International Conference on
*Secondary calculus and cohomological Physics*
Moscow, August 1997

# HOMOLOGICAL PERTURBATION THEORY AND COMPUTABILITY OF HOCHSCHILD AND CYCLIC HOMOLOGIES OF CDGAS

Álvarez V.*, Armario J.A.†, Real P.* and Silva B.*

Dpto. Matemática Aplicada I, Univ. de Sevilla
*E-mails:* valvarez@euler.fie.us.es, armario@cica.es,
real@cica.es, silva@cica.es

Abstract. We establish an algorithm computing the homology of commutative differential graded algebras (briefly, CDGAs). The main tool in this approach is given by the Homological Perturbation Theory particularized for the algebra category (see [21]). Taking into account these results, we develop and refine some methods already known about the computation of the Hochschild and cyclic homologies of CDGAs. In the last section of the paper, we analyze the $p$-local homology of the iterated bar construction of a CDGA ($p$ prime).

## 1. Introduction.

The description of efficient algorithms of homological computation might be considered as a very important question in Homological Algebra, in order to use those processes mainly in the resolution of problems on algebraic topology; but this subject also influence directly on the development of non so closedareas as Cohomological Physics (in this sense, we find useful references in [12], [24], [25]) and Secondary Calculus ([14], [27], [28]).

Working in the context of CDGAs, Homological Perturbation Theory ([9], [11]), supplies at once a general algorithm computing the homology of these objects (see [16]), which often bears high computational charges and actually restricts its application to the low dimensional homological calculus.

Our first goal consists in refining this algorithm, by means of preservation issues over the category of CDGAs when applying perturbation techniques (see [21]). Indeed, the process reveal a polynomial behavior when we deal with some concrete families of CDGAs; for instance, we compute in this paper the homology of the CDGAs $A^{s,(r_1,\cdots,r_{n-1})}_{e_1,\cdots,e_n}$ (with $r_i \in \mathbb{N}$, $\forall i = 1, \cdots, n-1$). These and similar positive results make us be sincerely expectant in the achievement of an effective general algorithm computing the homology of CDGAs.

Once the previous algorithm is outlined, our interest turns to the calculation of Hochschild and cyclic homologies.

*Key words and phrases:* Homology, CDGA, bar construction, (minimal) homological model, contraction, perturbation, twisted tensor product of CDGA.

The computation of Hochschild homology has already been studied by Guccione and Guccione in [8], where they set recursive formulae in order to determine the differential of a *small homological model* for a CDGA given. We construct here an alternative approach using the proper machinery of Homological Perturbation Theory; not only from that point of view of Lambe's (see [16]), but also attending to the preservation of algebra structures (as developed in [21]). The main difference between this method and Guccione's one is that we get here more explicit homological information, available in terms of homotopy equivalences. This fact allows an immediate translation of the attained results as the departure point for obtaining the cyclic homology, via perturbation.

Unlike the Hochschild homology one, the method computing cyclic homology is quite more complicated, since no algebraic preservation results can be applied. We study here an interesting particular instance, in which the difficulties decrease substantially. It seems to be a good idea to use the $\lambda$-operations described by Loday ([18], [19]) in order to solve the general problem of the computation of cyclic homology: a suggestive way appears in this sense.

The present paper is organized into the following sections. The first one is devoted to explain the basic algebraic concepts from which the process of homological computation is constructed; these ideas help to the fine understanding of the underlying algorithmic concepts. We shall use without further explanation the proper concepts of our framework, Algebraic Topology and Homological Algebra: we refer to some notions like *DGA-algebra*, *DGA-coalgebra*, *derivation*, *coderivation*, *Hopf DGA-algebra* and similar ones.

Along the second section, a positive answer is given about the computability problem of the homology of CDGAs; the algorithm developed here produces not only an explicit homotopy equivalence for the reduced bar construction of a CDGA, but also a comparison map with the bar resolution: i.e., an explicit homotopy equivalence with the "standard resolution".

Nevertheless, this is not a completely satisfactory solution as this result does not allow one to implement the algorithm into practise (without any memory or time limits): the scheme often displays a high complexity.

The subsequent subsection introduces a concrete example, treated with specific techniques in order to improve the general procedure. Arguments which might improve the algorithm of the homological computation are explained at the end of this paragraph.

Finally, the obtention of general algorithms computing Hochschild and cyclic homologies and the search of *minimal homological models* of iterated Bar constructions of CDGAs will be the subjects of the remaining sections.

## 2. Preliminaries.

Although relevant notions of Homological Algebra are explained through the exposition of the paper, most of common concepts are not explicitly given (which might be consulted for instance in [4] or [15]).

Let $\Lambda$ be a commutative ring with non zero unit and let us assume that $\Lambda$ is the ground ring. The notation $(A, d_A, *_A)$ means that $A$ is a differential graded algebra (briefly, DGA-algebra) endowed with a differential operator $d_A$ and an associative product $*_A$. If there is no further confusion, subindexes or even operators are to be omitted.

Notice that a DGA-algebra is said to be connected whenever $A_0$ is $\Lambda$. The degree of an element $a \in A_n$ is denoted by $|a| = n$.

Three particular algebras are of especial interest in the development of the paper, which are the exterior, polynomial and divided power algebras. Let $n$ be a fixed integer. The exterior algebra $E(x, 2n+1)$ is the graded algebra with generators 1 and $x$ of degrees 0 and $2n+1$, respectively; and trivial product (that is, $x^2 = 0$ and $x \cdot 1 = x$). The polynomial algebra $P(y, 2n)$ consists of the graded algebra with generators 1 of degree 0 and each power $y^i$ of degree $2ni$, with $i \in \mathbb{N}$; the product is given by the usual one of polynomials, i.e.: $y^i \cdot y^j = y^{i+j}$ for non negative integers $i$ and $j$. Finally, the divided power algebra $\Gamma(y, 2n)$ is the graded algebra with generators 1 and $y^{(i)}$ (of course, $y = y^{(1)}$) of respective degrees 0 and $2ni$, $i \in \mathbb{N}$; and product defined by the rules $y^{(i)} \cdot y^{(j)} = \begin{pmatrix} i+j \\ i \end{pmatrix} y^{(i+j)}$, whichever non negative integers $i$ and $j$ be. Each of these three types of algebras can be considered as a CDGA with trivial differential.

The reduced bar construction associated to a DGA-algebra given is a standard algebraic tool which allows one to save the product structure of the initial DGA-algebra along the process of homological computation.

The *reduced bar construction* associated to a DGA-algebra $A$ is defined to be as the DG-module $\bar{B}(A)$ consisting of the tensor products of $i$ copies of $A$ at each dimension $i$ (understanding $\Lambda$ at the initial level, i.e., in degree 0), and differential operator depending on the original differential (tensor differential) and product (simplicial differential) morphisms on $A$ (see [15]). A generic homogeneous element of degree $n$ is usually denoted by the expression $[a_{r_1}|a_{r_2}|\cdots|a_{r_s}]$, where $a_{r_i} \in A_{r_i}$ and $s + \sum_{i=1}^{s} r_i = n$; the *tensor degree* of such an element is $\sum_{i=1}^{s} r_i = n$, and the *simplicial degree* is $s$. Whenever the algebra $A$ is commutative, it is possible to define a multiplicative structure upon $\bar{B}(A)$ (via an operator called *shuffle product*), so that the reduced bar construction becomes also a CDGA.

A *resolution* of $\Lambda$ over a DGA-algebra A consists in a differential $A$-module $X$, which is projective as an $A$-module, such that the homology of $X$ is zero except for degree 0, where it coincides with $\Lambda$. If $X$ is actually a free $A$-module then $X$ is called a *free resolution*.

A relevant notion to introduce is the twisted tensor product of DGA-algebras [1] (briefly, $TTP$). Let $\{A_i\}_{i \in I}$ be a set of commutative DGA-algebras. A *twisted tensor product* $\tilde{\otimes}^{\rho}_{i \in I} A_i$ is a CDGA satisfying the following conditions:

**i)** $\tilde{\otimes}^{\rho}_{i \in I} A_i$ coincides with the tensor product $\otimes_{i \in I} A_i$ as a graded algebra.

**ii)** The differential operator consists in the sum of the differential of the banal tensor product and a derivation $\rho$.

Actually, a TTP of DGA-algebras is a "perturbed" DGA-algebra in which the "perturbation" only affects its differential and not its product. Notice that this concept reveals a little structural difference with respect to that of *multiplicative principal construction* of Cartan (see [3], [20]).

An interesting type of free resolutions of $\Lambda$ over a commutative DGA-algebra $A$ consists in TTPs of the form $X = A \tilde{\otimes}^{\rho} \bar{X}$, where $\bar{X}$ is a free CDGA. In this circumstances, $\bar{X}$ is known to be the *reduced complex* of the resolution. For instance, if $A$ is a CDGA, the "bar resolution" $B(A)$ (see [15]) becomes a twisted tensor product of DGA-algebras. More precisely, $B(A)$ is the commutative DGA-algebra $A\tilde{\otimes}^{\rho}\bar{B}(A)$, where

$$\rho(a \otimes [a_1|a_2|\cdots|a_n]) = (a *_A a_1) \otimes [a_2|\cdots|a_n].$$

The algorithmic techniques that are proposed in pursuit of solving the problem of the homology computation are fitted into the framework of constructive Homological Algebra. The main input data are the *contractions* ([5], [13]): a contraction $c$, $N \overset{c}{\Rightarrow} M$ (sometimes denoted by $(f, g, \phi)$) from a DG-module $(N, d_N)$ to a DG-module $(M, d_M)$ consists in an homotopy equivalence determined by three morphisms $f$, $g$ and $\phi$; being $f : N_* \to M_*$ (projection) and $g : M_* \to N_*$ (inclusion) two DG-module morphisms and $\phi : N_* \to N_{*+1}$ an homotopy operator. Moreover, these data are required to satisfy the following rules: $fg = 1_M$, $f\phi = 0$, $\phi g = 0$, $\phi d_N + d_N\phi + gf = 1_N$ and $\phi\phi = 0$.

Notice that a free resolution $K$ of $\Lambda$ over the DGA-algebra $A$ *splits off the bar construction* if there exists a contraction (called splitting) from $B(A)$ to $K$. This notion is due to Lambe ([17]) and is to be applied later on.

There is a relevant particular type of contractions of DGA-algebras, that we systematically use in the remainder of this paper. We refer to *semifull algebra contractions* ([21]), which consist of a multiplicative inclusion, a quasi-algebra projection and a quasi-algebra homotopy. Given a contraction $(f, g, \phi)$ between two DGAs $A$ and $A'$, we recall that:

1. The projection $f$ is said to be a *quasi-algebra projection* whenever the following conditions hold:
$$f(\phi *_A \phi) = 0, \quad f(\phi *_A g) = 0, \quad f(g *_A \phi) = 0. \tag{1}$$
2. The homotopy operator $\phi$ is said to be a *quasi-algebra homotopy* whenever the rules below are verified:
$$\phi(\phi *_A \phi) = 0, \quad \phi(\phi *_A g) = 0, \quad \phi(g *_A \phi) = 0. \tag{2}$$

For instance, the bar resolution $B(A)$ of a DGA-algebra $A$ generates the following semifull algebra contraction:

$$R_{B(A)} : \{B(A), \Lambda, \epsilon_{B(A)}, \eta_{B(A)}, s\},$$

where the homotopy operator $s : B(A) \to B(A)$ is given by

$$s(a \otimes [a_1|\cdots|a_n]) \;=\; [a|a_1|\cdots|a_n].$$

The restriction to this particular type of contractions is due to the fact that they are preserved by most of algebra operations: for instance, the set of all semifull algebra contractions is closed by composition and tensor product of contractions.

The basic procedure we state here consists in the establishment of an explicit contraction from an initial DG-module $N$ to a free finite type DG-module $M$, so that the homology of $N$ is actually computable from that of $M$: that is, $H_*(N) = H_*(M)$.

There exist other important tools from which one reaches extra homological information: under certain circumstances one may "perturb" a given contraction in order to obtain a new contraction between the initial underlying graded modules endowed with distinct differential operators (to know more about Homological Perturbation Theory, consult [9], [11], [13], [16] and [17]).

Notice that semifull algebra contractions are preserved via perturbation processes.

We recall the concept of a perturbation datum. Let $N$ be a graded module and let $f : N \to N$ be a morphism of graded modules. The morphism $f$ is defined to be *pointwise nilpotent* whenever for all $x \in N$, $x \neq 0$, a positive integer $n$ exists such that $f^n(x) = 0$. Notice that the integer $n$ generally depends on the element $x$. A *perturbation of a DGA-module $N$* consists in a morphism of graded modules $\delta : N \to N$ of degree $-1$, such that $(d_N + \delta)^2 = 0$ and $\xi_N \delta = 0$. A *perturbation datum of the contraction $c : \{N, M, f, g, \phi\}$* is a perturbation $\delta$ of the DGA-module $N$ verifying that the composition $\phi\delta$ is pointwise nilpotent. In the particular case that both of $N$ and $M$ are DGA-algebras, a new notion appears: an *algebra perturbation datum* $\delta$ of a contraction $c$ from $N$ to $M$ consists in a perturbation datum $\delta$ of $c$ which is also a derivation.

We deal with perturbation results particularized for the DGA-algebra category (see [21]). In this sense, we give below an analogous theorem to the classical "Algebra Perturbation Lemma" ([9], [11], [13]) appropriated for semifull algebra contractions:

Theorem 2.1 *(**SF-APL**) ([21])*

*Let $c : \{N, M, f, g, \phi\}$ be a semifull algebra contraction and $\delta : N \to N$ be an algebra perturbation datum of $c$. Then, a new semifull algebra contraction*

$$c_\delta : \{(N, d_N + \delta, \epsilon_N, \eta_N),\ (M, d_M + d_\delta, \epsilon_M, \eta_M),\ \ f_\delta,\ g_\delta,\ \phi_\delta\}$$

*is defined by the formulas: $d_\delta = f\delta\Sigma_c^\delta g$; $f_\delta = f(1 - \delta\Sigma_c^\delta\phi)$; $g_\delta = \Sigma_c^\delta g$; $\phi_\delta = \Sigma_c^\delta\phi$; where*

$$\Sigma_c^\delta \;=\; \sum_{i\geq 0}(-1)^i\,(\phi\delta)^i \;=\; 1 - \phi\delta + \phi\delta\phi\delta - \cdots + (-1)^i(\phi\delta)^i + \cdots .$$

Let us note that $\Sigma_c^\delta(x)$ is a finite sum for each $x \in N$, because of the pointwise nilpotency of the composition $\phi\delta$. Moreover, it is obvious that the morphism $d_\delta$ becomes a perturbation of the DGA-module $(M, d_M, \epsilon_M, \eta_M)$.

In order to calculate the homology of a DGA-algebra, the notion of "homological model" arises as a computational alternative to the reduced bar construction.

An *homological model* for a commutative DGA-algebra $A$ is a free DGA-algebra of finite type $HBA$ such that there exists a semifull algebra contraction from $\bar{B}(A)$ to $HBA$. In particular, it is verified that $H_*(\bar{B}(A)) = H_*(HBA)$.

At this stage, it is necessary to borrow Cartan's definition of the suspension and $p$-transpotency additive functions.

Let $A$ be a commutative DGA-algebra. The following morphisms of graded modules can be defined:

- The *suspension*, $\sigma : A \to \bar{B}(A)$; defined as $\sigma(a) = [a]$, for $a \in A$.
- The *p-transpotency* (being $p$ a prime integer), $\varphi_p : A \to \bar{B}(A)$; defined as $\varphi_p(a) = [a|a^{p-1}]$, with $a \in A$.

## 3. Computability of the homology of CDGAs. General algorithm.

It is commonly known that every commutative DGA-algebra $A$ "factorizes" into a tensor product of exterior and polynomial algebras endowed with a differential-derivation; in the sense that there exists an homomorphism connecting both structures, which induces an isomorphism in homology (consult [2] if necessary).

In fact, the object we start from is a generic commutative DGA-algebra $A$, factorized into the model above as a TTP of exterior and polynomial algebras. The studies explained below are reduced to the finitely generated case; that is, a twisted tensor product of algebras $\tilde{\otimes}^{\rho}_{i\in I} A_i$, where $I$ denotes a finite set of indices, $\rho$ is a differential-derivation and $A_i$ is an exterior or a polynomial algebra, for every $i \in I$.

Taking into account the ideas expressed in the section before, the principal goal that arises is to obtain a "chain" of contractions starting at the reduced bar construction $\bar{B}(A)$ and ending up at free DGA-algebra of finite type.

Three *almost-full algebra contractions* (i.e., semifull algebra contractions endowed with multiplicative projections) are used in order to find the structure of graded module of an homological model for the DGA-algebra $A$:

- The contraction defined in [6] from $\bar{B}(A \otimes A')$ to $\bar{B}(A) \otimes \bar{B}(A')$, being $A$ and $A'$ two commutative DGA-algebras; which is briefly denoted by $C_{\bar{B}\otimes}$.

  Though Eilenberg and Mac Lane only set explicit formulae for the projection and inclusion morphisms, an explicit formula of the recursive definition of the homotopy operator given by Eilenberg and Mac Lane in [5] is established in [21].

  A relevant property concerning the homotopy operator $\phi_{\bar{B}\otimes}$ of the contraction above is that
  $$\phi_{\bar{B}\otimes}([a_1 \otimes a'_1|a_2|\cdots|a_m]) \;=\; -[a'_1|a_1|a_2|\cdots|a_m], \tag{3}$$
  with $a_i \in A$, $\forall\ 1 \le i \le m$ and $a'_1 \in A'$.

  Given a tensor product $\otimes_{i\in I} A_i$ of commutative DGA-algebras, a contraction from $\bar{B}(\otimes_{i\in I} A_i)$ to $\otimes_{i\in I}\bar{B}(A_i)$ is easily determined, by applying $C_{\bar{B}\otimes}$ several times in a suitable way. This new contraction is also denoted by $C_{\bar{B}\otimes}$.

- The isomorphism of DGA-algebras (hence, a contraction) $C_{\bar{B}E}$ from $\bar{B}(E(u, 2n+1))$ to $\Gamma(\sigma(u), 2n+2)$, also described in [6]. Note that the generator $v$ of the previous divided power algebra is denoted by $\sigma(u)$, since $g_{\bar{B}E}(v) = \sigma(u)$, where $\sigma$ is the Cartan's suspension operator.

  Last isomorphism might be considered as a *full algebra contraction* (that is, an almost-full algebra contraction endowed with an algebra homotopy operator), in an obvious way.

- The contraction $C_{\bar{B}P}$ from $\bar{B}(P(y, 2n))$ to $E(\sigma(y), 2n)$ ([6]). The generator of the exterior algebra is denoted by $\sigma(y)$ since both elements correspond one to each other by the projection and inclusion of the contraction.

All the previous contractions have already been deeper described for decades, so we will not explain them further.

Thanks to these three contractions, it is possible to establish by composition the following semifull algebra contraction $C_0 = (f, g, \phi)$:

$$\bar{B}(\otimes_{i\in I} A_i) \;\Rightarrow\; \otimes_{i\in I}\bar{B}(A_i) \;\Rightarrow\; \otimes_{i\in I} HBA_i,$$

where $HBA_i$ represents an exterior or a divided polynomial algebra, depending on whether $A_i$ is a polynomial or an exterior algebra. Obviously, the product on $\otimes_{i\in I} HBA_i$ is the natural one.

The next step is perturbing $C_0$, in order to obtain an homological model of the initial TTP $\tilde{\otimes}^{\rho}_{i\in I} A_i$. The morphism $\rho$ produces a perturbation-derivation $\delta$ onto the tensor differential of $\bar{B}(\otimes_{i\in I} A_i)$. It seems necessary to emphasize that the pointwise nilpotency of $\phi\delta$ is guaranteed by the following facts:

- The homotopy operator $\phi$ increases the simplicial degree of $\bar{B}(\otimes_{i\in I} A_i)$ by one.
- The perturbation $\delta$ does not change the simplicial degree. Indeed, $\delta$ lowers the tensor degree by one.

Therefore, by applying **SF-APL** it is constructed a new semifull algebra contraction $(f_\delta, g_\delta, \phi_\delta)$:

$$\bar{B}(\tilde{\otimes}^{\rho}_{i\in I} A_i) \overset{(C_0)_\delta}{\Rightarrow} (\otimes_{i\in I} HBA_i, d_\delta),$$

where the differential $d_\delta$ is determined by the perturbation process. That means that $HBA = \otimes_{i\in I}(HBA_i, d_\delta)$ is an **homological model** of $A = \tilde{\otimes}_{i\in I} A_i$. Notice that the homotopy operator $\phi_\delta$ increases the simplicial degree at least by one. This fact is to be applied in the sequel.

THEOREM 3.1 *Given a finite type TTP $A$ of exterior and polynomial algebras, there exists a semifull algebra contraction from the reduced bar construction $\bar{B}(A)$ to a free DGA-algebra $H$ of finite type. That is, we have a homological model for $A$. Moreover, $H$ consists in a TTP of exterior and divided power algebras such that, at graded algebra level, it is verified that:*

- *each $E(u, 2n-1)$ factor in $A$ contributes with a $\Gamma(\sigma(u), 2n)$ factor in $H$.*
- *each $P(u, 2n)$ factor in $A$ contributes with an $E(\sigma(u), 2n+1)$ factor in $H$.*

In this way, it is described a general algorithm computing the homology of CDGAs. Obviously, the homology of the homological model obtained can be computed using an algorithm based upon the establishment of the Smith's normal form of the matrices representing the differentials at each degree ([26]).

The computational cost to construct the contraction $(C_0)_\delta$ is high: notice that both the inclusion and homotopy operators of the contraction $C_{\bar{B}\otimes}$ give an answer in exponential time, when evaluated on each element. In fact, the formula of the differential operator $d_\delta$ produced by the homological perturbation machinery is given by:

$$d_\delta \;=\; f\circ\delta\circ(1-\phi\circ\delta+\phi\circ\delta\circ\phi\circ\delta-\cdots)\circ g. \tag{4}$$

Attending to the previous remarks about the efficiency on the evaluation of morphisms, a first impression is that the evaluation of $d_\delta$ upon any generator becomes in general a process of exponential nature.

In spite of this, it is possible to take advantage of $d_\delta$ being a derivation (that is, a morphism compatible with the product of the homological model), in order to improve the complexity problem: indeed, the fact that $d_\delta$ is a derivation implies that it is only necessary to know the value of this morphism applied upon the generators of the model (notice that there are so many generators as the cardinal of the set of indices $I$ indicates). This is an enormous improvement in the computation of the differential on the small model.

Finally, Theorem 3.1 can be used to derive small free resolutions of $\Lambda$ over $A$, as it is shown in the following theorem (which is a graded-commutative version of Th. 8.1.3 of [16]):

THEOREM 3.2 *[1]*
*Let $A$ be a connected commutative DGA-algebra. Given a semifull algebra contraction from the reduced bar construction $\bar{B}(A)$ to a free commutative DGA-algebra $H$ (in which the homotopy operator increases the simplicial degree at least by one); there exists a free resolution $K$ which splits off of the bar construction. Moreover, this resolution $K$ is a twisted tensor product of the DGA-algebras $A$ and $H$ and the splitting is a semifull algebra contraction.*

In the light of the theorem above, it is possible to state the following one:

THEOREM 3.3 *Given a finite type TTP $A$ of exterior and polynomial algebras, there exists a free resolution $A\tilde{\otimes}^{\rho'}H$ that split off of the bar construction. More precisely, the splitting is a semifull algebra contraction and the reduced complex $H$ is determined by Theorem 3.1.*

Now, a particular case is examined in detail, in order to clarify the general method.

### 3.1. An example: Homology of the algebras $A^{s,(r_1,r_2,\dots,r_{n-1})}_{e_1,e_2,\dots,e_n}$.

This section is devoted to the description of the homological models of a particular family of CDGAs. We make use of properties of algebra structure preservation. The ground ring is supposed to be $Z$ in this subsection.

Let $A^{s,(r_1,r_2,\dots,r_{n-1})}_{e_1,e_2,\dots,e_n}$ denote the CDGA-algebra which consists of the DG-module

$$\begin{array}{l} E(x,2s+1)\tilde{\otimes}^{\rho_1}P(y_1,2s+2)\tilde{\otimes}^{\rho_2}P(y_2,(2s+2)(r_1+1))\tilde{\otimes}^{\rho_3} \\ \tilde{\otimes}^{\rho_3}\ \cdots\ \tilde{\otimes}^{\rho_n}P(y_n,(2s+2)(r_1+1)\cdots(r_{n-1}+1)), \end{array}$$

and the differential operator

$$\rho_i(y_i)=e_i\,x\otimes y_1^{r_1}\otimes y_2^{r_2}\otimes\cdots\otimes y_{i-1}^{r_{i-1}},\quad \forall 1\le i\le n,$$

where $e_i\in\mathbb{N}$ with $e_1>e_2>\cdots>e_n$ and $r_i\in\mathbb{N}\cup\{0\}$.

Let us consider the following contraction $C_0 = (f, g, \phi)$ as starting point:

$$\bar{B}(E(x, 2s+1) \otimes P(y_1, 2s+1) \otimes \cdots \otimes P(y_n, (2s+2)(r_1+1)\cdots(r_{n-1}+1)))$$
$$\overset{R_{\bar{B}\otimes}}{\Rightarrow}$$
$$\bar{B}(E(x, 2s+1)) \otimes \bar{B}(P(y_1, 2s+2)) \otimes \cdots \otimes \bar{B}(P(y_n, (2s+2)(r_1+1)\cdots(r_{n-1}+1)))$$
$$\overset{R_{\bar{B}E}\otimes R_{\bar{B}P}\otimes\cdots}{\Rightarrow}$$
$$\Gamma(\sigma(x), 2s+2) \otimes E(\sigma(y_1), 2s+3) \otimes \cdots \otimes E(\sigma(y_n), (2s+2)(r_1+1)\cdots(r_{n-1}+1)+1)).$$

The question is to apply the general method described before to this initial contraction. From now on, the degrees of the algebra generators are omitted in orden to simplify the notation.

The differentials $\rho_1$, $\rho_2$, ..., $\rho_n$ produce a perturbation-derivation $\delta$ in

$$\bar{B}(E(x) \otimes P(y_1) \otimes \otimes \cdots \otimes P(y_n).$$

Applying the perturbation machinery, the following semifull algebra contraction arises:

$$\bar{B}(A^{s,(\,{}_2r_1,\ldots,\,{}_nr_{n-1})}_{e_1,\ldots,e_n}) \overset{(C_0)_\delta}{\Rightarrow} (\Gamma(\sigma(x)) \otimes E(\sigma(y_1)) \otimes \cdots \otimes E(\sigma(y_n)), d_\delta),$$

where $d_\delta$ is the differential determined in the perturbation process.

Hence,

$$H\bar{B}A^{s,(\,{}_2r_1,\ldots,\,{}_nr_{n-1})}_{e_1,\ldots,e_n} = (\Gamma(\sigma(x)) \otimes E(\sigma(y_1)) \otimes \cdots \otimes E(\sigma(y_n)), d_\delta)$$

constitutes an **homological model** for this CDGA.

Therefore, it is only necessary to evaluate the differential on each generator $\sigma(y_1)$, $\sigma(y_2)$, ..., $\sigma(y_n)$.

Concretely, computing the homology of $A^{s,()}_{e_1}$, the formula (4) reduces to $f \circ \delta \circ g$, since $\phi \circ \delta \circ g = 0$. If we go on calculating the image of the differential, we get:

First at all, $d_\delta(\sigma(y_1)) = -e_1 \cdot \sigma(x)$.

Considering $\sigma(y_2)$,

$$\begin{aligned} d_\delta(\sigma(y_2)) &= f_\delta \circ \delta \circ g(\sigma(y_2)) \\ &= -e_2 \cdot f_\delta([x \otimes {y_1}^{r_1}]). \end{aligned}$$

Note that it is verified that $f_\delta([x \otimes {y_1}^{r_1}])$ is 0 if $r_1 \geq 1$ and $\sigma(x)$ otherwise ($r_1 = 0$). Attending to (3), we deduce that $\phi([x \otimes {y_1}^{r_1}]) = -[{y_1}^{r_1}|x]$. And applying $\delta$ upon this element generates $r_1 \cdot e_1[x \otimes {y_1}^{r_1-1}|x]$. By applying alternatively the morphisms $\phi$ and $\delta$ on the last element, it is obtained $r_1! \cdot {e_1}^{r_1}[x| \overset{r_1+1 \text{ times}}{\cdots} |x]$ after $r-1$ iterations. There is only one non zero possible action of a morphism over this element, which is $f$. This fact leads directly to the formula

$$f_\delta([x \otimes {y_1}^{r_1}]) = r_1! \cdot {e_1}^{r_1} \cdot \sigma(x)^{(r_1+1)}.$$

To sum up, the differential $d_\delta$ applied over $\sigma(y_2)$ is

$$d_\delta(\sigma(y_2)) = -r_1! \cdot {e_1}^{r_1} \cdot e_2 \cdot \sigma(x)^{(r_1+1)}.$$

There is another outstanding fact we have to take into account: the perturbed projection preserves certain divided powers. Indeed, $f_\delta$ is a quasi-algebra projection (see (1)) and it is verified that

$$f_\delta([x\otimes {y_1}^{r_1}]*_{\bar B} \overset{m \text{ times}}{\cdots} *_{\bar B}[x\otimes {y_1}^{r_1}]) \;=\; f_\delta([x\otimes {y_1}^{r_1}])*_{HB} \overset{m \text{ times}}{\cdots} *_{HB} f_\delta([x\otimes {y_1}^{r_1}]), \qquad (5)$$

where $*_{\bar B}$ denotes the shuffle product of the Bar construction and $*_{HB}$ denotes the natural product on the homological model.

In an analogous way, it is easy to prove that $f_\delta$ preserves divided powers of the type: $[x\otimes y_1^{r_1}\otimes\cdots\otimes y_s^{r_s}| \overset{m \text{ times}}{\cdots} |x\otimes y_1^{r_1}\otimes\cdots\otimes y_s^{r_s}]$.

In particular,

$$f_\delta([x\otimes {y_1}^{2r_1}]*_{\bar B} \overset{m \text{ times}}{\cdots} *_{\bar B}[x\otimes {y_1}^{2r_1}]) \;=\; m!\, f_\delta([x\otimes {y_1}^{2r_1}| \overset{m \text{ times}}{\cdots} |x\otimes {y_1}^{2r_1}]),$$

since $|[x\otimes y_1^{r_1}]|$ is even.

Therefore, the following formula is obtained:

$$f_\delta([x\otimes {y_1}^{r_1}| \overset{m \text{ times}}{\cdots} |x\otimes {y_1}^{r_1}]) \;=\; \frac{(mr_1+m)!}{m!(r_1+1)^m}\cdot {e_1}^{mr_1}\cdot \bar{x}^{mr_1+m}. \qquad (6)$$

This fact is essential for achieving good homological results in this case. Evaluating the differential operator $d_\delta$ over the generator $\sigma(y_3)$,

$$\begin{array}{rcl} g(\sigma(y_3)) &=& [y_3],\\ \delta([y_3]) &=& -e_3\cdot[x\otimes {y_1}^{r_1}\otimes {y_2}^{r_2}],\\ f([x\otimes {y_1}^{r_1}\otimes {y_2}^{r_2}]) &=& 0,\\ \phi([x\otimes {y_1}^{r_1}\otimes {y_2}^{r_2}]) &=& -[{y_2}^{r_2}|x\otimes {y_1}^{r_1}],\\ \bar\delta([{y_2}^{r_2}|x\otimes {y_1}^{r_1}]) &=& r_2\cdot e_2\cdot[x\otimes {y_1}^{r_1}\otimes {y_2}^{r_2-1}|x\otimes {y_1}^{r_1}],\\ (\delta\circ\phi)^{r_2}([x\otimes {y_1}^{r_1}\otimes {y_2}^{r_2}]) &=& r_2!\cdot {e_2}^{r_2}\cdot[x\otimes {y_1}^{r_1}| \overset{r_2 \text{ times}}{\cdots} |x\otimes {y_1}^{r_1}|x\otimes {y_1}^{r_1}]. \end{array}$$

Taking into account (6) and the results before:

$$d_\delta(\sigma(y_3)) \;=\; -r_2!\cdot\frac{[(r_2+1)(r_1+1)]!}{(r_2+1)!(r_1+1)^{r_2+1}}\cdot e_3\cdot {e_2}^{r-2}\cdot {e_1}^{r_1(r_2+1)}\cdot\sigma(x)^{(r_2+1)(r_1+1))}.$$

Working in this way, it is easy to explicit the action of $d_\delta$ upon each generator $\sigma(y_i)$:

$$\begin{array}{rcl} d_\delta(\sigma(y_i)) &=& -r_{i-1}!\cdot\dfrac{[\prod_{k=1}^{i-1}(r_k+1)]!}{(r_{i-1}+1)!\prod_{k=1}^{i-2}(r_k+1)^{\prod_{j=k+1}^{i-1}(r_j+1)}}\cdot e_i\cdot {e_{i-1}}^{r_{i-1}}\cdot \\ && \cdot\prod_{k=1}^{i-2} {e_k}^{r_k\prod_{j=k+1}^{i-1}(r_j+1)}\cdot \quad \sigma(x)^{\left(\prod_{k=1}^{i-1}(r_k+1)\right)}, \end{array} \qquad (7)$$

where $i=1,2,\ldots,n$.

This completes the study of the homology calculation of $A^{s,(r_1,\ldots,r_{n-1})}_{e_1,\ldots,e_n}$.

Note that though coefficients are big enough, working with $\mathbb{Z}_{(p)}$ as ground ring simplifies enormously the difficulties on the real computations of the formulae above.

## 4. Hochschild homology of CDGAs. General Method.

Given a commutative DGA-algebra $A$, it is possible to construct a semifull algebra contraction $C_1$ from $\bar{B}(A)$ to a twisted tensor product $HBA$ consisting of exterior and divided power algebras, as it is shown in the previous section. Of course, $HBA$ is endowed with its natural product. Now, the idea of how to determine the Hochschild homology of $A$ is straightforward: using the contraction above as a starting point, a new semifull algebra contraction $C_2$ can be constructed from the Hochschild complex $HC_*(A)$ to a free finitely generated DGA-algebra (via algebra perturbation machinery). In this way, a real algorithm computing Hochschild homology of commutative DGA-algebras appears. We may say that the work of Lambe in [16] has provided inspiration for this scheme. Here, we enrich this method, making use of algebra preservation results in the perturbation machinery (essencially, the **SF-APL** Theorem).

Recall that whenever there is no further confusion, subindexes or even operators are to be omitted.

The *Hochschild complex of a CDGA* (see [29]) is defined as the graded algebra

$$HC_*(A) = A \otimes \bar{B}(A),$$

endowed with the differential operator $b = b_0 + b_1$. The $b_0$ morphism verifies the following rule:

$$\begin{aligned} b_0(a_0 \otimes [a_1|\cdots|a_n]) &= da_0 \otimes [a_1|\cdots|a_n] \\ &\quad - \textstyle\sum_{i=1}^{n}(-1)^{\epsilon_{i-1}} a_0 \otimes [a_1|\cdots|a_{i-1}|da_i|a_{i+1}|\cdots|a_n], \end{aligned}$$

where $\epsilon_i = i + \sum_{k=0}^{i} |a_k|$. The formula for the $b_1$ operator is given by:

$$\begin{aligned} b_1(a_0 \otimes [a_1|\cdots|a_n]) &= a_0a_1 \otimes [a_2|\cdots|a_n] \\ &\quad + \textstyle\sum_{i=0}^{n-1}(-1)^{\epsilon_i} a_0 \otimes [a_1|\cdots|a_{i-1}|a_ia_{i+1}|a_{i+2}|\cdots|a_n] \\ &\quad -(-1)^{(|a_n|+1)\epsilon_{n-1}} a_na_0[a_1|\cdots|a_{k-1}]. \end{aligned}$$

It is easy to check that the following identities are hold:

$$b_0b_0 = 0, \qquad b_1b_1 = 0, \qquad b_0b_1 + b_1b_0 = 0.$$

Notice that the product to be considered in $HC_*(A)$ is the natural one, that is

$$*_{HCA} = (*_A \otimes *_{HBA})(1 \otimes T \otimes 1)),$$

where $T(a \otimes b) = (-1)^{|a|\cdot|b|} b \otimes a$.

The only difference between the Hochschild complex and the banal tensor product $A \otimes \bar{B}(A)$ lies on the operator $b_1$, and it is given by the first and last summands which does not appear in the differential of the simple tensor product. Indeed, it is not hard to deduce that the Hochschild complex of a CDGA becomes a twisted tensor product of the CDGAs $A$ and $\bar{B}(A)$. More precisely, $HC_*(A) = A \tilde{\otimes}^{\delta_h} \bar{B}(A)$, where

$$\delta_h(a_0 \otimes [a_1|\cdots a_n]) = (-1)^{|a_0|} a_0a_1 \otimes [a_2|\cdots|a_n] - (-1)^{(|a_n|+1)\epsilon_{n-1}} a_na_0[a_1|\cdots|a_{n-1}]. \qquad (8)$$

First of all, let us consider a semifull algebra contraction

$$(f_1, g_1, \phi_1) : (\bar{B}(A), d_{\bar{B}(A)}, *_{\bar{B}}) \overset{C_1}{\Rightarrow} (HBA, d_{HBA}, *_{HBA}),$$

where $HBA$ is a free finitely generated CDGA (actually, a TTP of exterior and divided power algebras).

It is immediate to construct the tensor product contraction $(1_A \otimes f_1, 1_A \otimes g_1, 1_A \otimes \phi_1)$:

$$(A \otimes \bar{B}(A), d_A \otimes 1_{\bar{B}(A)} + 1_A \otimes d_{\bar{B}(A)}, *_{HCA}) \overset{1_A \otimes C_1}{\Rightarrow} (A \otimes HBA, 1_A \otimes d_{HBA} + d_A \otimes 1_{HBA}, (*_A \otimes *_{HBA})(1_A \otimes T \otimes 1_{HBA})).$$

From now on, the product and differential operators of the small DGA-algebra of the above contraction are denoted as $*_{hHA}$ and $d_{hHA}$, respectively.

It is an easy exercise verifying that $1_A \otimes C_1$ is a semifull algebra contraction.

The $\delta_h$ morphism (see (8)) becomes a perturbation datum for the contraction $1_A \otimes C_1$; moreover, $\delta_h$ is a derivation.

Now, the question boils down to apply **SF-APL** to this contraction, taking as algebra perturbation datum the morphism $\delta_h$. It remains to prove the pointwise nilpotency of the composition $(1_A \otimes \phi_1)\delta$. The DGA-algebra $A \otimes \bar{B}(A)$ inherits a filtration given by the tensor degree of the reduced bar construction $\bar{B}(A)$. It is easy to see that $\delta$ lowers filtration, at least, by one (notice that $A$ is connected). On the other hand, since $1_A \otimes \phi_1$ increases the simplicial degree by one, this homotopy operator is filtration preserving. Then, $(1_A \otimes \phi_1)\delta$ lowers the filtration, at least, by one; and this means that this composition is pointwise nilpotent. This completes the sketch of the proof.

Therefore, using **SF-APL** it follows that the pertubed contraction $C_2 = (1_A \otimes C_1)_{\delta_h}$ constitutes a semifull algebra contraction:

$$(f_2, g_2, \phi_2) : (HC(A), b, *_{HC}) \overset{C_2}{\Rightarrow} (A \otimes HBA, d_{hHA} + d_{\delta_h}, *_{hHA}).$$

Hence, a “homological model” for the Hochschild complex of any CDGA is established in this way.

Since the perturbed differential $d_{\delta_h}$ is a derivation, this morphism is only needed to be evaluated acting on the generators of $A \otimes HBA$ (as an algebra). It is immediate to conclude that $d_{\delta_h} = 0$. In fact, this is a natural consecuence from the following identities:

$$\delta_h(1_A \otimes g_1)(a \otimes 1) = \delta_h(a \otimes [\,]) = 0,$$

$$\delta_h(1_A \otimes g_1)(1 \otimes z) = \delta_h(1 \otimes [z]) = z \otimes [\,] - z \otimes [\,] = 0,$$

where $a \otimes 1$ and $1 \otimes z$ are the only two types of generic generators of $A \otimes HBA$, being $a$ and $z$ generators of $A$ and $HBA$, respectively.

In the same way, since $g_2 = (g_1)_{\delta_h}$ is a morphism of DGA-algebras, this inclusion is completely set knowing its images on the generators of $A \otimes HBA$. Now, applying $g_2$ on a generator $t$ of $A \otimes HBA$, it is obtained that $g_2(t) = (1_A \otimes g_1)_{\delta_h}(t) = (1_A \otimes g_1)(t)$. That is, the perturbation machinery does produce no changes on the inclusion of the contraction. However, the projection $f_2$ and the homotopy operator $\phi_2$ are, in general, different from the morphisms $1_A \otimes f_1$ and $1_A \otimes \phi_1$, respectively.

In this sense, we state the following theorem:

THEOREM 4.1 *Given a finite type TTP $A$ of exterior and polynomial algebras, there exists a semifull algebra contraction from the Hochschild complex of $A$, $HC_*(A)$, to a tensor product $A \otimes H$, where $H$ is the homological model of the reduced bar construction of $A$ described in Theorem 3.1.*

Though theorem above shows that the Hochschild homology of a CDGA does not provide new homological information distinct from that generated by the reduced bar construction, the contraction explained before is to be essential in the sequel, in order to compute the cyclic homology (via perturbation).

## 5. Cyclic homology. General Algorithm.

The perturbation machinery provides us, in an easy way, an algorithm calculating the cyclic homology of commutative DGA-algebras. The main references associated to this section are [7], [22] and [29].

The *cyclic complex* of $A$ ([7]) consists of the graded module

$$CC_*(A) \quad = \quad P(u,2) \otimes HC_*(A),$$

endowed with the following differential operator:

$$d_c(u^i \otimes w) = \begin{cases} u^i \otimes b(w) + u^{i-1} \otimes B(w) & \text{if } i > 0, \\ 1 \otimes b(w) & \text{if } i = 0. \end{cases}$$

The morphism $B$ is given by the formula:

$$B(a_0 \otimes [a_1|\cdots|a_n]) \quad = \quad \textstyle\sum_{i=0}^{n}(-1)^{(\epsilon_{i-1}+1)(\epsilon_n-\epsilon_{i-1})} 1 \otimes [a_i|\cdots|a_n|a_0|a_1|\cdots|a_{i-1}].$$

The following product operation may be defined on the cyclic complex associated to every commutative DG-algebra $A$ (notice that it is not the natural one):

$$(u^i \otimes w_1) *_{CC} (u^j \otimes w_2) \quad = \quad \begin{cases} u^i \otimes (w_1 *_{HC} B(w_2)) & \text{if } j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the natural product of the cyclic complex is not compatible with its differential structure. Due to this fact, the cyclic complex of a CDGA is not a twisted tensor product, in the sense described on section 2.

Now then, starting from the semifull algebra contraction

$$(f_2, g_2, \phi_2) : (HC(A), b, *_{HC}) \overset{C_2}{\Rightarrow} (A \otimes HBA, d_{hHA}, *_{hHA}),$$

which determines the Hochschild homology of a CDGA $A$, it is possible to establish the following contraction $(1_P \otimes f_2, 1_P \otimes g_2, 1 \otimes \phi_2)$ at a DG-module level:

$$(CC(A), 1 \otimes b) \overset{1_{P(u,2)} \otimes C_2}{\Rightarrow} (P(u,2) \otimes (A \otimes HBA), 1_P \otimes d_{hHA}). \tag{9}$$

Let consider now the morphism

$$\delta_c(u^i \otimes w) \quad = \quad \begin{cases} u^{i-1} \otimes B(w) & \text{if } i > 0, \\ 0 & \text{if } i = 0. \end{cases}$$

It is very easy to prove that $\delta_c$ becomes a perturbation of the DG-module $P(u,2) \otimes HC_*(A)$, as well as the pointwise nilpotency of $(1 \otimes \phi_2) \circ \delta_c$. So, $\delta_c$ is a perturbation datum for the contraction (9). The following perturbed contraction $((1_P \otimes f_2)_{\delta_c}, (1_P \otimes g_2)_{\delta_c}, (1_P \otimes \phi_2)_{\delta_c})$ is obtained by applying the Basic Perturbation Lemma:

$$[1_P \otimes C_2]_{\delta_h} : (CC_*(A), d_c) \overset{C_3}{\Rightarrow} (P(u,2) \otimes (A \otimes HBA), 1_P \otimes d_{hHA} + d_{\delta_c}).$$

The small DG-module of the last contraction is denoted by $CBA$. It is important to emphasize that the contraction above is determined at DG-module level.

THEOREM 5.1 *Given a finite type TTP $A$ of exterior and polynomial algebras, there exists a contraction from the cyclic complex of $A$, $CC_*(A)$, to a free DG-module of finite type.*

Notice now that the algebra perturbation technique SF-APL can not be applied to this situation (either for the product $*_{CC}$ or for the natural product of the underlying tensor product of the cyclic complex). There is no chance to using the preservation of algebra structures in the contraction $C_3$. This fact makes the computation of cyclic homology extremely difficult.

This situation can be remedied in one particular case: considering banal tensor products of exterior and polynomial algebras.

Let $A$ be a non-twisted tensor product $E(x_1, \ldots, x_m) \otimes P(y_1, \ldots, y_n)$. The contraction

$$(f_1, g_1, \phi_1) : (\bar{B}(A), d_{\bar{B}}, *_{\bar{B}}) \overset{C_1}{\Rightarrow} (\Gamma(\bar{x}_1, \ldots, \bar{x}_m) \otimes E(\bar{y}_1, \ldots, \bar{y}_n), 0, *_{HBA})$$

becomes an almost-full algebra contraction.

Now, $(1_A \otimes C_1) + \delta_h$ provides the appearance of the contraction

$$(f_2, g_2, \phi_2) = (1_A \otimes C_1)_{\delta_h} : \ (HC_*(A), b, *_{HC}) \overset{C_2}{\Rightarrow} (A \otimes HA, d_{hHA}, *_{hHA}).$$

The contraction $C_3 = (f_3, g_3, \phi_3)$ satisfies the following property. Let $z_i$ be a monomial of length 1 in $\{x_1, \ldots, x_m, y_1, \ldots, y_n\}$, $\forall i$. Notice that $g_1([z_i]) = 0$ whenever $z_i = y_j^k$, with $k > 1$. Hence,

$$\begin{aligned} &(1_P \otimes g_2)(u^i \otimes (z_{j_1} *_A \cdots *_A z_{j_r}) \otimes (\bar{z}_{k_1} *_{hHA} \cdots *_{hHA} \bar{z}_{k_s}) = \\ &\quad u^i \otimes (g_2(z_{j_1}) *_A \cdots *_A g_2(z_{j_r})) \otimes g_2(\bar{z}_{k_1}) *_{HC} \cdots *_{HC} g_2(\bar{z}_{k_s}) = \\ &\quad u^i \otimes (z_{j_1} *_A \cdots *_A z_{j_r}) \otimes [z_{k_1}] *_{HC} \cdots *_{HC} [z_{k_s}] = \\ &\quad u^i \otimes (z_{j_1} *_A \cdots *_A z_{j_r}) \otimes B(z_{k_1} \otimes [\,]) *_{HC} \cdots *_{HC} B(z_{k_s} \otimes [\,]). \end{aligned}$$

Then,

$$\begin{aligned} &\delta_c(1_P \otimes g_2)(u^i \otimes (z_{j_1} *_A \cdots *_A z_{j_r}) \otimes (\bar{z}_{k_1} *_{hHA} \cdots *_{hHA} \bar{z}_{k_s})) = \\ &u^{i-1} \otimes [B((z_{j_1} *_A \cdots *_A z_{j_r}) \otimes [\,]) *_{HC} B(z_{k_1} \otimes [\,]) *_{HC} \cdots *_{HC} B(z_{k_s} \otimes [\,])]. \end{aligned}$$

The last equality is proved using a property due to Loday ([18]):

$$B(x * B(y)) = B(x) * B(y).$$

Since $\phi_2$ is a quasi algebra homotopy (see (2)), it is proved in [21] that $\phi_2$ is, indeed, a morphism of DGA-algebras (i.e., multiplicative morphism).

Then, it verified that

$$\phi_2 *_{HC} (Bg_2 \otimes g_2) \; = \; (\phi_2 B g_2) *_{HC} g_2.$$

Therefore,

$$\begin{array}{l}[(1\otimes\phi_2)\delta_c(1\otimes g_2)](u^i\otimes(z_{j_1} *_A \cdots *_A z_{j_r})\otimes(\bar z_{k_1} *_{hHA}\cdots *_{hHA}\bar z_{k_s})) = \\ u^{i-1}\otimes\phi_2[B((z_{j_1} *_A\cdots *_A z_{j_r})\otimes[\,])] *_{HC} g_2(\bar z_{k_1} *_{HC}\cdots *_{HC}\bar z_{k_s}).\end{array}$$

Carrying on in this way, it follows that

$$\begin{array}{l}d_{\delta_c}(u^i\otimes[(z_{j_1} *_A\cdots *_A z_{j_r})\otimes(\bar z_{k_1} *_{hHA}\cdots *_{hHA}\bar z_{k_s})]) = \\ u^{i-1}\otimes[d_c(z_{j_1} *_A\cdots *_A z_{j_r}\otimes[\,]) *_{hHA}\bar z_{k_1} *_{hHA}\cdots *_{hHA}\bar z_{k_s}].\end{array}$$

That is, the work is confined to know the image of the elements in $\bar{B}(A)$ of simplicial degree 1 upon the action of $f_3$.

Notice that the general "twisted" case is much more complicated.

Loday studies in [19] some natural filtrations of the Hochschild and cyclic homologies of commutative algebras. The method consists in establishing certain $\lambda$-operations "á la Grothendieck", and consequently, a $\gamma$-filtration for each theory. These filtrations exist with no hypothesis over the characteristic of the ground ring. These operations can be studied in the present context: the trick lies on considering these operations as endomorphisms of the DG-module $HC_*(\;)$ or $CC_*(\;)$. In this way, the homology generators of $CC(A)$ might be determined.

## 6. $p$-Minimal homological models of iterated reduced bar constructions of CDGAs.

We are mainly concerned in this section with the construction of a homological model for an iterated reduced bar construction of a TTP of exterior and polynomial algebras.

We need to enunciate some new definitions to obtain such a result.

Definition 6.1 *[1]* A twisted tensor product $TTP = \tilde{\otimes}^{\rho}_{i\in I} A_i$ is called *decomposable* whenever there is a non-trivial partition $I = I_1 \cup I_2$ of $I$, such that $TTP$ decomposes into a tensor product of twisted tensor products $TTP_1 = \tilde{\otimes}^{\rho_1}_{i\in I_1} A_i$ and $TTP_2 = \tilde{\otimes}^{\rho_2}_{i\in I_2} A_i$ with $\rho_m = \rho|_{TTP_m}$ for $m = 1, 2$. Otherwise, $TTP$ is called *indecomposable of length $\ell$* (or *$\ell$-indecomposable*), where $\ell$ is the cardinal of the set of indices $I$.

If you take a finite TTP $A$ of exterior and polynomial algebras (may be constituted by $n$ algebras) and apply the general method developed the sections before (Theorem 3.1), a semifull algebra contraction from $\bar{B}(A)$ to a TTP of $n$ exterior and divided power algebras is obtained. It seems interesting to study the computability of the homology of $\bar{B}^n(A)$, with $n \in \mathbb{N}$.

It is necessary to recall a theorem proved by Gugenheim, Lambe and Stasheff in which a contraction from the reduced bar construction of $A$ to the bar tilde construction of $M$ (see [10], [23]) arises, taking as input datum a contraction from a DG-algebra $A$ to a DG-module $M$.

THEOREM 6.2 *[11]*

*Let $A$ and $M$ be a DGA-algebra and a DGA-module respectively, and $c : \{A, M, f, g, \phi\}$ be a contraction. Hence, the following full coalgebra contraction can be established:*

$$\bar{B}(c) : \{\bar{B}(A), \tilde{B}(M), \bar{B}(f), \bar{B}(g), \bar{B}(\phi)\}, \tag{10}$$

*where $d_s^A$ is the simplicial differential operator of $\bar{B}(A)$ and the DG-module $\tilde{B}(M)$ is the Stasheff's bar tilde construction.*

Whenever $M = A'$ is a DGA-algebra and $c$ is an algebra contraction, then the contraction $\bar{B}(c)$ "connects" two bar constructions (see [11]). For instance, if $g$ is a morphism of DGA-algebras then $\bar{B}(c)$ is a contraction from $\bar{B}(A)$ to $\bar{B}(A')$, and its inclusion $\bar{B}(g)$ coincides with $T(S(\bar{g}))$. Since this morphism preserves shuffle products, $\bar{B}(c)$ becomes an algebra contraction with multiplicative inclusion. With no further assumptions, it can be proved that $\bar{B}(g)$ always preserves shuffle products (see [21]). In other words, the previous theorem might be enriched, by establishing the behavior of the contraction $\bar{B}(c)$ regarding to the algebra structures determined by the shuffle product.

THEOREM 6.3 *[21]*

*Let $c : \{A, M, f, g, \phi\}$ be a contraction, where $A$ is a commutative DGA-algebra and $M$ is a DGA-module. Then*

$$\bar{B}(c) : \{\bar{B}(A), \tilde{B}(M), \bar{B}(f), \bar{B}(g), \bar{B}(\phi)\}$$

*constitutes a semifull algebra contraction.*

Applying the reduced bar construction to an exterior algebra generates a divided power algebra; due to this fact and in order to iterate the process above, it is necessary to establish a contraction for the bar construction of a divided power algebra with one generator of even degree. Working in $\mathbb{Z}_{(p)}$, being $p$ a prime number, a "$p$-minimal homological model" of this algebra is obtained, consisting in a tensor product of an exterior algebra and an infinite number of "$p$-minimal" 2-indecomposable TTPs of exterior and divided power algebras.

Let us suppose, therefore, that the ground ring is given by $\mathbb{Z}_{(p)}$, that is, $\mathbb{Z}$ localized on a prime integer $p$:

$$\mathbb{Z}_{(p)} = \{\frac{r}{s}, \quad \text{such that} \quad m.c.d.(p, s) = 1\}.$$

DEFINITION 6.4 *[13]* Let $M$ be a DGA-module over $\mathbb{Z}_{(p)}$. We say that a morphism of DGA-modules $h : M \to M$ is $p$-minimal whenever $h(M) \subseteq p \cdot M$. We say that a DGA-module $M$ is *$p$-minimal* whenever it is free, of finite type as graded module over $\mathbb{Z}_{(p)}$ and its differential $d_M$ is $p$-minimal.

Notice that a contraction between two $p$-minimal DGA-modules constitutes an isomorphism of DGA-modules.

Taking into account the previous definitions, a commutative DGA-algebra $H$ is said to be a *$p$-minimal homological model* of a given CDGA $A$ whenever $H$ is minimal and there exists a contraction from $\bar{B}(A)$ to $H$.

Hence, the following result states:

THEOREM 6.5 *[1]*

*Let $\mathbb{Z}_{(p)}$ be the ground ring. There exists a semifull algebra contraction from the reduced bar construction of a $p$-minimal $\ell$-indecomposable twisted tensor product $A$ of exterior and divided power algebras (with $\ell \geq 2$) to a tensor product $H$ of $p$-minimal $k$-indecomposable twisted tensor products (with $k \leq \ell$) of exterior and divided power algebras (equipped with the natural product). That is, $H$ is a $p$-minimal homological model of $A$.*

*Moreover, attending only to the graded algebra level, it is verified that*

- *each $E(u, 2n-1)$ factor in $A$ contributes with a $\Gamma(\sigma(u), 2n)$ factor in $H$.*
- *each $\Gamma(u, 2n)$ factor in $A$ contributes with $E(\sigma\gamma_{p^i}(u), 2np^i + 1)$ (taking $i$ values upon the non negative integers) and $\Gamma(\varphi_p\gamma_{p^{i-1}}(u), 2np^i + 2)$ ($i \geq 1$), factors in $H$.*

What the previous theorem really means is that there exists an homological preservation (or non-degeneration) result of the $\ell$-indecomposability of TTPs of that kind.

Finally, the following theorem arises:

THEOREM 6.6 *Let $\mathbb{Z}_{(p)}$ be the ground ring. Let $A$ be a TTP of $n$ exterior and polynomial algebras and let $m$ be a natural number. There is a semifull algebra contraction from the iterated bar construction $\bar{B}^m(A)$ to a tensor product $H$ of $i$-indecomposable TTPs of exterior and divided power algebras, with $i \leq n$.*

In fact, the departure point above is constituted by a semifull algebra contraction from $\bar{B}(A)$ to a TTP of $n$ algebras (exterior and divided power algebras), which has already been established before. Combining appropriately Theorem 6.3 and Theorem 6.5, the result follows.

Theorems above admits an immediate translation to the language of free resolutions, via Theorem 6.3:

THEOREM 6.7 *Let $A$ be a TTP of $n$ exterior and polynomial algebras and let $m$ be a natural number. There exists a free resolution $\bar{B}^{m-1}(A)\tilde{\otimes}^{\nu} H$ that splits off of the bar resolution $B(\bar{B}^{m-1}(A))$. More precisely, the splitting is a semifull algebra contraction and the reduced complex $H$ is determined by Theorem 6.6.*

## References


[1] Armario J.A., Real P. and Silva B.: *A generalization of a result of J.C. Moore about Cartan's little constructions*, communication presented at *Conference about Higher Homotopy Structures in Topology and Mathematical Physics*, Vassar college, New York (June, 1996).

[2] Burghelea D. and Vigué Poirrier M.: *Cyclic homology of commutative algebras I*, Lecture Notes in Mathematics, Algebraic Topology Rational Homotopy, **1318**, pp. 51-72 (1986).

[3] Cartan H.: *Algèbres d'Eilenberg-Mac Lane*, Séminaire H. Cartan 1954/55, (exposé **2** à **11**), E. Normale Superiere, Paris (1.956).

[4] Cartan H. and Eilenberg S.: *Homological Algebra*, Princeton University Press, Princeton (1956).

[5] Eilenberg S. and Mac Lane S.: *On the groups $H(\pi, n)$, I*, Annals of Math., **58**, pp. 55-139 (1953).

[6] Eilenberg S. and MacLane S.: *On the groups $H(\pi, n)$, II*, Annals of Math., **vol. 60**, pp. 49-139 (1954).

[7] Getzler E., Jones J.D.S. and Petrack S.: *Differential forms on loop spaces and the cyclic bar complex*, Topology, **vol. 30**, n. 3, pp. 339-371 (1991).

[8] Guccione J.A. and Guccione J.J.: *Hochschild homology of commutative algebras in positive characteristic*, Communications in algebra, **22** (15), pp. 6037-6046 (1994).

[9] Gugenheim V.K.A.M. and Lambe L.A.: *Perturbation theory in Differential Homological Algebra, I*, Illinois J. Math., **vol. 33**, pp. 556-582 (1989).

[10] Gugenheim V.K.A.M., Lambe L.A. and Stasheff J.D.: *Algebraic aspects of Chen's twisting cochain*, Illinois J. Math., **34** (2), pp. 485-502 (1990).

[11] Gugenheim V.K.A.M., Lambe L.A. and Stasheff J.D.: *Perturbation theory in differential homological algebra, II*, Illinois J. Math., **35** (3), pp. 357-373 (1991).

[12] Henneaux M. and Teitelboim C.: *Generalitation of gauges systems*, Princenton Univ. Press (1992).

[13] Huebschmann J. and Kadeishvili T.: *Small models for chain algebras*, Math. Zeit., **207**, pp. 245-280 (1991).

[14] Krasil'shchik I.S.: *Some new cohomological invariants for nonlinear differential equations*, Diff. Geom. Appl. **2**, pp. 307-350 (1992).

[15] MacLane S.: *Homology*, Classics in Mathematics, Springer (1995).

[16] Lambe L.A.: *Homological perturbation theory, Hochschild homology and formal groups*, Contemp. Math., **134**, pp. 183-218 (1992).

[17] Lambe L.A.: *Resolutions which split off of the bar construction*, J. Pure Appl. Algebra **84**, pp. 311-329 (1993).

[18] Loday J-L. y Quillen D.: *Cyclic homology and the Lie algebra of matrices*, Comment. Math. Helv., **59**, pp. 565-591 (1984).

[19] Loday J-L.: *Opérations sur l'homologie cyclique des álgebres commutatives*, Invent. Math., **96**, pp. 205-230 (1989).

[20] Prouté A.: *Algèbres différentielles fortement homotopiquement associatives ($A_\infty$-algèbres)*, Ph. D. Thesis, Université Paris VII.

[21] Real P.: *Homological Perturbation Theory and associativity*, Prepublicación: MA1-05-96I. Enviado a *J. Pure App. Alg.*

[22] Seibt P.: *Cyclic homology of algebras*, World Scientific Publishing, Singapore (1987).

[23] Stasheff J.D.: *Homotopy associativity of H-spaces, II*, Transactions of A.M.S., **vol 108**, pp. 293-312 (1963).

[24] Stasheff J.D: *Constrained Hamiltonians, BRS and homological algebra*, Proc. Conf. on Elliptic Functions in Algebraic Topology (1986).

[25] Stasheff J.D: *Cohomological Physics*, Lecture Notes in Mathematics, Algebraic Topology Rational Homotopy, **1318**, Springer Verlag, pp. 228-237 (1988).

[26] Veblen O.: *Analysis Situs*, A.M.S. Publications, **vol. 5** (1931).

[27] Vinogradov A.M., Krasil'shchik I.S. and Lychagin V.V.: *Introduction to the Geometry of Nonlinear Differential Equations*, Nauka, Moscú (1986); English transl. *Geometry of Jet Spaces and Nonlinear Partial Differential Equations*, Gordon and Breach, New York (1986).

[28] Vinogradov A.M.: *From symmetries of partial differential equations towards secondary ("quantized") calculus*, J. Geom. Phys., **14**, pp. 146-194 (1994).

[29] Weibel C.A.: *An introduction to homological algebra*, Cambridge studies in advanced mathematics, **38**, Cambridge University Press (1994).