Cartan's contructions and the twisted Eilenberg-Zilber theorem [1]

V. Álvarez, J.A. Armario, M.D. Frau and P. Real

Dpto. Matematica aplicada I, Universidad de Sevilla, Spain

Dedicated to Professor Tornike Kadeishvili for his 60 anniversary

Abstract

Let $G \times_\tau G'$ be the principal twisted Cartesian product with fibre G, base G and twisting function $\tau\colon G'_* \to G_{*-1}$ where G and G' are simplicial groups as well as $G\times_\tau G'$; and $C_N(G) \otimes_t C_N(G')$ be the twisted tensor product associated to $C_N(G \times_\tau G')$ by the twisted Eilenberg-Zilber theorem. Here we prove that the pair $(C_N(G) \otimes_t C_N(G'), \mu)$ is a multiplicative Cartan's construction where μ is the standard product on $C_N(G) \otimes C_N(G')$. Furthermore, assuming that a contraction from $C_N(G')$ to HG' exists and using techniques from homological perturbation theory, we extend the former result to other "twisted" tensor products of the form $C_N(G) \otimes HG'$.

Key Words: Simplicial groups; Twisted cartesian product; Eilenberg-Zilber theorem; Cartan's construction; Contraction; Homological perturbation lemma.
MSC: 55R20; 18D99.

1 Introduction

The twisted Eilenberg-Zilber theorem [Bro67, Shi62] establishes a contraction (a special chain homotopy equivalence) from the normalized canonical chain complex $C_N(F \times_\tau B)$ of the twisted cartesian product $F \times_\tau B$ to the twisted tensor product (in the sense of [B59]) $C_N(F) \otimes_t C_N(B)$. As a module, $C_N(F) \otimes_t C_N(B)$ is the ordinary tensor product of $C_N(F)$ with $C_N(B)$; both of which are DGA-algebras, when F and B are simplicial groups. In a recent paper [AAFR07], the authors proved that if F, B and $F \times_\tau B$ are groups, then $C_N(F) \otimes_t C_N(B)$ is a DGA-algebra with respect to the module map

$$\mu\colon C_N(F) \otimes_t C_N(B) \ \otimes\ C_N(F) \otimes_t C_N(B) \to C_N(F) \otimes_t C_N(B)$$

by $\mu = (\mu_{C_N(F)} \otimes \mu_{C_N(B)})(1 \otimes T \otimes 1)$ where $\mu_{C_N(F)}$ and $\mu_{C_N(B)}$ are the products in $C_N(F)$ and $C_N(B)$ respectively, and $T(x \otimes y) = (-1)^{|x|\,|y|} y \otimes x$.

Cartan introduces in [Car56] the notion of construction as an important tool for homology computations and to the study of cohomology operations (for further details, see [Moo76]). It is well-known that if F is a group then $C_N(F) \otimes_t C_N(B)$ is a Cartan's construction. Furthermore, Proute [Pro84] proved that if G is an abelian simplicial group then $C_N(G) \otimes_t C_N(\overline{W}(G))$, associated to normalized chain complex of the universal G-bundle

[1] Address correspondence to Prof. José Andrés Armario, Departamento de Matemática Aplicada I, ETSII, Universidad de Sevilla, Avda. Reina Mercedes, S.N. 41012 Sevilla (Spain); Fax: +34-954557878; E-mail: armario@us.es

$G \times_\tau \overline{W}(G)$ (see [May67, p.88]) by the twisted Eilenberg-Zilber theorem, is a multiplicative Cartan's construction. Here we extend this result to a wider class of principal twisted cartesian product of simplicial groups (TCP). Before stating this result we recall the notion of TCP. Consider two simplicial sets F, B and a simplicial group G which operates on F from the left. A *Twisted Cartesian Product* E with fibre F, base B and structural group G consists of a simplicial set $E_n = F_n \times B_n$ and

$$\begin{array}{rcl} \partial_0(f,b) & = & (\tau b * \partial_0 f, \partial_0 b) \\ \partial_i(f,b) & = & (\partial_i f, \partial_i b), \quad \text{for } \ i > 0 \\ s_i(f,b) & = & (s_i f, s_i b), \quad \text{for } \ i \geq 0; \end{array}$$

as face and degeneracy operators. Here $*\colon G \times F \to F$ is the action of G on F and τ is a *twisting function*, i.e., $\tau_n\colon B_n \to G_{n-1},\ n \geq 1$ satisfies

$$\begin{array}{rcl} \partial_0\tau(b) & = & [\tau(\partial_0 b)]^{-1} \cdot \tau(\partial_1 b) \\ \partial_i\tau(b) & = & \tau(\partial_{i+1} b), \quad \text{for } \ i > 0 \\ s_i\tau(b) & = & \tau(s_{i+1} b), \quad \text{for } i \geq 0 \\ \tau(s_0 b) & = & e_n, \end{array}$$

where e_n denotes the identity element of the corresponding group G_n. We write $E = F \times_\tau B$. If $F = G$ then we say that this PCT is principal. Here are our main results.

Theorem 1.1. *Let F and B be simplicial groups and $\tau\colon B \to F$ be a twisting function, such that, the principal twisted Cartesian product (PTCP), $F \times_\tau B$, with fibre F and base B is a simplicial group. Then, the pair*

$$(C_N(F) \otimes_t C_N(B),\ \mu)$$

associated to $C_N(F \times_\tau B)$ by the twisted Eilenberg-Zilber theorem, is a multiplicative Cartan's construction.

If we also suppose that F is reduced and there exist a contraction c from $C_N(B)$ to a DGA-module HB,

$$c\colon {}^{\phi}\!\!\circlearrowright C_N(B) \underset{g}{\overset{f}{\rightleftarrows}} H(B).$$

Then, using, the techniques of homological perturbation theory, it is able to construct (see [LS87, AAFR09]) a contraction

$${}^{\phi}\!\!\circlearrowright C_N(F \times_\tau B) \underset{g}{\overset{f}{\rightleftarrows}} (C_N(F) \otimes HB,\ D)$$

where D denotes the differential of the complex on the left and has the form $d \otimes 1 + 1 \otimes d + d_{t\cap}$ ("terms of higher order").

In the case that HB is small enough so that the computation of its homology can actually be carried out, we say that the pair (c, HB) is a homological model for B. Let us observe that, in this situation, $(C_N(F) \otimes HB,\ D)$ is similar to the dual of Hirsch complex [Hir53].

Theorem 1.2. *Under the hypotheses of the theorem 1.1 and assuming that F is reduced, HB is a DGA-algebra and c is a semi-full algebra contraction (a notion of contraction between algebras weaker than algebra contraction). It is able to state that*

$$C_N(F) \otimes H(B),\, D$$

is a multiplicative Cartan's construction as well.

2 The proof of the theorem 1.1

Firstly, we will quickly review basic notions of Homological Algebra, and introduce the notation and terminology that we use throughout the remainder of this article. More details can be found in [McL95]. Let Λ be a commutative ring with non zero unit, taken henceforth as ground ring and fixed throughout, and A be an augmented differential graded algebra over Λ, briefly a DGA-algebra. The differential, product, augmentation and coaugmentation of A will be denoted respectively by d_A, μ_A, ϵ_A and η_A. Nevertheless, we will sometimes write them simply as d, μ, ϵ and η when no confusion can arise. In what follows, the Koszul sign conventions will be used. A morphism $\rho : A_* \to A_{*-1}$ is called *derivation* if it is compatible with the algebra structures on A. The degree of an element $a \in A$ is denoted by $|a|$. In addition, we recall that if B is also a DGA-algebra, then $A \otimes B$ has canonically associated an algebra structure by means of the morphism $\mu_{A\otimes B} = (\mu_A \otimes \mu_B)(1_A \otimes T \otimes 1_B)$, where $T(b \otimes a) = (-1)^{|b|\,|a|} a \otimes b$. If the DG-algebra A is connected, that is $A_0 = \Lambda$ and $d_1 : A_1 \to A_0$ is zero, then there is a canonical augmentation $\epsilon_A = 1_\Lambda : A_0 \to \Lambda$. If A is a DG-algebra,then $A^\sharp$ will denote the graded algebra obtained from A by setting the differential of A equal to zero (i.e. forgetting the differential), and if M is an A-module, then $M^\sharp$ will denote $A^\sharp$-module obtained by setting the differential equal to zero.

We will use here the *twisted tensor product* structure. Let A be a DG-algebra and C be a DG-coalgebra (we denote by Δ_C its coproduct). A *twisting cochain* is a morphism of graded modules $t\colon C_* \to A_{*-1}$ such that

$$d_A t + t d_C + t \cup t = 0, \quad \epsilon_A t = 0, \quad t\eta_C = 0;$$

where $t \cup t = -\mu_A(t \otimes t)\Delta_C$. It is well-known that $d^t = d_A \otimes 1 + 1 \otimes d_C + t\cap$ is a differential on $A \otimes C$, where the morphism $t\cap$ is defined by:

$$t\cap \;=\; (\mu_A \otimes 1)(1 \otimes t \otimes 1)(1 \otimes \Delta_C). \tag{1}$$

The DG-module $(A \otimes C, d^t)$ is called *the twisted tensor product (or TTP) of A and C along t*. We will also use the notation $A \otimes_t C$ for such a DG-module.

A construction is a triple (A, N, M) where

1. A is a DGA-algebra.

2. M is an augmented A-module.

3. N is a DGA-module such that $N = \overline{M} = \Lambda \otimes_A M = M/I(A)M$ where $I(A)$ is the augmentation ideal of A.

satisfying that $M^\sharp = A^\sharp \otimes N^\sharp$.

Example 2.1. *The twisted tensor product $A \otimes_t C$ gives rise to a construction $(A, C, A \otimes_t C)$.*

A *multiplicative construction* is a construction (A, N, M) together with the structure of an algebra on M and also on N such that $A^\sharp \otimes N^\sharp \to M^\sharp$ is an isomorphism of algebras.

Hence, the proof of Theorem 1.1 follows at once from the fact that the morphism

$$\mu = (\mu_{C_N(F)} \otimes \mu_{C_N(B)})(1 \otimes T \otimes 1)$$

endows to $C_N(F) \otimes_t C_N(B)$ of a DGA-algebra structure (see [AAFR07, Theorem 3.9.]).

□

3 The proof of the theorem 1.2

We assume throughout this section that M and N denote two DGA-modules such that a contraction from N to M exists.

We recall that a *contraction* (see [EM53], [HK91]) is a data set $c : \{N, M, f, g, \phi\}$ where $f : N \to M$ and $g : M \to N$ are morphisms of DGA-modules (respectively, called *the projection* and *the inclusion*) and $\phi : N \to N$ is a morphism of graded modules of degree $+1$ (called *the homotopy operator*). These data are required to satisfy the rules: **(c1)** $fg = 1_M$, **(c2)** $\phi d_N + d_N \phi + gf = 1_N$ **(c3)** $\phi\phi = 0$, **(c4)** $\phi g = 0$ and **(c5)** $f\phi = 0$. The last three are called the side conditions [LS87]. In fact, these may always be assumed to hold, since the homotopy ϕ can be altered to satisfy these conditions [GL89]. These formulas imply that both chain complexes N and M have the same homology. We will also denote a contraction c by ${}^{\phi:} N \underset{g}{\overset{f}{\rightleftharpoons}} M$. The Eilenberg-Zilber theorem [EZ53] provides the most classic example of a contraction of chain complexes.

Now we add an additional structure: N is a DGA-A-module with product $\mu_N \colon A \otimes N \to N$. No such assumption is made on M (M is a DGA-module) but the question will arise if (M, μ_M) where $\mu_M = \phi\mu_N(1 \otimes g) \colon A \otimes M \to M$ becomes a DGA-A-module. Under the hypothesis that $\phi\mu_N(1 \otimes g) = 0$ we give an affirmative answer. Moreover, the injection g is A-lineal. The proof of this result is a simple inspection.

Now, we recall the concept of a perturbation datum. let $f : N \to N$ be a morphism of graded modules. The morphism f is *pointwise nilpotent* if for all $x \in N$ ($x \neq 0$), a positive integer n exists (in general, the number n depends on the element x) such that $f^n(x) = 0$. A *perturbation of a DGA-module N* is a morphism of graded modules $\delta : N \to N$ of degree -1, such that $(d_N + \delta)^2 = 0$ and $\epsilon_A \delta_1 = 0$. A *perturbation datum of the contraction* $c : \{N, M, f, g, \phi\}$ is a perturbation δ of the DGA-module N verifying that the composition $\phi\delta$ is pointwise nilpotent.

A *Transference Problem* consists of a contraction $c\colon \{M,\ N,\ f,\ g,\ \phi\}$ together with a perturbation δ of the DGA-module N. The problem is to determine new morphisms d_δ, f_δ, g_δ and ϕ_δ such that $c_\delta : \{(N, d_N + \delta),\ (M, d_M + d_\delta),\ \ f_\delta,\ g_\delta,\ \phi_\delta\}$ is a contraction.

The Basic Perturbation Lemma ([Bro67, GL89, GLS91, Rea00]) gives an explicit solution to the Transference Problem, assuming that δ is a perturbation datum of c.

Theorem 3.1. *(***BPL***)*

Let $c : \{N, M, f, g, \phi\}$ be a contraction and $\delta : N \to N$ a perturbation datum of c. Then, a new contraction

$$c_\delta : \{(N, d_N + \delta),\ (M, d_M + d_\delta),\ \ f_\delta,\ g_\delta,\ \phi_\delta\}$$

is defined by the formulas: $d_\delta = f\delta\Sigma_c^\delta g$; $f_\delta = f(1 - \delta\Sigma_c^\delta \phi)$; $g_\delta = \Sigma_c^\delta g$; $\phi_\delta = \Sigma_c^\delta \phi$; where

$$\Sigma_c^\delta \ = \ \sum_{i\geq 0}(-1)^i\ (\phi\delta)^i \ = \ 1 - \phi\delta + \phi\delta\phi\delta - \cdots + (-1)^i(\phi\delta)^i + \cdots .$$

Let us note that $\Sigma_c^\delta(x)$ is a finite sum for each $x \in N$, because of the pointwise nilpotency of the composition $\phi\delta$. Moreover, it is obvious that the morphism d_δ is a perturbation of the DG-module (M, d_M).

The twisted Eilenberg-Zilber theorem can be seen as an important example of the usefulness of this lemma (see [Shi62]). It solves the Transference Problem for twisted cartesian products.

In the theorem below we assume that N is a DGA-A-module. This theorem gives conditions under which the BPL works preserving the DGA-A-module category.

Theorem 3.2. *Let $\delta\colon N \to N$ be a perturbation datum of ${}^{\phi\colon}N \underset{g}{\overset{f}{\rightleftharpoons}} M$ such that δ is compatible with the A-module structure on N (i.e., $\mu_N(1 \otimes \delta) = \delta\mu_N$). If*

$$\phi\mu_N(1 \otimes g) = 0 \quad \text{and} \quad \phi\mu_N(1 \otimes \phi) = 0,$$

then the DGA-module $M_\delta = (M, d{+}d_\delta)$, obtained by applying BPL, is a DGA-A-module with regards to the module map $\mu_{M_\delta} = f\mu_N(1 \otimes g)$ and the injection of the perturbed contraction, g_δ, is A-lineal.

Proof. This is again seen by inspection.

□

Let us recall that the DGA-module $A \otimes N$ has a trivial structure of A-module with regards to the module map

$$\begin{array}{rccc} \mu_{A\otimes N}\colon & A \otimes (A \otimes N) & \to & A \otimes N \\ & a_1 \otimes (a_2 \otimes n) & \to & \mu_A(a_1 \otimes a_2) \otimes n. \end{array}$$

From ${}^{\phi\colon}N \underset{g}{\overset{f}{\rightleftharpoons}} M$, it is well-known that we can establish this new contraction

$${}^{1\otimes\phi\colon}A \otimes N \underset{(1\otimes g)}{\overset{(1\otimes f)}{\rightleftharpoons}} A \otimes M. \qquad (2)$$

It may be readily verified that the following identities hold:

$$(1\otimes\phi)\mu_{A\otimes N}(1\otimes(1\otimes g))=0,\quad (1\otimes\phi)\mu_{A\otimes N}(1\otimes(1\otimes\phi))=0.$$

$$\mu_{A\otimes M}=(1\otimes f)\mu_{A\otimes N}(1\otimes(1\otimes g)).$$

With these identities at hand, we can write the following consequence of Theorem 3.2.

Corollary 3.3. *If $\delta\colon A\otimes N\to A\otimes N$ is a perturbation datum for (2) such that it is compatible with the A-module structure on $A\otimes N$, then the DGA-module $(A\otimes M)_\delta=(A\otimes M, d+d_\delta)$, obtained by applying BPL, is a DGA-A-module with regards to the module map $\mu_{A\otimes M}$ and the injection of the perturbed contraction, $(1\otimes g)_\delta$, is A-lineal.*

In the sequel, we focus on the case $A=C_N(F)$ and $N=C_N(B)$ where F and B are simplicial groups. If we assume that a contraction, c,

$$^{\phi}{:}C_N(B)\underset{g}{\overset{f}{\rightleftharpoons}}H(B) \tag{3}$$

exists, then we can establish the following contraction

$$C_N(F)\otimes C_N(B)\rightleftharpoons C_N(F)\otimes HB. \tag{4}$$

Let $\tau\colon B\to F$ be a twisting function, such that, the principal twisted Cartesian product (PTCP), $F\times_\tau B$, with fibre F and base B is a simplicial group. The complex $C_N(F)\otimes_t C_N(B)$ denotes the TTP associated to $C_N(F\times_\tau B)$ by the twisted Eilenberg-Zilber theorem.

Lemma 3.4. *The morphism $t\cap=(\mu_{C_N(F)}\otimes 1)(1\otimes t\otimes 1)(1\otimes\Delta_{C_N(B)})$ (see (1)) satisfies the following properties:*

1. *If F is reduced, then $t\cap$ is a perturbation datum of the contraction (4).*
2. *$t\cap$ is compatible with the $C_N(F)$-module structure on $C_N(F)\otimes_t C_N(B)$ (i.e., $\mu_{F\otimes B}(1\otimes t\cap)=t\cap\mu_{F\otimes B}$).*

Proof.

1. See [LS87, Lemma 3.4] or [AAFR09, proposition 5.3].
2. This is again seen by inspection.

□

Hence, if F is reduced, we can perturb the contraction (4) using $t\cap$ as a perturbation datum. With these inputs, the BPL gives as output the following contraction:

$$C_N(F)\otimes_t C_N(B)\rightleftharpoons(C_N(F)\otimes HB,\, D). \tag{5}$$

where D denotes its differential.

Now, we can state:

Proposition 3.5. *If F is reduced, then $(C_N(F) \otimes HB,\, D)$ becomes a DGA-$C_N(F)$-module with regards to the trivial module structure on $C_N(F) \otimes HB$.*

Proof.
The proof follows from Corollary 3.3 and lemma 3.4.

□

Focusing only on the underlying graded module structure. We have the following identity:

$$I(C_N(F))(C_N(F) \otimes HB) = I(C_N(F)) \otimes HB.$$

It may be readily verified that the following properties hold:

1. $t \cap (I(C_N(F)) \otimes C_N(B)) \subseteq I(C_N(F)) \otimes C_N(B)$.
2. The correspondent restrictions of the morphisms composing the contraction (5) form the following contraction:

$$I(C_N(F)) \otimes C_N(B) \rightleftharpoons I(C_N(F)) \otimes HB.$$

The formula for the differential of the complex $(C_N(F) \otimes HB,\, D)$, given by the BPL, is $D = d \otimes 1 + 1 \otimes d + d_{t\cap}$ where

$$d_{t\cap} = (1 \otimes f)t \cap (1 \otimes g) - (1 \otimes f)t \cap (1 \otimes \phi)t \cap (1 \otimes g) + \cdots \tag{6}$$

As immediate consequence of this formula and the properties above, we have

$$D(I(C_N(F)) \otimes HB) \subseteq I(C_N(F)) \otimes HB.$$

Hence, $(I(C_N(F)) \otimes HB, D)$ is a DGA-$C_N(F)$-submodule of $(I(C_N(F)) \otimes HB, D)$.

Proposition 3.6. *We have the following identity of DGA-modules*

$$HB = (C_N(F) \otimes HB, D)/(I(C_N(F))(C_N(F) \otimes HB, D))$$

Proof.
On the one hand, it is easy to see that this two complexes are isomorphic as graded module.

On the other hand, taking into account that t vanishes on one-simplices since F is reduced (see [May67]) and the formula (6), we have

$$d_{t\cap}(f \otimes b) = \begin{cases} 0 & |b| \leq 1 \\ 0 \mod I(C_N(F)) \otimes HB & |b| > 1 \end{cases}$$

Hence, we have that the differential of both complexes are the same. □

Proposition 3.7. *Under the hypotheses of the Preposition 3.5 and 3.6. We can state that $(C_N(F) \otimes HB,\, D)$ is a construction.*

Finally, If we assume that the contraction (3) is semi-full, we will get the desired result. Now, we recall that c is a semi-full algebra contraction ([Rea00]) if the injection, g, is a morphism of DGA-algebras and the projection, f, and the homotopy operator, ϕ, satisfy the following properties:

$$f\mu_{C(B)}(\phi\otimes\phi)=0,\quad f\mu_{C(B)}(\phi\otimes g)=0,\quad f\mu_{C(B)}(g\otimes\phi)=0,$$

$$\phi\mu_{C(B)}(\phi\otimes\phi)=0,\quad \phi\mu_{C(B)}(\phi\otimes g)=0,\quad \phi\mu_{C(B)}(\phi\otimes\phi)=0.$$

Under these conditions we can say that these contractions (4) and (5) are semi-full (see [Rea00, Theorem 4.18]). Moreover, $(C_N(F)\otimes HB,\, D)$ is a DGA-algebra with regards to the standard product $C_N(F)\otimes HB$. From this fact, we conclude that $(C_N(F)\otimes HB,\, D)$ is a multiplicative construction.

□

4 Conclusions

In this paper we have faced the problem of transferring the multiplicative construction structure up to contraction. The problem of transferring other structures (e.g. algebra, coalgebra, TTP) up to contraction has been widely treated in the literature [GLS91, Rea00, AAFR05]. In general, we have that (co)algebra and TTP become A_∞-(co)algebra and A_∞-TTP via contraction, respectively.

We proved that the property of being a multiplicative construction on $C_N(F)\otimes_t C_N(B)$ has been transferred to $(C_N(F)\otimes HB,\, D)$ via contraction. However, if we focus on the structure of TTP, we only have that $(C_N(F)\otimes HB,\, D)$ is an A_∞-twisted tensor product of $C_N(F)$ (algebra) and HB (A_∞-coalgebra) along $\bar{t}=tg$ (A_∞-twisted cochain), see [AAFR05, Theorem 2.1].

Acknowledgment

We would like to thank Kristeen Cheng for her reading of this paper. This work was partially supported by the PAICYT research project FQM–296 from Junta de Andalucía (Spain).

References

[AAFR05] Álvarez, V., Armario, J.A., Frau, M.D., Real, P. (2005). Transferring TTP-structures via contraction. *Homology, Homotopy and Applications* 7(2):41–54.

[AAFR07] Álvarez, V., Armario, J.A., Frau, M.D., Real, P. (2007). Algebra structures on the twisted Eilenbeg-Zilber theorem. *Comm. Alg.* 35:3273–3291.

[AAFR09] Álvarez, V., Armario, J.A., Frau, M.D., Real, P. Algebra structures on the comparison of the reduced bar construction and the reduced W-construction. *To appear in Comm. Alg.*

[B59] Brown, E. H. (1959). Twisted tensor products. *Ann. Math* 69:223–242.

[Bro67] Brown, R. (1967). The twisted Eilenberg-Zilber theorem. *Celebrazioni Archimedee del secolo XX. Simposio di topologia* 34–37.

[Car56] Cartan, H. (1956). Algèbres d'Eilenberg-Mac Lane. *Séminaire H. Cartan 1954/55*, (exposé 2 à 11). Ecole Normale Superière, Paris.

[EM53] Eilenberg, S., Mac Lane, S. (1953). On the groups $H(\pi, n)$, I. *Annals of Math.* 58:55-106.

[EZ53] Eilenberg, S., Zilber, J.A. (1953). On products of complexes. *Am. J. Math.* 75:200-204.

[GL89] Gugenheim, V.K.A.M., Lambe, L.A. (1989). Perturbation theory in Differential Homological Algebra I, *Illinois J. Math.* 33 (4):566–582.

[GLS91] Gugenheim, V.K.A.M., Lambe, L.A., Stasheff, J.D. (1991). Perturbation theory in Differential Homological Algebra II, *Illinois J. Math.* 35 (3):357–373.

[Hir53] Hirsch, G. (1953). Sur les groups d'homologies des espaces fibres. *Bull. Soc. Math. de Belg.* 6:79–96.

[HK91] Huebschmann, J., Kadeishvili, T. (1991). Small models for chain algebras. *Math. Z.,* 209:245–280.

[LS87] Lambe, L.A., Stasheff, J.D. (1987). Applications of perturbation theory to iterated fibrations. *Manuscripta Math.,* 58:367–376.

[McL95] Mac Lane, S. (1995). *Homology.* Classics in Mathematics Springer-Verlag, Berlin. Reprint of the 1975 edition.

[May67] May, J.P. (1967). *Simplicial Objects in Algebraic Topology*. Princenton: Van Nostrad.

[Moo76] Moore, J.C. (1976). Cartan's constructions. *Asterisque 32-33*, 173-212.

[Pro84] Prouté, A. (1984). *Algebrès diffeérentielles fortement homotopiquement associatives (A_∞-algèbre).* Ph. D. Thesis, Université Paris VII.

[Rea00] Real, P. (2000). Homological perturbation theory and associativity. *Homology, Homotopy and Applications* 2:51–88.

[Shi62] Shih, W. (1962). Homologie des espaces fibrés. *Inst. Hautes Etudes Sci.* 13: 93-176.