ALGORITHMS IN ALGEBRAIC TOPOLOGY AND HOMOLOGICAL ALGEBRA: PROBLEM OF COMPLEXITY

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This review tackles the problem of the high computational complexity lying in most algorithms in algebraic topology and homological algebra. Three particular algorithms are considered: the computation of the homology of commutative differential graded algebras, the homology of principal twisted Cartesian products of Eilenberg–MacLane spaces, and a combinatorial method of computing Steenrod squares. Bibliography: 54 *titles.*

§1. Introduction

Since the notion of homeomorphism was clearly defined, classification of topological spaces up to homeomorphism has become the main problem in topology. This soon became a hopeless job, and so what we know today as *algebraic topology* was born, where topological problems are studied by casting them into a simpler form by means of some algebraic structures. An example of this is given by *invariant algebraic structures* associated with topological spaces, such as homology, cohomology, or homotopy groups.

This review fits into the field of algebraic topology and, more explicitly, of homological algebra, which studies "the pure algebraic properties of algebraic objects in algebraic topology." A principal problem in homological algebra is to find efficient algorithms for homological computations applicable for solving problems on algebraic topology, though this subject directly influences the development of not so close areas such as cohomological physics [22, 45, 46] or secondary calculus [24, 53, 52].

The literature contains some papers in which the foundations for building up constructive algebraic topology are given (see, e.g., [41, 42]). In the present paper, we follow some ideas of [42]. More precisely, we adopt the Eilenberg–MacLane philosophy sketched in the remarkable papers [14, 15, 27], i.e., we work in the combinatorial setting (provided by simplicial topology [31]) and take explicit homotopy equivalences between differential graded modules as initial data. All this is complemented by the homological perturbation machinery [19, 20, 23]. In this way, a suitable framework for computing homology in algebraic topology is raised.

Whereas the question of computability in algebraic topology is delicate (see [7, 30, 54, 29, 33, 49, 27, 35, 50, 41, 42], etc.), the first approach to the problem of complexity is disappointing. In most algorithms in this area, there are high computational costs and the problems of efficiency amount to improving parts of the algorithms, so that the number of steps is minimized.

We analyze three algorithms that we have already designed in previous papers from the complexity point of view. The first one is an algorithm for computing the homology of commutative differential graded algebras (briefly, CDGAs) [2]. This process can be regarded as a refined version of the method outlined in [25]. In the second one, combinatorial fibrations are treated. More specifically, a "constructive" version of the Serre spectral sequence for twisted

Cartesian products (briefly, TCPs) of Eilenberg–MacLane spaces is established (see [1]). Finally, starting from a manageable combinatorial formulation of the morphisms measuring the strong homotopy commutativity of the cup product in cohomology (see [37, 21]), we discuss the complexity of computing the Steenrod squares of cochains.

§2. Preliminaries

Although relevant notions of homological algebra are explained through the exposition of this survey, most of the common concepts are not explicitly given and are used without further explanations (for details, see [10, 28]).

We recall the notions of DG-module and DG-algebra.

DG-modules and DG-algebras. Our ground ring Λ is $\mathbb{Z}$ or $\mathbb{Z}$ localized at a prime p. A *differential graded module* (M, d) is a module M graded on the positive integers and endowed with a differential operator d of degree -1 such that $d \circ d = 0$. The *homology* $H_*(M)$ of (M, d) consists of the quotient groups $H_n(M) = \operatorname{Ker} d_n / \operatorname{Im} d_{n+1}$. If this object is also endowed with an inner product which is compatible with both the grading and the differential, then a new structure arises, a *DG-algebra*.

For instance, we can consider the exterior algebra with one generator u of degree $2n+1$ and the trivial product, or the polynomial algebra with one generator v of degree $2n$ and the natural product of monomials.

An *augmented graded differential algebra* (DGA-algebra) A is a DG-algebra endowed with two graded morphisms $\xi_A \colon A \to \Lambda$ (augmentation) and $\eta_A \colon \Lambda \to A$ (co-augmentation) such that $\xi_A \circ \eta_A = 1_\Lambda$ and $\xi_A \circ d_1 = 0$.

In what follows, we use the designation $(A, d_A, *_A)$ for a DGA-algebra A endowed with a differential operator d_A and an associative product $*_A$. If there is no confusion, subindices or operators are omitted. The degree of an element $a \in A_n$ is denoted by $|a| = n$ or with the index: a_n.

Simplicial sets. Algebraic topology analyzes transformations from continuous data to discrete data. Usually, these transformations involve combinatorial methods, one of the earliest and most significant of which is a triangulation of a topological space.

A *simplicial set* [31] is a graded set indexed on the nonnegative integers together with two families of morphisms (face operators and degeneracy operators) satisfying the following commutativity relations:

$$\partial_i \partial_j = \partial_{j-1} \partial_i \quad \text{if } i < j; \qquad s_i s_j = s_{j+1} s_i \quad \text{if } i \leq j;$$

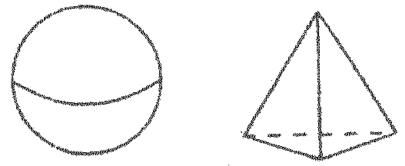
$$\partial_i s_j = s_{j-1} \partial_i \quad \text{if } i < j; \qquad \partial_i s_j = s_j \partial_{i-1} \quad \text{if } i > j+1;$$

$$\partial_j s_j = 1_K = \partial_{j+1} s_j.$$

Depending on the initial category, many other simplicial objects arise (for instance, simplicial modules, simplicial groups, etc.) if both face and degeneracy operators belong to the same category.

For instance, here is a combinatorial description of the 2-sphere.

TOPOLOGY SIMPLICAL TOPOLOGY



degree 2: faces

degree 1: edges

degree 0:vertices

S^2

COMBINATORIAL VERSION OF S^2

Homology. The next step towards an algebra is that every simplicial set X yields an algebraic object $C_\star(X)$ canonically associated with X and endowed with the structure of a DG-module. Namely, the n-component of $C_\star(X)$ is the free Λ-module $C_n(X) = \Lambda[X_n]$ generated by the n-simplices of X, and the differential operator $d_n : C_n(X) \to C_{n-1}(X)$ is given by the formula $d_n = \sum_{0 \le i \le n} (-1)^i \partial_i$.

The subset $s(C_\star(X))$ consisting of all degenerate simplices of $C_\star(X)$ is a graded submodule preserved by the action of the differential (i.e., $d_n(s(C_\star(X))_n) \subset s(C_\star(X))_{n-1}$), so that the quotient complex, simply denoted by $C(X) = \{C_\star(X)/s(C_\star(X))\}$, also becomes a DG-module, which is called the *normalized DG-module* canonically associated with X.

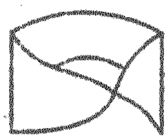
The homology $H_\star(X)$ of X is defined by the formula $H_\star(X) = H_\star(C(X))$.

In the example above, the n-simplices are the weakly increasing $(n+1)$-sequences formed by given ordered vertices; the ith face of an n-simplex is obtained by removing the ith vertex; similarly, the ith degeneracy operator arises from duplicating the ith vertex.

Fiber bundles. In the context of simplicial topology, the fiber bundles simplify their structures and are regarded in a factorized fashion as a *twisted Cartesian product* $F \times_\tau B$ of two simplicial sets, F (fiber) and B (base), where the τ operator means a "twisting" action on the 0-face operator of $F \times B$.

For instance, consider the fiber bundle whose fiber is S^1 and whose base is the unit segment I, i.e., the fiber bundle is a cylinder. The *Möbius band* is obtained from this cylinder by applying a suitable twisting function τ:

TOPOLOGY SIMPLICAL TOPOLOGY ALGEBRAIC TOPOLOGY



$I \times_\tau S^1$

$H(C(I \times_\tau S^1)) =$

$= H(C(I) \otimes_t C(S^1))$

MÖEBIUS BAND TWISTED CARTESIAN PRODUCT $C(I) \otimes_t C(S^1)$

We note that homology is preserved in a suitable way.

Contractions. The main tool for reducing the homology of a twisted Cartesian product to that of a twisted tensor product is a special type of homotopy equivalence, which is due to

Eilenberg and Zilber [17], the so-called *Eilenberg–Zilber contraction*.

In our framework, the main input data are the *contractions* [14, 23]. A contraction c from a DG-module N to a DG-module M, sometimes simply denoted by $N \overset{c}{\Rightarrow} M$ or (f, g, ϕ), is a homotopy equivalence determined by three morphisms f, g, and ϕ, where $f\colon N_\star \to M_\star$ (projection) and $g\colon M_\star \to N_\star$ (inclusion) are DG-module morphisms and $\phi\colon N_\star \to N_{\star+1}$ is a homotopy operator. Moreover, we assume that the following conditions are fulfilled:

$$fg = 1_M, \quad f\phi = 0, \quad \phi g = 0, \quad \phi d_N + d_N\phi + gf = 1_N, \quad \text{and} \quad \phi\phi = 0;$$

thus, the DG-modules involved have isomorphic homology.

Using explicit contractions as data, we note that the basic procedure for computing the homology of an object (see [42]) consists of the construction of an explicit contraction from an initial DG-module N to a free DG-module M of finite type, so that the homology of N is actually computable from that of M via a matrix algorithm due to Veblen [51] (based upon the normal Smith form of the matrices representing the differentials at each degree).

Now we present a typical type of contraction. Let X and Y be simplicial sets, and let AW, EML, and SHI be the following morphisms:
Alexander–Whitney,

$$AW\colon C(X \times Y) \longrightarrow C(X) \otimes C(Y),$$

Eilenberg–MacLane,

$$EML\colon C(X) \otimes C(Y) \longrightarrow C(X \times Y),$$

and Shih,

$$SHI\colon C(X \times Y) \longrightarrow C(X \times Y),$$

which are defined on positive degrees by the formulas

$$AW(a_n \times b_n) = \sum_{i=0}^{n} \partial_{i+1} \cdots \partial_n a_n \otimes \partial_0 \cdots \partial_{i-1} b_n,$$

$$EML(a_p \otimes b_q) = \sum_{(\alpha,\beta)\in\{(p,q)\text{-shuffles}\}} (-1)^{\mathrm{sg}(\alpha,\beta)} (s_{\beta_q} \cdots s_{\beta_1} a_p \times s_{\alpha_p} \cdots s_{\alpha_1} b_q),$$

$$\begin{aligned} SHI&(a_n \times b_n) \\ &= -\sum (-1)^{m+\mathrm{sg}(\alpha,\beta)} (s_{\beta_q+m} \cdots s_{\beta_1+m} s_{m-1} \partial_{n-q+1} \cdots \partial_n a_n s_{\alpha_{p+1}+m} \\ &\qquad\qquad \cdots s_{\alpha_1+m} \partial_m \cdots \partial_{m+p-1} b_n), \end{aligned}$$

where

$$m = n - p - q, \quad \mathrm{sg}(\alpha,\beta) = \sum_{i=1}^{p} (\alpha_i - (i-1)),$$

and the latter sum is taken over the indices $0 \le q \le n-1$, $0 \le p \le n-q-1$ and $(\alpha, \beta) \in \{(p+1, q)\text{-shuffles}\}$.

We note that in degree one the morphisms AW and EML coincide with the identity map and SHI is zero.

Then, the collection

$$EZ_{X,Y}\colon \{C(X \times Y),\, C(X) \otimes C(Y),\, AW_{X,Y},\, EML_{X,Y},\, SHI_{X,Y}\} \tag{1}$$

is a contraction, which is often called an *Eilenberg–Zilber contraction.*

The above explicit formula for the operator SHI was stated by Rubio [40] and proved by Morace (see the appendix in [38]). It is quite surprising that until now there has not been an active interest in the study of the homotopy operator involved in an Eilenberg–Zilber contraction, not only from the point of view of getting an explicit formula for it (a recursive formula was determined by Eilenberg and MacLane [15]) but also from the point of view of obtaining algebraic preservation results of this operator with regard for the underlying coalgebra structure on $C(X \times Y)$.

A deep analysis of the complexity involved in the above contraction shows that the projection morphism AW acts in polynomial time $O(n^2)$, though the inclusion and homotopy operators act in exponential time $O(2^n)$ since the number of (p, q)-shuffles is given by the expression $\binom{p+q-1}{p-1}$. We note that the size of the instance is given by the degree of the elements and the elementary operations are the simplicial operators.

In some sense, the problem of computational complexity in our algorithmic approach is summarized in the previous example. Our main goal is to avoid the categorically exponential complexity of EML and SHI in our algorithms. Thus, an interesting problem arises: to determine the lack of certain classes of elements such that the above morphisms act in "reasonable" time and the homological information is completely localized over these elements.

In fact, the tool provided by contractions suggests, in general, new perspectives and approaches to classical problems of algebraic topology and homological algebra: Eilenberg and MacLane fit an unusual way of working into algebraic topology, which is summarized in [14–16]. Here, we follow these characteristic procedures rather than the Cartan ones. Indeed, the deep analysis of the morphisms, products, and algebra structures involved in most of the *contractions* in algebraic topology leads to a new and unexplored field, which is the direction in which we are movingin: we refer to the homological perturbation theory applied to the algebra or coalgebra category.

Perturbations. Shih [43] first established a perturbation process on Brown's theorem [6] in 1961, but it was not until 1989 that perturbation theory was finally described. Gugenheim, Lambe, and Stasheff [19, 20] found that, under certain circumstances, the process of "perturbing" a given contraction stops and leads to a new contraction that connects the homologies of the respective modules. This technique is called the "basic perturbation lemma," which we describe next.

First, we recall the concept of a perturbation datum. Let N be a graded module, and let $f\colon N \to N$ be a morphism of graded modules. The morphism f is *pointwise nilpotent* if for all $x \in N$, $x \neq 0$, there exists a positive integer n such that $f^n(x) = 0$. Note that, generally, the integer n depends on the element x.

A *perturbation* of a DGA-module N is a morphism of graded modules $\delta\colon N \to N$ of degree -1 such that $(d_N + \delta)^2 = 0$ and $\xi_N \delta = 0$. A *perturbation datum of the contraction* $c = \{N, M, f, g, \phi\}$ is a perturbation δ of the DGA-module N such that the composition $\phi\delta$ is pointwise nilpotent.

Basic perturbation lemma (BPL). We now state the main tool for handling contractions. The *basic perturbation lemma* [19, 20] is an algorithm which, for a contraction $c = \{N, M, f, g, \phi\}$ and a perturbation datum δ of c, generates a new contraction

$$c_\delta\colon \{(N, d_N + \delta), (M, d_M + d_\delta), f_\delta, g_\delta, \phi_\delta\},$$

defined by the formulas

$$d_\delta = f\delta\Sigma_c^\delta g, \quad f_\delta = f(1 - \delta\Sigma_c^\delta\phi), \quad g_\delta = \Sigma_c^\delta g, \quad \phi_\delta = \Sigma_c^\delta\phi,$$

where

$$\Sigma_c^\delta = \sum_{i\geq 0}(-1)^i\,(\phi\delta)^i = 1 - \phi\delta + \phi\delta\phi\delta - \cdots + (-1)^i(\phi\delta)^i + \cdots .$$

The stop condition of this theorem is guaranteed by the pointwise nilpotency of the composition $\phi\delta$. We note that the main difficulty in establishing algorithms in our setting lies in determining that the stop condition of the BPL applied to a specified case be satisfied.

In the particular case where both N and M are DGA-algebras, a new notion appears: an *algebra perturbation datum* δ of a contraction c from N to M is a perturbation datum δ of c that is also a derivation, i.e., satisfies the relations

$$\delta *_N = *_N(1 \otimes \delta + \delta \otimes 1), \qquad \xi_N\delta = 0.$$

We can deal with perturbation results particularized to the DGA-algebra category, often applied to special algebra contractions (called *semifull algebra contractions*), in which the injection preserves products and the projection and homotopy operators f and ϕ satisfy the relations

$$f *_A (\phi \otimes \phi) = 0, \quad f *_A (\phi \otimes g) = 0, \quad f *_A (g \otimes \phi) = 0 \tag{2}$$

and

$$\phi *_A (\phi \otimes \phi) = 0, \quad \phi *_A (\phi \otimes g) = 0, \quad \phi *_A (g \otimes \phi) = 0. \tag{3}$$

In this sense, we give the following more specific version of the "basic perturbation lemma."

Theorem 2.1 (SF-APL [38]). *Let c be a semifull algebra contraction, and let δ be an algebra perturbation datum of c. Then the perturbed contraction c_δ is a semifull algebra contraction.*

Thus, the general procedure that we state here in order to compute the homology of a DGA-algebra concerns the existence of "small models" [23].

However, the technique described above is not good enough: the main obstacle is the high complexity of the procedures often involved. For instance, if $F \times_\tau B$ is a twisted Cartesian product, Shih [43] proved that if we perturb the Eilenberg–Zilber contraction $C(F \times B) \Rightarrow C(F)\bigcirc C(B)$, where

$$\delta(f, b) = (\tau b * \partial_0 f,\ \partial_0 b) - (\partial_0 f,\ \partial_0 b)$$

is the perturbation datum, we recover Brown's theorem [6], in which a contraction from the normalized DG-module associated with the twisted Cartesian product $F \times_\tau B$ onto the twisted tensor product $C(F)\bigcirc_t C(B)$ is established. Then, the computation of the homology of the combinatorial fiber bundle $F \times_\tau B$ reduces to determining a Brown cochain t and the knowledge of some homological information (in terms of contractions) for the factors F and B.

Of course, the Brown cochain t is determined by the differential d_δ resulting from the perturbation process.

Now, we recall the formula for the perturbed differential operator d_δ in terms of the proper morphisms AW, EML, and SHI of the initial Eilenberg–Zilber contraction:

$$d_\delta = AW \circ \delta \circ (1 - SHI \circ \delta + SHI \circ \delta \circ SHI \circ \delta - \cdots) \circ EML.$$

The formula reveals an exponential character of the action of the differential operator, since EML and SHI act so. This is a common fact in general perturbation processes of algebraic topology.

We note that in the early sixties, Szczarba [48] gave an explicit formula for the cochain t involved in the differential operator d_δ.

In order to simplify the high computational charges of the techniques above, it is convenient to use the algebraic structures present in the contractions. This approach allows us to get interesting results in our research, as we show in this paper.

§3. The first algorithm: homology of CDGAs

There are various methods of computing the homology groups of commutative differential graded algebras (CDGAs), though they often involve high complexity processes.

There is an approximation which deals with the recursive construction of a resolution of the ground ring Λ over an initial algebra (see [26]). A *resolution* of Λ over a DGA-algebra A is a differential A-module X such that X is projective as an A-module and the homology of X vanishes except in degree 0, where it is Λ. If X is actually a free A-module, then X is called a *free resolution*.

Working in the context of CDGAs, homological perturbation theory [19, 20] supplies a general algorithm for computing the homology of these objects (see [25]), which often bears high computational charges and actually restricts its application to the low-dimensional homological calculus.

It is possible to improve this general method by applying SF-APL: the refining of the above algorithm, by means of preservation issues over the category of CDGAs when applying perturbation techniques [38], is partially shown in [2].

It is commonly known that every commutative DGA-algebra A "factorizes" into the tensor product of exterior and polynomial algebras endowed with a differential-derivation in the sense that there exists a weak homotopy equivalence, i.e., a homomorphism connecting both structures that induces an isomorphism in homology (see [8]).

Twisted tensor products. Next, we recall the definition of a "twisted tensor product of algebras."

A *twisted tensor product* (TTP) $\tilde{\bigotimes}^{\rho}_{i\in I} A_i$ is a CDGA satisfying the following conditions:

(i) It is the usual tensor product $\bigotimes_{i\in I} A_i$ as a graded algebra.

(ii) The differential operator is the sum of the differentials corresponding to the banal tensor product and a derivation ρ.

Actually, a TTP of DGA-algebras is a perturbed DGA-algebra in which the perturbation only affects the differential and not the product.

Now we recall that the reduced bar construction associated with a CDGA-algebra A is a DGA-algebra

$$\overline{B}(A) = \left(\bigoplus_{n \geq 0} A^n,\ |\ |_s + |\ |_t,\ d_s + d_t \right)$$

with grading $|\ | = |\ |_s + |\ |_t$, where

$$|[a_1|\cdots|a_n]|_t = \sum_{i=1}^{n} |a_i|, \qquad |[a_1|\cdots|a_n]|_s = n,$$

and with differential operator $d = d_s + d_t$, where

$$d_t([a_1|\cdots|a_n]) = -\sum_{i=1}^{n} (-1)^{e_i - 1} [a_1|\cdots|d_A(a_i)|\cdots|a_n],$$

$$\begin{aligned} &d_s([a_1|\cdots|a_n]) \\ &\quad = \xi_A(a_1)[a_2|\cdots|a_n] + \sum_{i=1}^{n-1} (-1)^{e_i} [a_1|\cdots|a_i *_A a_{i+1}|\cdots|a_n] + (-1)^{e_n} [a_1|\cdots|a_{n-1}]\xi_A(a_n), \end{aligned}$$

and $e_i = i + |a_1| + \cdots + |a_i|$.

The product on $\overline{B}(A)$ is the *shuffle product* $*_B$:

$$[a_1|\cdots|a_p] *_{\overline{B}} [b_1|\cdots|b_q] = \sum_{(\alpha,\beta)} (-1)^{\epsilon(\pi,a,b)} [c_{\pi(1)}|\cdots|c_{\pi(p+q)}],$$

where (α, β) runs through the (p, q)-shuffles, π is the permutation which determines them, $(c_1, \ldots, c_{p+q}) = (a_1, \ldots, a_p, b_1, \ldots, b_q)$, and

$$\epsilon(\pi, a, b) = \sum_{\pi(i) > \pi(p+j)} |[a_i]||[b_j]|.$$

Small models. The importance of the reduced bar construction is clear from the following definition of "small models" for DGA-algebras.

A *homological model* for a commutative DGA-algebra A is a free DGA-algebra HBA of finite type and a semifull algebra contraction $\overline{A} \overset{c}{\Rightarrow} HBA$ onto HBA (see [2]).

The object we start from is a generic commutative DGA-algebra A, factorized into the model above as a TTP of exterior and polynomial algebras, $A = \tilde{\bigotimes}^{\rho}_{i \in I} A_i$.

Taking into account the ideas expressed in the previous section, we see that the main goal is to obtain a "chain" of contractions starting from the reduced bar construction $\overline{B}(A)$ of A and ending up at a free DGA-algebra of finite type.

Three *almost-full algebra contractions* (i.e., semifull algebra contractions endowed with multiplicative projections) are mainly used in order to find the structure of a graded module on a homological model of the DGA-algebra A:

- The contraction $C_{\overline{B}\otimes}$ from $\overline{B}(A \otimes A')$ to $\overline{B}(A) \otimes \overline{B}(A')$, where A and A' are two commutative DGA-algebras (see [15]).

The morphisms (f, g, ϕ) that determine the contraction $C_{\overline{B}\otimes}$ are given by the following explicit formulas:

$$f[x_1 \otimes y_1|\cdots|x_n \otimes y_n] = \sum_{i=0}^{n} \xi_A(x_1 \cdots x_i)\xi_{A'}(y_{i+1} \cdots y_n)[x_1|\cdots|x_i] \otimes [y_{i+1}|\cdots|y_n],$$

$$g([x_1|\cdots|x_p] \otimes [y_1|\cdots|y_q]) = [x_1 \otimes 1|\cdots|x_p \otimes 1] \star [1 \otimes y_1|\cdots|1 \otimes y_q],$$

and the homotopy operator ϕ strongly depends on the SHI operator of a particular Eilenberg–Zilber contraction (up to sign, $C_{\overline{B}\otimes}$ coincides with an Eilenberg–Zilber contraction [15]).

For a given tensor product $\bigotimes_{i\in I} A_i$ of commutative DGA-algebras, a contraction

$$\overline{B}\Big(\bigotimes_{i\in I} A_i\Big) \Rightarrow \bigotimes_{i\in I} \overline{B}(A_i)$$

can easily be determined by applying $C_{s\overline{B}\otimes}$ several times in a suitable way. This new contraction is also denoted by $C_{\overline{B}\otimes}$.

- The isomorphism $C_{\overline{B}E}\colon \overline{B}(E(x, 2n+1)) \to \Gamma(\sigma(x), 2n+2)$ of DGA-algebras (also described in [15]). We note that the generator v of the previous divided power algebra is denoted by $\sigma(u)$, since $g_E(v) = \sigma(u)$, where σ is the Cartan suspension operator, $\sigma\colon A \to \overline{B}(A)$, $\sigma(a) = [a]$. The homotopy operator ϕ_E obviously becomes zero since $C_{\overline{B}E}$ is an isocontraction.

The latter isomorphism can be regarded, in a natural way, as a *full algebra contraction* (i.e., an almost-full algebra contraction endowed with an algebra homotopy operator).

- The contraction $C_{\overline{B}P}$ from $\overline{B}(P(y, 2n))$ to $E(\sigma(y), 2n+1)$ (see [15]). The generator of the exterior algebra is denoted by $\sigma(y)$ so that both elements y and $\sigma(y)$ correspond to each other by the projection f_P and the inclusion g_P of the contraction.

All of the previous contractions have already been extensively described in recent decades, and so we do not dwell on them.

Composing these three contractions, we can introduce the following semifull algebra contraction $C_0 = (F, G, \Phi)$:

$$\overline{B}\Big(\bigotimes_{i\in I} A_i\Big) \Rightarrow \bigotimes_{i\in I} \overline{B}(A_i) \Rightarrow \bigotimes_{i\in I} HBA_i,$$

where HBA_i represents an exterior or a divided polynomial algebra, depending on whether A_i is a polynomial or an exterior algebra, respectively. Note that the product on $\bigotimes_{i\in I} HBA_i$ is the natural one.

In the next step, we perturb C_0 in order to obtain a homological model of the initial twisted tensor product $\tilde{\bigotimes}^{\rho}_{i\in I} A_i$. The morphism ρ produces a perturbation-derivation δ onto the tensor differential of $\overline{B}(\bigotimes_{i\in I} A_i)$. It is necessary to emphasize that the pointwise nilpotence of $\phi\delta$ is guaranteed by the following facts:

- The homotopy operator ϕ increases the simplicial degree of $\overline{B}(\bigotimes_{i\in I} A_i)$ by one.

- The perturbation δ does not change the simplicial degree. Indeed, δ lowers the tensor degree by one.

Therefore, applying SF-APL (see Theorem 2.1), we obtain a new semifull algebra contraction $(F_\delta, G_\delta, \Phi_\delta)$:

$$\overline{B}\Big(\tilde{\bigotimes}^{\rho}_{i\in I} A_i\Big) \stackrel{(C_0)_\delta}{\Rightarrow} \Big(\bigotimes_{i\in I} HBA_i, d_\delta\Big),$$

where the differential d_δ is determined by the perturbation process. This means that $HBA = \bigotimes_{i\in I}(HBA_i, d_\delta)$ is a *homological model* of $A = \tilde{\bigotimes}_{i\in I} A_i$. We note that the homotopy operator Φ_δ increases the simplicial degree at least by one.

Theorem 3.1 [2]**.** *Let A be a finite-type twisted tensor product of exterior and polynomial algebras. Then there exists a semifull algebra contraction from the reduced bar construction $\overline{B}(A)$ to a free DGA-algebra H of finite type. That is, we have a homological model for A. Moreover, H is a TTP of exterior and divided power algebras such that, at the graded algebra level, we have:*

- *each $E(u, 2n-1)$ factor in A yields a $\Gamma(\sigma(u), 2n)$ factor in H.*
- *each $P(u, 2n)$ factor in A yields an $E(\sigma(u), 2n+1)$ factor in H.*

Thus, a general algorithm for computing the homology of CDGAs is described. We note that it is connected with the classical notion of homology, since, following Lambe's notation, for a given twisted tensor product A of exterior and polynomial algebras (see [26]), there exists a free resolution $A\tilde{\otimes}^{\rho'} H$ that splits off from the reduced bar construction, and this splitting is a semifull algebra contraction [2].

The computational cost for constructing the contraction $(C_0)_\delta$ is significant. Note that both the inclusion and homotopy operators of the contractions $C_{\overline{B}\otimes}$ give an answer in exponential time, when evaluated on each element, since several Eilenberg–Zilber contractions are involved at this stage. Thus, the first impression is that the evaluation of d_δ at an element becomes, in general, a process of exponential nature.

In spite of the discourse before, we can use the fact that d_δ is a derivation in order to improve the complexity problem. Indeed, since d_δ is a derivation, it is only necessary to know the value of this morphism applied to the generators of the model (note that the number of generators is equal to the cardinality of the index set I). This is an enormous improvement in the computation of the differential on the small model.

In order to clarify the general method, we study in detail what happens when we start from the particular case of the following CDGA-algebra. In what follows, we assume that the ground ring is $\mathbb{Z}$.

Let $A^s_{e_1}$ denote the CDGA-algebra that consists of the DG-module

$$E(x, 2s+1)\tilde{\otimes}^{\rho_1} P(y_1, 2s+2)$$

and the differential operator

$$\rho_1(y_1) = e_1\, x,$$

where $e_1 \in \mathbb{N}$. We start with the following contraction $C_0 = (F, G, \Phi)$:

$$\begin{aligned}\overline{B}(E(x, 2s+1) \otimes P(y_1, 2s+1)) &\stackrel{C_{\overline{B}\otimes}}{\Longrightarrow} \overline{B}(E(x, 2s+1)) \otimes \overline{B}(P(y_1, 2s+2)) \\ &\stackrel{C_{\overline{B}E}\otimes C_{\overline{B}P}}{\Longrightarrow} \Gamma(\sigma(x), 2s+2) \otimes E(\sigma(y_1), 2s+3).\end{aligned}$$

In the previous notation, this means that

$$C_0 = \{(f_E \otimes f_P) \circ f, g \circ (g_E \otimes g_P), \phi + g \circ (1_\Gamma \otimes \phi_P) \circ f\}.$$

Now we want to apply the general method described above to this initial contraction. From now on, the degrees of the algebra generators are omitted in order to simplify the notation.

The differential ρ_1 produces a perturbation-derivation δ in

$$\overline{B}(E(x) \otimes P(y_1))$$

in an obvious way.

Applying the perturbation machinery, we obtain the following semifull algebra contraction:

$$\overline{B}(A^s_{e_1}) \overset{(C_0)_\delta}{\Rightarrow} (\Gamma(\sigma(x)) \otimes E(\sigma(y_1)), d_\delta),$$

where d_δ is the differential determined in the perturbation process.

Hence,

$$H\overline{B}A^s_{e_1} = (\Gamma(\sigma(x)) \otimes E(\sigma(y_1)), d_\delta)$$

is a *homological model* for this CDGA, where the differential operator is given by the formula

$$d_\delta = F \circ \delta \circ \left(\sum (-1)^i (\Phi \circ \delta)^i\right) \circ G$$

$$= (f_E \otimes f_P) \circ f \circ \delta \circ \left(\sum (-1)^i \left((\phi + g \circ (1_\Gamma \otimes \phi_P) \circ f) \circ \delta\right)^i\right) \circ g \circ (g_E \otimes g_P).$$

Since d_δ is a derivation, it is only necessary to evaluate the differential at each of the generators $\sigma(x)$ and $\sigma(y_1)$. In this particular case, the formula of d_δ reduces to $F \circ \delta \circ G$ since $\Phi \circ \delta \circ G = 0$. Calculating the image of the differential, we obtain $d_\delta(\sigma(x)) = 0$ and $d_\delta(\sigma(y_1)) = -e_1 \cdot \sigma(x)$.

This procedure may be generalized in a proper way to every CDGA belonging to the family $A^{s,(r_1,r_2,\ldots,r_{n-1})}_{e_1,e_2,\ldots,e_n}$ of CDGA-algebras consisting of the DG-module

$$E(x, 2s+1)\tilde{\otimes}^{\rho_1} P(y_1, 2s+2)\tilde{\otimes}^{\rho_2} P(y_2, (2s+2)(r_1+1))\tilde{\otimes}^{\rho_3} \cdots \tilde{\otimes}^{\rho_n} P(y_n, (2s+2)(r_1+1)\cdots(r_{n-1}+1))$$

and the differential operator

$$\rho_i(y_i) = e_i\, x \otimes y_1^{r_1} \otimes y_2^{r_2} \otimes \cdots \otimes y_{i-1}^{r_{i-1}}, \quad 1 \le i \le n,$$

where $e_i \in \mathbb{N}$ with $e_1 > e_2 > \cdots > e_n$ and $r_i \in \mathbb{N} \cup \{0\}$.

Taking, as perturbation datum, the derivation morphism δ induced by ρ_i, we can consider a similar perturbation process for the contraction corresponding to C_0.

We should take into account the following important fact: for a particular class of elements, the perturbed projection F_δ (a quasi-algebra projection) is a multiplicative morphism. For example:

$$F_\delta(\overbrace{[x \otimes {y_1}^{r_1}] *_{\overline{B}} \cdots *_{\overline{B}} [x \otimes y_1{}^{r_1}]}^{m \text{ times}}) = \overbrace{F_\delta([x \otimes {y_1}^{r_1}]) *_{HB} \ldots *_{HB} F_\delta([x \otimes {y_1}^{r_1}])}^{m \text{ times}}, \tag{4}$$

where $*_{HB}$ denotes the natural product on the homological model.

In this way, it is easy to find the action of d_δ on each generator $\sigma(y_i)$:

$$d_\delta(\sigma(y_i)) \\ = -r_{i-1}!\frac{\left[\prod_{k=1}^{i-1}(r_k+1)\right]!}{(r_{i-1}+1)!\prod_{k=1}^{i-2}(r_k+1)^{\prod_{j=k+1}^{i-1}(r_j+1)}} e_i {e_{i-1}}^{r_{i-1}} \prod_{k=1}^{i-2} {e_k}^{r_k \prod_{j=k+1}^{i-1}(r_j+1)} \sigma(x)^{\prod_{k=1}^{i-1}(r_k+1)}, \qquad (5)$$

where $i = 1, 2, \dots, n$.

This completes the study of the calculation of the homology of algebras in the family $A^{s,(r_1,\dots,r_{n-1})}_{e_1,\dots,e_n}$. We note that, although the coefficients are sufficiently large, taking $\mathbb{Z}_{(p)}$ as the ground ring, we enormously simplify the difficulties of the real computation of the above formulas, which have already been implemented in Mathematica [5].

In view of these and similar positive results, we may anticipate the achievement of an "efficient" general algorithm for computing the homology of CDGAs.

§4. The second algorithm: p-local homology of the $K(\pi, n) \times_\tau K(\pi', n')$ spaces

A classical result in algebraic topology is that every simplicial set factorizes, up to homotopy, into a twisted Cartesian product of irreducible prime factors. These prime spaces $K(\pi, n)$, called Eilenberg–MacLane spaces, have nontrivial homotopy group π in only one dimension n. On the other hand, a common technique in homotopy theory is to "localize" the homological information, i.e., the homology is sensitive to only one prime.

Our second algorithm computes the p-local homology of a principal twisted Cartesian product (PTCP)

$$X = X(\pi, m, \tau, \pi', n) = K(\pi, m) \times_\tau K(\pi' n),$$

where both $K(\pi, m)$ and $K(\pi', n)$ are Eilenberg–MacLane spaces and π and π' are finitely generated Abelian groups, $m, n \in \mathbb{N}$. This study is partially represented in [1].

If π is an ordinary (discrete) Abelian group, we can regard $G\pi$ as a simplicial group with π in each degree and the identity mappings as the face and degeneracy operators. Then the simplicial group $K(\pi, n)$ is defined as follows: $K(\pi, 0) = G\pi$ and $K(\pi, n+1) = \overline{W}K(\pi, n)$, where $\overline{W}$ is the classifying construction [31]. We recall that the classifying construction $\overline{W}(G)$, which is canonically associated with a simplicial group G, is the following simplicial group:

$$\begin{aligned}
\overline{W}_0(G) &= \{[\,]\}; \\
\overline{W}_n(G) &= G_{n-1} \times \dots \times G_0, \quad n > 0; \\
s_0[\,] &= [e_0]; \\
\partial_i[g_0] &= [\,], \quad i = 0, 1; \\
\partial_0[g_n, \dots, g_0] &= [g_{n-1}, \dots, g_0], \\
\partial_{i+1}[g_n, \dots, g_0] &= [\partial_i g_n, \dots, \partial_1 g_{n-i+1}, g_{n-i-1} + \partial_0 g_{n-i}, g_{n-i-2}, \dots, g_0], \\
s_0[g_{n-1}, \dots, g_0] &= [e_n, g_{n-1}, \dots, g_0], \\
s_{i+1}[g_n, \dots, g_0] &= [s_i g_n, \dots, s_0 g_{n-i}, e_{n-i}, g_{n-i-1}, \dots, g_0],
\end{aligned}$$

where $[\,]$ is the unique element of G_0, e_n is the identity element of $(G, +)$, and $[g_{n-1}, \dots, g_0]$ is a typical element of G_n, for $n > 0$.

Now we recall several results that allow us to determine the p-local homology of a PTCP of Eilenberg–MacLane spaces.

On the one hand, for any commutative simplicial group G, there exists a semifull algebra contraction $C(\overline{W}(G)) \Rightarrow \overline{B}(C(G))$ (see [39, 3]).

On the other hand, taking $\mathbb{Z}_{(p)}$ as the ground ring, we can define a semifull algebra contraction from the reduced bar construction of a Cartan elementary complex [9] to a tensor product of Cartan's elementary complexes [38].

This "endogamic" process allows us to completely determine the p-local homology of an Eilenberg–MacLane space. Note that, for a given contraction from the CDGA A to the CDGA A' with multiplicative inclusion or projection, the contraction connecting the reduced bar constructions $\overline{B}(A)$ and $\overline{B}(A')$ is, indeed, a semifull algebra contraction [38].

In order to construct a homological model for $C(K(\pi, n))$, where π is a finitely generated Abelian group, we use two particular almost full algebra contractions, which are established in [15]. On the one hand, the contraction $\overline{B}(\Lambda[\mathbb{Z}]) \Rightarrow E(u, 1)$, where $E(u, 1)$ is the exterior algebra; on the other hand, the contraction $\overline{B}(\Lambda[\mathbb{Z}_{p^r}]) \Rightarrow E(u, 1) \otimes_\rho \Gamma(v, 2)$, where ρ is the derivation with $\rho(v) = p^r \cdot u$ and $\rho(u) = 0$.

Combining the above results, we obtain the following chain of DGA-algebra contractions:

$$C(K(\pi, 1)) \Rightarrow C(\overline{W}(K(\pi, 0))) \Rightarrow \overline{W}_N(C(K(\pi, 0))) \Rightarrow \overline{B}(\Lambda[\pi]) \Rightarrow G(\pi, 1);$$

and hence,

$$C(K(\pi, n+1)) \Rightarrow C(\overline{W}(\overline{W}^n(K(\pi, 0)))) \Rightarrow \overline{W}_N(\overline{W}^n_N(C(K(\pi, 0))))$$

$$\Rightarrow \overline{B}(\overline{W}^n_N(C(K(\pi, 0))) \Rightarrow \overline{B}(\overline{B}^n(\Lambda[\pi])) \Rightarrow \overline{B}(G(\pi, n)) \Rightarrow G(\pi, n+1).$$

Using them, we easily prove the following theorem from [36]:

Theorem 4.1 [1]**.** *Let $\Lambda = \mathbb{Z}_{(p)}$ be the ground ring. Let n be a nonnegative integer, and let π be a finitely generated Abelian group. There is a semifull algebra contraction $c_{\pi,n}$ from $C(K(\pi, n))$ to a tensor product $G(\pi, n)$. The latter is composed of Cartan's elementary complexes corresponding to admissible sequences* [9].

Obviously, the p-local homology of $K(\pi, n)$ is the p-local homology of $G(\pi, n)$, which can easily be computed since $G(\pi, n)$ is a free DGA-algebra of finite type.

Once the p-local homology of an Eilenberg–MacLane space is already found from our point of view, we inquire for the p-local homology of a PTCP of two Eilenberg–MacLane spaces.

Actually, this homology is given by an extension of Brown's theorem.

Theorem 4.2 [1]**.** *There is a contraction c_τ from $C(X(\pi, m, \tau, \pi', n))$ to a free DG-module M of finite type that can be represented in the form*

$$(G(\pi, m) \otimes G(\pi', n), d_{G(\pi,m)} \otimes 1 + 1 \otimes d_{G(\pi',n)} + d_\tau).$$

However, we can deal with particular PTCPs of Eilenberg–MacLane spaces, namely, those that are naturally endowed with a group structure from that of their component factors. Furthermore, the twisting function τ itself is a group homomorphism.

A classical result [31] asserts that if X is a commutative simplicial group, then X is homotopically equivalent to a banal Cartesian product of Eilenberg–MacLane spaces. We assume that $X(\pi, m, \tau, \pi', n)$ is a simplicial group, considering, as inner product, the natural inner product of the Cartesian product $K(\pi, m) \times K(\pi', n)$. In light of the previous comment, it seems possible to transform the simplicial group $X(\pi, m, \tau, \pi', n)$ into a simple Cartesian product by means of some simplicial techniques. Here, we use a different approach in order to compute the p-local homology of X in this case.

Thus, we can apply an improved version of Brown's theorem concerning the algebra category via SF-APL, so that the differential operator d_τ obtained by the perturbation process is completely determined if we know its images acting on the generators of the final algebra $G(\pi, m) \bigcirc G(\pi', n)$.

Theorem 4.3 [1]. *Assume that $X(\pi, m, \tau, \pi', n)$, endowed with the natural inner product of the Cartesian product $X(\pi, m, 0, \pi', n)$, is a commutative simplicial group. Then c_τ is a semifull algebra contraction.*

Thus, the improvement in the computation of the differential is enormous in comparison with the general case. We try to develop new methods for reducing the high exponential nature of the general algorithm.

In this sense, an interesting challenge is to obtain a universal "twisting cochain" $\bar{t}$ in the sense of Prouté [34] such that there exists a contraction

$$C(\overline{W}(G) \times \tau \overline{W}(G')) \Rightarrow \overline{B}(C(G)) \bigcirc_{\bar{t}} \overline{B}(C(G')).$$

This result can be regarded as an extension of Brown's theorem in the special case where we start from a TCP of classifying spaces.

§5. The third algorithm: a new method for computing the Steenrod squares

The third example is provided by a combinatorial method for computing Steenrod squares of a simplicial set.

Different constructions of Steenrod squares. There are several well-known methods for constructing the Steenrod squares [47].

One of them is via the cohomology of the Eilenberg–MacLane spaces (see, e.g., [31], p. 107).

Another method involves a family of morphisms $\{D_i\}$ measuring the lack of commutativity of the cup product on the cochain level. This sequence of morphisms is called a *higher diagonal approximation*, and its existence is always guaranteed by the method of acyclic models [13]. In this way, we can derive a recursive procedure to obtain the formula for any D_i (see [12] and [32, Sec. 7]).

This latter method can be improved by using some simplicial objects [21]. (In this way, new explicit formulas expressing the component morphisms of a higher diagonal approximation in terms of the face operators are finally established.) This leads to a new and powerful algebro-topological tool.

We start from a higher diagonal approximation determined in [37], where the following formula for D_i is obtained in terms of the component morphisms of an Eilenberg–Zilber contraction

$$C_\star(X \times X) \Rightarrow C_\star(X) \bigcirc C_\star(X),$$

where $C_\star(X)$ denotes the normalized chain complex of a given simplicial set X:

$$D_i = AW \circ (t \circ SHI)^i \circ \Delta,$$

where the morphisms Δ and t are defined by the formulas

$$\Delta\colon C_\star(X) \to C_\star(X \times X), \quad \Delta(x) = (x, x),$$

$$t\colon C_\star(X \times X) \to C_\star(X \times X), \quad t(x_1, x_2) = (x_2, x_1).$$

On the one hand, the component morphisms of the contraction above are defined in terms of the face operators (∂_i) and the degeneracy operators (s_i) of the simplicial set X. On the other hand, the formula for any morphism D_i always involves the homotopy operator of the Eilenberg–Zilber contraction mentioned above, and the explicit formula for this morphism is determined by shuffles (some special types of permutations) of degeneracy operators. Consequently, if we directly express the morphisms D_i in terms of the face and degeneracy operators of X, then the number of summands appearing in the formulas of any morphism D_i evaluated for an element of degree n is at least 2^n, in general. Therefore, any algorithm designed to start from these formulas is too slow and actually useless for practical implementation.

In spite of this, the idea of simplifying these formulas arises in a natural way. This simplification (or normalization) is based on the fact that any composition of the face and degeneracy operators of every simplicial set can be factorized in a "canonical way" (see [31, p. 4]). More precisely, any composition of this type can be uniquely expressed in the form

$$s_{j_t} \cdots s_{j_1} \partial_{i_1} \cdots \partial_{i_s}, \tag{6}$$

where $j_t > \cdots > j_1 \geq 0$, $i_s > \cdots > i_1 \geq 0$ and all indices exceed the degree of the elements over which such a morphism is defined to act.

Moreover, since the image of a morphism D_i lies in the tensor product $C_\star(X) \otimes C_\star(X)$, we see that the summands of the simplified formula for D_i that include factors involving any degeneracy operators in their expressions must be eliminated, because they vanish at the normalized chain complex associated with X. Thus, the formulas for the morphisms D_i can be represented as follows.

Theorem 5.1. *Let $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ be the ground ring. Then, the morphism*

$$h_n = AW \circ (t \circ SHI)^n\colon\ C_m(X \times X) \to (C_\star(X) \otimes C_\star(X))_{m+n}$$

has the following form:

- *If n is odd, then*

$$AW(t\,SHI)^n = \sum_{i_n=n}^{m} \sum_{i_{n-1}=n-1}^{i_n-1} \cdots \sum_{i_0=0}^{i_1-1} \partial_{i_0+1} \cdots \partial_{i_1-1} \partial_{i_2+1} \cdots \partial_{i_{n-2}-1} \partial_{i_{n-1}+1} \cdots \partial_{i_n-1}$$
$$\otimes\, \partial_0 \cdots \partial_{i_0-1} \partial_{i_1+1} \cdots \partial_{i_{n-1}-1} \partial_{i_n+1} \cdots \partial_m.$$

- *If n is even, then*

$$AW(t\,SHI)^n = \sum_{i_n=n}^{m} \sum_{i_{n-1}=n-1}^{i_n-1} \cdots \sum_{i_0=0}^{i_1-1} \partial_{i_0+1} \cdots \partial_{i_1-1} \partial_{i_2+1} \cdots \partial_{i_{n-1}-1} \partial_{i_n+1} \cdots \partial_m$$
$$\otimes\, \partial_0 \cdots \partial_{i_0-1} \partial_{i_1+1} \cdots \partial_{i_{n-2}-1} \partial_{i_{n-1}+1} \cdots \partial_{i_n-1}.$$

Thus, the classical definition of Steenrod squares,

$$\mathrm{Sq}^i(c) = \begin{cases} (c \otimes c) D_{j-i}, & i \le j, \\ 0, & i > j, \end{cases}$$

where $c \in \mathrm{Hom}(C_j(X), \mathbb{Z}_2)$, is complemented by a manageable combinatorial definition of the higher diagonal approximation.

Theorem 5.2. *Let the ground ring be $\mathbb{Z}_2$. If $c \in \mathrm{Hom}(C_j(X), \mathbb{Z}_2)$ and $x \in C_{j+i}(X)$, then the Steenrod squares $\mathrm{Sq}^i \colon H^j(X; \mathbb{Z}_2) \to H^{j+i}(X; \mathbb{Z}_2)$ are defined by the following formulas:*

- *If $i \le j$ and $j \equiv i \pmod 2$, then*

$$\mathrm{Sq}^i(c)(x)$$
$$= \sum_{i_n=S(n)}^{m} \sum_{i_{n-1}=S(n-1)}^{i_n-1} \cdots \sum_{i_1=S(1)}^{i_2-1} c\,\partial_{i_0+1} \cdots \partial_{i_1-1} \partial_{i_2+1} \cdots \partial_{i_{n-1}-1} \partial_{i_n+1} \cdots \partial_m(x)$$
$$\otimes\, c\,\partial_0 \cdots \partial_{i_0-1} \partial_{i_1+1} \cdots \partial_{i_{n-2}-1} \partial_{i_{n-1}+1} \cdots \partial_{i_n-1}(x).$$

- *If $i \le j$ and $j+1 \equiv i \pmod 2$, then*

$$\mathrm{Sq}^i(c)(x)$$
$$= \sum_{i_n=S(n)}^{m} \sum_{i_{n-1}=S(n-1)}^{i_n-1} \cdots \sum_{i_1=S(1)}^{i_2-1} c\,\partial_{i_0+1} \cdots \partial_{i_1-1} \partial_{i_2+1} \cdots \partial_{i_{n-2}-1} \partial_{i_{n-1}+1} \cdots \partial_{i_n-1}(x)$$
$$\otimes\, c\,\partial_0 \cdots \partial_{i_0-1} \partial_{i_1+1} \cdots \partial_{i_{n-1}-1} \partial_{i_n+1} \cdots \partial_m(x).$$

- *If $i > j$, then $\mathrm{Sq}^i(c)(x) = 0$. In the formulas above, $n = j - i$, $m = j + i$, and*

$$S(k) = i_{k+1} - i_{k+2} + \cdots + (-1)^{k+n-1} i_n + (-1)^{k+n} \left\lfloor \frac{m+1}{2} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor,$$

for all $0 \le k \le n$, where $i_0 = S(0)$.

It is clear that, at least in the case where X has a finite number of nondegenerate simplices in each dimension, the explicit formulas above constitute an actual algorithm for calculating the Steenrod squares.

If we assume that the face operators are evaluated in constant time, then the computational complexity of these formulas is measured by the number of face operators involved in the de
nition of $\mathrm{Sq}^i(c)$, which is $O((i+j)^3)$.

It is necessary to discuss several facts on the computation of these formulas. First of all, if X presents a finite number of nondegenerate simplices at each degree, our method can be regarded as an actual algorithm. For instance, if the number of nondegenerate simplices in every X_ℓ is $O(\ell^2)$, then, assuming that each face operator of X is an elementary operation, we see that the complexity of our algorithm is $O((i+j)^5)$.

Nevertheless, the most interesting examples appearing in algebraic topology show, in general, a high complexity in the number of nondegenerate simplices at each degree, so that our method is only useful in low dimensions. Perhaps, appropriately combining these combinatorial formulas with classical properties of Steenrod squares (see [11, 14]) and well-known studies for calculating cocycles, we could substantially improve the method.

The general strategy in our research is as follows: we start from some algorithms in algebraic topology and homological algebra and improve them by applying refined perturbation techniques. Although the complexity of these algorithms often bears exponential costs, in many cases we are able to reduce this exponential nature by considering some particular properties.

A complementary work to be performed is the real implementation of the algorithms designed.

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