \le n-8$ together with a change of scale lead to $a_{2,n-7} = 1$ and $a_{2,j} = 0$ for $1 \le i \le n-8$. Now, changes of basis similar to some of those already seen (subcase 1.1.1) lead to $\mu^{3,r}_{(5,1,\dots,1)}$ providing pairwise non-isomorphic algebras. This result follows from

$$dim(\mathcal{Z}(\mathfrak{g})) = n - 2r - 4, \qquad 1 \le r \le E\left(\frac{n-6}{2}\right)$$

<u>Case 2</u>: $c \ne 0$
The law of the algebra can be expressed by

$$[X_0, X_i] = X_{i+1} \qquad 1 \le i \le 4$$
$$[X_1, X_2] = cX_4 + Y_{n-7}$$
$$[X_1, X_3] = cX_5$$
$$[X_1, X_4] = Y_{n-6}$$
$$[X_2, X_3] = -Y_{n-6}$$
$$[Y_i, Y_j] = b_{ij}X_5 \qquad 1 \le i < j \le n-8$$

implementing a simple change of scale it can be assumed $c = 1$.

By a process of finite induction, similar to the one used in the preceding argument, (subcase 1.1), the family of algebras $\mu^{4,r}_{(5,1,\dots,1)}$ is obtained. We now have that $dim(\mathcal{Z}(\mathfrak{g})) = n - 2r - 3$, which implies that the algebras corresponding to $\mu^{4,r}_{(5,1,\dots,1)}$ are pairwise non-isomorphic. $\qquad\qquad\square$

Remark 3.4. It can be easily seen that there are only three non-split $(n-5)$-filiform algebras for n even $(\mu^{1,E(\frac{n-6}{2})}_{(5,1,\dots,1)}, \mu^{3,E(\frac{n-6}{2})}_{(5,1,\dots,1)}, \mu^{4,E(\frac{n-6}{2})}_{(5,1,\dots,1)})$, and only one for n odd, $\mu^{2,E(\frac{n-7}{2})}_{(5,1,\dots,1)}$.

4 Naturally graded $(n-5)$-filiform Lie algebras of maximal derived algebra

In this section, all the naturally graded $(n-5)$-filiform Lie algebras of maximal derived algebra are provided.

It can be easily seen that, in this case, the corresponding gradation contains only five non-null homogeneous subspaces , $\mathfrak{g}_1$, $\mathfrak{g}_2$, $\mathfrak{g}_3$, $\mathfrak{g}_4$ and $\mathfrak{g}_5$, verifying X_0, $X_1 \in \mathfrak{g}_1$, $X_i \in \mathfrak{g}_i$, $2 \leq i \leq 5$, $Y_i \in \mathfrak{g}_1$, $1 \leq i \leq n-8$, $Y_{n-7} \in \mathfrak{g}_3$ and $Y_{n-6} \in \mathfrak{g}_5$ with $\{X_0, X_1, \ldots, X_5, Y_1, \ldots Y_{n-6}\}$ an adapted basis of $\mathfrak{g}$.

¿From now on, $dim(\mathfrak{g}) = n \geq 8$, which follows from $dim[\mathfrak{g}, \mathfrak{g}] = 6$.

Proposition 4.1. *There exist only two naturally graded non-split $(n-5)$-filiform Lie algebras with $dim[\mathfrak{g}, \mathfrak{g}] = 6$, $\mu_{(5,1,1,1)}^{1,1}$ and $\mu_{(5,1,1,1)}^{3,1}$, both corresponding to dimension 8.*

Proof: There are no naturally graded algebras in $\mu_{(5,1,\ldots,1)}^{2,r}$, this follows from $Y_{n-8} \in \mathfrak{g}_1$ and $X_i \in \mathfrak{g}_i$ with $1 \leq i \leq 5$, which is in contradiction with the bracket $[X_1, Y_{n-8}] = X_4$ that always appears in the family. In the same way, it can be easily seen that there are no naturally graded algebras in $\mu_{(5,1,\ldots,1)}^{4,r}$ which follows from the existence of the bracket $[X_1, X_3] = X_5$.

As the previous reasoning showed, if $r > 1$ there are no naturally graded algebras in the families of laws $\mu_{(5,1,\ldots,1)}^{1,r}$ and $\mu_{(5,1,\ldots,1)}^{3,r}$ because in these cases the bracket $[Y_1, Y_2] = X_5$ appears.

At this point, it only remains to study the algebras $\mu_{(5,1,\ldots,1)}^{1,1}$ and $\mu_{(5,1,\ldots,1)}^{3,1}$, but they correspond to split algebras, $\mu_{(5,1,1,1)}^{1,1} \oplus \mathbb{C}^{n-8}$ and $\mu_{(5,1,1,1)}^{3,1} \oplus \mathbb{C}^{n-8}$ respectively; save for dimension 8, in which case the algebras $\mu_{(5,1,1,1)}^{1,1}$ and $\mu_{(5,1,1,1)}^{3,1}$ are obtained. These two algebras that are obtained in dimension 8 are evidently naturally graded [4]. $\square$

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