

ON PERMANENTS OF SYLVESTER HADAMARD MATRICES

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ABSTRACT. It is well-known that the evaluation of the permanent of an arbitrary $(-1, 1)$ -matrix is a formidable problem. Ryser's formula is one of the fastest known general algorithms for computing permanents. In this paper, Ryser's formula has been rewritten for the special case of Sylvester Hadamard matrices by using their cocyclic construction. The rewritten formula presents an important reduction in the number of sets of r distinct rows of the matrix to be considered. However, the algorithm needs a preprocessing part which remains time-consuming in general.

1. INTRODUCTION

A Hadamard matrix H of order n is an $n \times n$ matrix with elements ± 1 and $HH^T = nI$. A Hadamard matrix is said to be normalized if it has its first row and column all 1's. We can always normalize a Hadamard matrix by multiplying rows and columns by -1 . It is well-known that n is either 2 or a multiple of 4 and it is conjectured that Hadamard matrices exist for every $n \equiv 0 \pmod{4}$ (see [5]).

Sylvester in 1867 noted that given a Hadamard matrix H of order n , then

$$\begin{bmatrix} H & H \\ H & -H \end{bmatrix}$$

is a Hadamard matrix of order $2n$. Matrices of this form are called *Sylvester Hadamard* and defined for all powers of 2. Below is given the Sylvester Hadamard matrix of order 2

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Starting with H_2 , Sylvester Hadamard matrices of order 2^k can be formed by $H_2 \times \cdots \times H_2$ the Kronecker product of k copies of H_2 and are denoted H_{2^k} .

Two Hadamard matrices H and H' are called *equivalent* (or Hadamard equivalent, or H -equivalent) if one can be obtained from the other by a sequence of row and/or column interchanges and row and/or column negations. The question of classifying Hadamard matrices of order $n > 32$ remains unanswered and only partial results are known.

We recall that the original interest in Hadamard matrices stemmed from the fact that a Hadamard matrix $H = [h_{ij}]$ of order n satisfies equality in Hadamard's inequality

$$(\det H)^2 \leq \prod_{j=1}^n \sum_{i=1}^n |h_{ij}|^2$$

for entries in the unit circle.

The permanent is a matrix function introduced (independently) by Cauchy and Binet in 1812.

Definition 1.1. Let N be the set $\{1, \dots, n\}$, ($n \in \mathbf{Z}^+$). The symmetric group S_n is the group of all $n!$ permutations of N . The *permanent* of an $n \times n$ matrix $A = [a_{ij}]$ is defined by

$$\text{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^n a_{i, \sigma(i)}.$$

At first sight it seems to be a simplified version of the determinant, but this impression is misleading. For instance, the determinant of an arbitrary matrix can be evaluated efficiently using Gaussian elimination, however the computation of the permanent is much more complicated. Valiant [9] proved that it belongs to the class of $\#P$ -complete problems, which basically means that there is almost no possibility of finding a polynomial time deterministic algorithm for computing the permanent in general.

Hadamard matrices of order n are characterized by attaining the maximal possible absolute value of the determinant, $n^{n/2}$, among all square matrices of order n with entries from the unit disc, but few things are known on their permanents, aside from $|\text{per}(H)| \leq |\det(H)| = n^{n/2}$. Moreover, the permanent of a Hadamard matrix has hardly been worked on. For what we know, the permanents for all Hadamard matrices of orders up to and including 28 were calculated in [11], but for orders greater than 28 the permanents remains unknown.

In 1974 Wang [10] posed the following question:

Problem 1 Can the permanent of a Hadamard matrix H of order n vanish for $n > 2$?

This problem, which is listed as problem $\#5$ in Minc's catalogue of open problems [7] has in any case been answered in the negative for $n < 32$ by Wanless [11]. Moreover, let $f(n) = \sum_{k \geq 1} \lfloor 2^{-k} n \rfloor$ be the greatest exponent e such that 2^e divides $n!$. Wanless conjectured that $2^{f(n)}$ divides $\text{per}(H)$ and $2^{f(n)+1}$ does not.

The Sylvester matrices have revealed to be the most treatable class of Hadamard matrices. For instance, the eigenvalues, the eigenvectors and the growth factor of Hadamard matrices are unknown in general but they are well-known for Sylvester Hadamard matrices [8, 3].

In the following result, we provide a sufficient condition to give a negative answer to Problem 1 for Sylvester Hadamard matrices.

Proposition 1.2. *If*

$$(1) \quad \text{per}((H_{2^n} \times I_{2^m})(I_{2^n} \times H_{2^m})) \geq \text{per}(H_{2^n} \times I_{2^m}) \text{per}(I_{2^n} \times H_{2^m})$$

for $n \geq 2$ and $m \geq 2$, then the permanent of a Sylvester Hadamard matrix of order 2^k never vanishes for $k > 2$.

Proof. It is well-known that

$$H_{2^{n+m}} = H_2 \overset{2^{n+m}-\text{copies}}{\times \cdots \times} H_2$$

(using the associative law for the Kronecker product)

$$= H_{2^n} \times H_{2^m}$$

(taking into account that $A \times B = (A \times I_m)(I_n \times B)$ for A and B squares matrices of order n and m , respectively)

$$= (H_{2^n} \times I_{2^m})(I_{2^n} \times H_{2^m}).$$

Let us recall the following basic properties for permanents [2],

- (1) $\text{per}(I_n \times A) = (\text{per}(A))^n$.
- (2) $\text{per}(A \times B) = \text{per}(B \times A)$.

Therefore, if the inequality (1) occurs then

$$\text{per}(H_{2^{n+m}}) \geq (\text{per}(H_{2^n}))^{2^m} (\text{per}(H_{2^m}))^{2^n}, \quad \text{for } n \geq 2 \text{ and } m \geq 2.$$

Taking into consideration the inequality above and that $\text{per}(H_4) = 8$ and $\text{per}(H_8) = 384$, it follows the desired result. \square

The following open problem arises naturally from Proposition 1.2.

Problem 2 For each $n \geq 2$ and $m \geq 2$ is the inequality (1) always true?

It is well-known that it is not in general true that

$$\text{per}(AB) = \text{per}(A) \text{per}(B)$$

for A and B square matrices. This is a devastating blow, since most of the nice properties of determinants follow from $\det(AB) = \det(A)\det(B)$. In particular, we cannot use Gaussian elimination to calculate permanents. For the special case when the matrices are nonnegative, Brualdi [2] proved that

$$\text{per}(AB) \geq \text{per}(A) \text{per}(B).$$

On the other hand, Problem 2 is conceivably much more difficult.

In the early 90s, a surprising link between homological algebra and Hadamard matrices [4] led to the study of cocyclic Hadamard matrices. Hadamard matrices of many classes are revealed to be (equivalent to) cocyclic matrices [5]. Among them are Sylvester Hadamard matrices, Williamson-type Hadamard matrices and Paley Hadamard matrices.

Let $G = \{g_1 = 1, g_2, \dots, g_{4t}\}$ be a multiplicative group, not necessarily abelian. Functions $\psi: G \times G \rightarrow \langle -1 \rangle \cong \mathbf{Z}_2$ which satisfy

$$\psi(g_i, g_j)\psi(g_i g_j, g_k) = \psi(g_j, g_k)\psi(g_i, g_j g_k), \quad \forall g_i, g_j, g_k \in G$$

are called (*binary*) *cocycles (over G)*[6].

A cocycle ψ is naturally displayed as a *cocyclic matrix* M_ψ ; that is, the entry in the (i, j) th position of the cocyclic matrix is $\psi(g_i, g_j)$, for all $1 \leq i, j \leq 4t$. i.e.,

$$M_\psi = [\psi(g_i, g_j)]_{g_i, g_j \in G},$$

where rows and columns of M_ψ are indexed by the elements of G .

Example 1.3. Let $G = \mathbf{Z}_2^k$. The vector inner product $\langle g_i, g_j \rangle$ determines a cocycle ψ , where $\psi(g_i, g_j) = (-1)^{\langle g_i, g_j \rangle}$ for all $g_i, g_j \in G$, and M_ψ is the Sylvester Hadamard matrix of order 2^k .

The main purpose of this paper is to rewrite Ryser's formula to evaluate the permanent of Sylvester Hadamard matrices using their cocyclic properties which may lead to improvements. In [1], an analogous approach has been done for computing the profile of cocyclic Hadamard matrices. As an example, we will focus on H_8 .

Notation. Throughout this paper we use $-$ for -1 and 1 for $+1$. We write H_{2^p} for a Sylvester Hadamard matrix of order 2^p . The cardinality of a set S is denoted $\#S$. The notation $(0, 1)$ -matrix means a matrix whose entries are either 0 or 1. We use I_n for the identity matrix of order n and M^T for the transpose of M .

2. RYSER'S FORMULA FOR SYLVESTER HADAMARD MATRICES

H.J. Ryser found an alternative method to evaluate the permanent of a matrix A of order n , which is one of the fastest known general algorithms for computing permanents. By counting multiplications it has an efficiency of $O(2^n n)$. See [12] for some of the theories of permanents. See [5] for some elementary notions from cocyclic Hadamard matrices that we use here.

If $A = [a_{ij}]$ is any $n \times n$ matrix

$$(2) \quad \text{per}(A) = \sum_{r=1}^n (-1)^r \sum_{\alpha \in Q_{r,n}} \prod_{j=1}^n \sum_{i \in \alpha} a_{i,j},$$

where $Q_{r,n}$ denotes the set of all strictly increasing sequences of r integers chosen from the set $\{1, 2, \dots, n\}$.

Here are three ways to calculate the permanent of

$$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & 1 & - & - \\ 1 & - & - & 1 \end{bmatrix}.$$

The Classical formula, using all the permutations S_4 , needs 23 additions and 72 multiplications. Ryser's method needs 51 multiplications and 101 additions. Finally, Algorithm 2.10, described below, needs only 9 multiplications and 4 additions. The last two methods are faster than the first one for larger matrices.

Definition 2.1. Let M_1 and M_2 be two $(0, 1)$ -matrices. We say that M_1 and M_2 are *P-equivalent* if one is obtained from the other by a sequence of the operations:

- permutations of rows and/or columns.
- "complementation" of columns.

We mean by complementation of a column that every one of its entries change from 0 to 1 and from 1 to 0.

Definition 2.2. We say that a $(0, 1)$ -matrix M is in *Strictly increasing order by row* (or briefly SIOR) if its i -th row is less than its j -th row (as binary numbers) when $i < j$.

Algorithm 2.3. Searching for a distribution of *P-equivalent* classes.

Input: Two positive integers r and p .

Output: The *P-equivalent* classes and the size of every class.

$$\Omega_{r,p} \leftarrow \emptyset$$

For each SIOR matrix X

1. Check if X is P -equivalent to one element of $\Omega_{r,p}$. If no, go to 3; otherwise go to 2.
2. $\#[Y] \leftarrow \#[Y] + 1$ where Y and X are P -equivalent. End.
3. $\Omega_{r,p} \leftarrow \Omega_{r,p} \cup [X]$ and $\#[X] \leftarrow 1$. End.

$$\Omega_{r,p} = \{[X_1], \dots, [X_k]\} \text{ and } \Omega_{r,p}^\# = \{\#[X_1], \#[X_2], \dots, \#[X_k]\}.$$

VERIFICATION: By construction, the uniqueness of $[Y]$ in the Step 2 is guaranteed, $\Omega_{r,p}$ is a set of equivalence classes and $\Omega_{r,p}^\#$ gives the size of every orbit.

The following result can be seen as an immediate consequence of the procedure described above.

Proposition 2.4. *Assuming that $\Omega_{r,p} = \{[X_1], \dots, [X_k]\}$ is a distribution of equivalence classes for the pair (r, p) . We have:*

$$\bullet \sum_{i=1}^n \#[X_i] = \binom{2^p}{r}, \quad \bullet \#[X_i] \leq 2^p p!, \quad \bullet \#\Omega_{r,p} \geq \frac{\binom{2^p}{r}}{2^p p!}.$$

Proof. The first identity follows from the fact that the number of SIOR matrices $A \in \{0, 1\}^{r \times p}$ is $\binom{2^p}{r}$. On the other hand, attending to Definition 2.1, it is clear that the size of the P -orbit for any X_i is bounded by $2^p p!$. Finally, the last inequality is a straightforward consequence of the two previous results in this proposition. \square

Now we study the connection between the permanent of Sylvester matrices and P -equivalence. Before that, we need some definitions and notations. From now on, we write g_i for the element of \mathbf{Z}_2^p which is the binary representation of the integer $i - 1$, where $i \in N$. Then M_α denotes the SIOR $r \times p$ matrix with entries $\{0, 1\}$ where its l -th row is g_{i_l} . That is,

$$M_\alpha = \begin{pmatrix} g_{i_1} \\ g_{i_2} \\ \vdots \\ g_{i_r} \end{pmatrix}$$

where $\alpha = \{i_1 < i_2 < \dots < i_r\} \in Q_{r,n}$.

Remark 2.5. There is a one to one correspondence between SIOR $r \times p$ matrices M with entries $\{0, 1\}$ and subsets $\alpha = \{i_1 < i_2 < \dots < i_r\} \in Q_{r,n}$ where $n = 2^p$. Sometimes, we denote α by α_M . Let us observe that the i_k -row/column of H_{2^p} is indexed by g_{i_k} as a cyclic matrix (see Example 1.3).

Theorem 2.6. *Assuming that $H = [h_{i,j}]$ is a Sylvester Hadamard matrix of order $n = 2^p$ and $M \in \{0, 1\}^{r \times p}$ is a SIOR matrix. Let us define*

$$\Phi(\alpha_M) = \prod_{j=2}^n \sum_{i \in \alpha_M} h_{i,j}.$$

Then,

- (1) $\Phi(\alpha_M)$ is invariant under rows permutation of M .
- (2) $\Phi(\alpha_M)$ is invariant under columns permutation of M .
- (3) $\Phi(\alpha_M)$ is invariant under complementations of columns of M .

Proof. Let us observe that,

$$h_{i,j} = \psi(g_i, g_j), \quad 1 \leq i, j \leq n \quad (\text{where } \psi \text{ was defined in Example 1.3}).$$

Given P and Q permutation matrices of order n . Then:

- (1) Obviously, any permutation in the rows of M corresponds with the same subset of r distinct rows of H_{2^p} , i.e., $\alpha_M = \alpha_{PM}$. Hence, $\Phi(\alpha_M) = \Phi(\alpha_{PM})$
- (2) A permutation of columns of M may correspond with a different subset of r distinct rows of H_{2^p} , i.e., $\alpha_M \neq \alpha_{MQ}$ in general. Since

$$\psi(g_i, g_j) = \psi(g_i Q, g_j Q),$$

thus $\Phi(\alpha_M) = \Phi(\alpha_{MQ})$.

- (3) Again, a complementation of the k -th column of M may correspond with a different subset of r distinct rows of H_{2^p} , i.e., $\alpha_M \neq \alpha_{MI(k)}$ in general. But

$$\psi(g_i I(k), g_j) = \begin{cases} \psi(g_i, g_j) & \text{if } k\text{-th entry of } g_j \text{ is } 0 \\ -\psi(g_i, g_j) & \text{if } k\text{-th entry of } g_j \text{ is } 1, \end{cases}$$

$I(k)$ denotes a diagonal matrix obtained from the identity matrix by negating the k -th diagonal entry and $g_i I(k)$ means the element of \mathbf{Z}_2^p where the k -th entry of g_i changes from 0 to 1 or viceversa and the other entries remain the same. From the fact that the number of elements of \mathbf{Z}_2^p with 1 in the k -th entry is 2^{p-1} , it follows that $\Phi(\alpha_M) = \Phi(\alpha_{MI(k)})$. □

Corollary 2.7. Let $M_1, M_2 \in \{0, 1\}^{r \times p}$ be SIOR matrices and $H = [h_{i,j}]$ be the Sylvester Hadamard matrix of order $n = 2^p$. If M_1 is P -equivalent to M_2 then

$$\Phi(\alpha_{M_1}) = \Phi(\alpha_{M_2}).$$

Proposition 2.8. Let $H = [h_{i,j}]$ be the Sylvester Hadamard matrix of order $n = 2^p$. Ryser's formula for H can be rewritten as

$$(3) \quad \sum_{r=1}^n (-1)^r r \sum_{i=1}^{\#\Omega_{r,p}} \#[X_i^r] \Phi(\alpha_{X_i^r})$$

where $\Omega_{r,p} = \{[X_1^r], \dots, [X_{k_r}^r]\}$ is the output of Algorithm 2.3.

Proof. By definition, Ryser's formula for H is

$$\sum_{r=1}^n (-1)^r r \sum_{\alpha \in Q_{r,n}} \Phi(S).$$

Taking into account Corollary 2.7, it takes a simple inspection to see that

$$\sum_{\alpha \in Q_{r,n}} \Phi(S) = \sum_{i=1}^{\#\Omega_{r,p}} \#[X_i^r] \Phi(\alpha_{X_i^r}), \quad \forall r = 1, \dots, n.$$

□

Remark 2.9. Let H be a normalized Hadamard matrix of order n . Then every column/row sum vanishes but the first which is n .

As a consequence,

$$\begin{aligned} \sum_{i \in \alpha} h_{i,1} &= r, \\ \sum_{i \in \alpha} h_{i,j} &= - \sum_{i \in \bar{\alpha}} h_{i,j}, \quad j \geq 2; \end{aligned}$$

for all $\alpha \in Q_{r,n}$ where $\bar{\alpha} \in Q_{n-r,n}$ and $\alpha \cup \bar{\alpha} = \{1, 2, \dots, n\}$. Taking into account the previous identities, the formula (3) can be rewritten as

$$(4) \quad \sum_{r=1}^{\frac{n}{2}} (-1)^r (2r - n) \sum_{i=1}^{\#\Omega_{r,p}} \#[X_i^r] \Phi(\alpha_{X_i^r}).$$

Algorithm 2.10. Evaluating the permanent of H_{2^p} by means of Ryser's formula

Input: H be a Sylvester Hadamard matrix of order $n = 2^p$.

Output: $\text{per}(H)$.

Step 1. Preprocess: Use Algorithm 2.3 to compute $\Omega_{r,p}$ and $\Omega_{r,p}^\sharp$ for $r = 1, \dots, 2^{p-1}$.

Step 2. Evaluate (4).

3. EXAMPLE: H_8

In this section we deal with the problem of computing the permanent of Sylvester Hadamard matrix of order 8,

$$H_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & - & 1 & - & 1 & - \\ 1 & 1 & - & - & 1 & 1 & - & - \\ 1 & - & - & 1 & 1 & - & - & 1 \\ 1 & 1 & 1 & 1 & - & - & - & - \\ 1 & - & 1 & - & - & 1 & - & 1 \\ 1 & 1 & - & - & - & - & 1 & 1 \\ 1 & - & - & 1 & - & 1 & 1 & - \end{bmatrix}.$$

The aim of this section is to show the difference in the number of elementary operations required in Ryser's formula (2) and in Algorithm 2.10 for computing $\text{per}(H_8)$.

First, a useful lemma.

r	inequivalent matrices	#orbits
1	$(0, 0, 0), (0, 0, 1)$	2, 6
3	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$	24, 24, 8

TABLE 1. Output of Algorithm 1 for $p = 3$ and $r = 1, 3$.

	Ryser's formula	Algorithm 2
#Oper.	9913	Step 1. 2688 Step 2. 91 2779

TABLE 2. # of operations that it is needed to compute $\text{per}(H_8)$.

Lemma 3.1. *Let H be a Sylvester Hadamard matrix of order $n = 2^p$ ($p=1,2,3,4$). Let $H_{k,p}$ be a $2^k \times 2^p$ matrix obtained from H by removing $2^p - 2^k$ of its rows (where $k = 1, \dots, p-1$). Then for at least one column of $H_{k,p}$, its column sum vanishes.*

Proof. Until $p = 3$ the proof is by induction, for $p = 4$ it is seen by inspection. \square

Corollary 3.2. *If $p = 1, 2, 3$ or 4 and $r = 2^k$ with $k = 1, \dots, p-1$ then $\Phi(\alpha_{X_i^r}) = 0$ for all $[X_i^r] \in \Omega_{r,p}$.*

Now, we are going to use Algorithm 2.10 to compute $\text{per}(H_8)$.

Step 1. Preprocess: Use Algorithm 2.3 to compute $\Omega_{r,p}$ and $\Omega_{r,p}^\#$. Due to Corollary 3.2 is only necessary for $r = 1, 3$. See Table 1 for the Output.

Step 2. Evaluate formula 4.

$$\text{per}(H_8) = (-1)^1(2-8) \sum_{i=1}^{\#\Omega_{1,3}} \# [X_i^1] \Phi(\alpha_{X_i^1}) + (-1)^3(6-8) \sum_{i=1}^{\#\Omega_{3,3}} \# [X_i^3] \Phi(\alpha_{X_i^3})$$

(Using Table 1, we get)

$$= 6 \cdot (2 \cdot 1 + 6 \cdot 1) + 2 \cdot (24 \cdot 3 + 24 \cdot 3 + 8 \cdot 3) = 384.$$

Finally, Table 2 compares the number of elementary operations (additions and multiplications) that it is needed to compute the permanent of H_8 by Ryser's formula (see (2)) and Algorithm 2.10.

4. CONCLUSIONS AND FURTHER WORK

In this article, we have rewritten Ryser's formula for computing the permanent of Sylvester Hadamard matrices attending to their cocyclic properties. The rewritten formula presents

an important reduction in the number of sets of r distinct rows of H_{2^p} which are involved. However, from a more practical point of view, Algorithm 2.10 is only appropriate for numerical calculations in low orders because the method relies on Algorithm 2.3, and this is generally time-consuming. For low orders, Algorithm 2.10 seems to be faster than Ryser's. Whereas in greater orders, both algorithms present similar limitations. Hence, analytical formulas for the permanents of Sylvester Hadamard are still far away.

Some improvements of our method could be:

- (1) Study the spectrum of the function $\Phi(\alpha)$ and search for a more efficient characterization of the set of all $\alpha \in Q_{r,n}$ with the same value for Φ . This could speed up our Algorithm.
- (2) Study if for each $p > 4$ Lemma 3.1 is true.

Let us remark that for $H_{1,p}$ and $H_{2,p}$ the thesis of Lemma 3.1 holds true $\forall p \geq 3$. It is a straightforward consequence of the orthogonality of H_{2^p} . Moreover, an affirmative answer to the second improvement implies the following result.

Proposition 4.1. *Let G be a $p \times 2^{p-1}$ matrix where its columns are different binary vectors of length p . If G is a generator matrix of a binary $(2^{p-1}, p)$ code C , then there exists at least one codeword of C with weight equal to 2^{p-2} .*

Proof. Let $H_{p-1,p}$ be the $2^{p-1} \times 2^p$ matrix obtained from $H_{2^p} = [\psi(g_i, g_j)]_{g_i, g_j \in \mathbf{Z}_2^p}$ by removing 2^{p-1} of its rows. Concretely, those rows index by the elements of \mathbf{Z}_2^p different from the columns of G . Let g_i be the element of \mathbf{Z}_2^p indexing the column of $H_{p-1,p}$ such that its column sum vanishes. Then, $g_i G$ is a codeword of C with weight equal to 2^{p-2} . Since swapping the 0 entries of $(g_i G)^T$ to -1 , it becomes the i -th column of $H_{p-1,p}$. \square

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