The maximal determinant of cocyclic $(-1, 1)$-Matrices over D_{2t}

V. Álvarez , J.A. Armario * , M.D. Frau , F. Gudiel

Department of Applied Math I, University of Seville, Avda. Reina Mercedes s/n, 41012 Seville, Spain

A B S T R A C T

Cocyclic construction has been successfully used for Hadamard matrices of order n. These $(-1, 1)$-matrices satisfy that $HH^T = H^T H = nI$ and give the solution to the maximal determinant problem when $n = 1, 2$ or a multiple of 4. In this paper, we approach the maximal determinant problem using cocyclic matrices when $n \equiv 2 \pmod 4$. More concretely, we give a reformulation of the criterion to decide whether or not the $2t \times 2t$ determinant with entries ± 1 attains the Ehlich–Wojtas' bound in the D_{2t}-cocyclic framework. We also provide some algorithms for constructing D_{2t}-cocyclic matrices with large determinants and some explicit calculations up to $t=19$.

AMS classification:
05B20
05B30
15A15

Keywords:
Cocyclic matrices
Maximal determinants
D-optimal designs
Gram matrices

1. Motivation of the problem – introduction

A *D-optimal design of order n* is a $n \times n$ $(1, -1)$-matrix having maximal determinant. Here and throughout this paper, for convenience, when we say determinant of a matrix we mean the absolute value of the determinant. The question of finding the determinant of a D-optimal design of order n is an old one which remains unanswered in general.

In 1893 Hadamard proved in [15] that for every $(-1, 1)$-matrix M,

$$det(M) \leqslant n^{\frac{n}{2}}. \qquad (1)$$

Furthermore, Hadamard proved that equality holds if and only if $MM^T = nI$. Matrices satisfying this condition are termed *Hadamard matrices*, and must have order 1, 2 or a multiple of 4. It is conjectured

* Corresponding author.
E-mail addresses: valvarez@us.es (V. Álvarez), armario@us.es (J.A. Armario), mdfrau@us.es (M.D. Frau), gudiel@us.es (F. Gudiel).

that Hadamard matrices exist for every $n \equiv 0 \pmod 4$. Although no proof of this fact is known, there is much evidence about its validity (see [19] and the references there cited).

Tighter bounds for the maximal determinant for all $(-1, 1)$-matrices of order $n \not\equiv 0 \pmod 4$ are known (see [5,11,12,35,23], for instance). For $n \equiv 1 \pmod 4$, Ehlich proved in [11] that

$$det(M) \leqslant (2n-1)^{\frac{1}{2}}(n-1)^{\frac{n-1}{2}}. \tag{2}$$

Moreover, equality holds if and only if there exists a $(-1, 1)$-matrix M of order n such that $MM^T = (n-1)I_n + J_n$ (see [5]). Here, as usual, I_n denotes the identity matrix of order n, and J_n denotes the $n \times n$ matrix all of whose entries entries are equal to one. If equality holds, $2n-1$ is a perfect square $(2k+1)^2$ (or equivalently, n is the summation of two consecutive squares, $n = k^2 + (k+1)^2$). It has been conjectured that a matrix attaining the bound exists whenever this is the case. However, order $85 = 6^2 + 7^2$ is the smallest for which this has not been proven.

For $n \equiv 2 \pmod 4$, Ehlich in [11] and independently Wojtas in [35] proved that

$$det(M) \leqslant (2n-2)(n-2)^{\frac{n-2}{2}}. \tag{3}$$

In order for equality to hold, it is required that there exists a $(-1, 1)$-matrix M of order n such that $MM^T = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}$, where $L = (n-2)I_{\frac{n}{2}} + 2J_{\frac{n}{2}}$. In these circumstances, it may be proved that, in addition, $2n-2$ is the sum of two squares, a condition which is believed to be sufficient (order 138 is the lowest for which the question has not been settled yet [13]). To be more precise, Ehlich proved in [11] that $2n-2 = (\frac{n}{2}-2r)^2 + (\frac{n}{2}-2s)^2$, where r (resp. s) is the number of rows in M from 1 to $\frac{n}{2}$ (resp. $\frac{n}{2}+1$ to n) for which the first entry is positive. Alternatively, Cohn proved in [7] that M can be chosen of the type $M = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$, so that $L = XX^T + YY^T = ZZ^T + WW^T = X^TX + Z^TZ = Y^TY + W^TW$, $0 = XZ^T + YW^T = X^TY + Z^TW$, and $2n-2 = x^2 + y^2$, for $x \geqslant y \geqslant 0$, where every row sum and column sum of each of X and W is x, each row sum and each column sum of Y is y and each row sum and column sum of Z is $-y$.

The case $n \equiv 3 \pmod 4$ appears to be the most difficult one. In spite of the fact that the bound (2) also holds for these matrices, Ehlich derived a tighter one in [12],

$$(n-3)^{\frac{n-s}{2}}(n-3+4r)^{\frac{u}{2}}(n+1+4r)^{\frac{v}{2}}\sqrt{1 - \frac{ur}{n-3+4r} - \frac{v(r+1)}{n+1+4r}}, \tag{4}$$

where $s = 3$ for $n = 3$, $s = 5$ for $n = 7$, $s = 5$ or 6 for $n = 11$, $s = 6$ for $n = 15, 19, \ldots, 59$, and $s = 7$ for $n \geqslant 63$, $r = \lfloor \frac{n}{s} \rfloor$, $n = rs + v$ and $u = s - v$. Cohn showed in [9] that this number is an integer only when $n = 112t^2 \pm 28t + 7$ for some integer t. Nevertheless, many orders allowed by Cohn's criterion are ruled out by the Hasse–Minkowski theorem on rational equivalence of quadratic forms (see [34]). In particular, Ehlich's bound is not achievable for order 91. The smallest order for which it is potencially attainable is 511.

It is well known that the Hadamard bound (1) is attained infinitely often, and has to be considered sharp in this sense. In [23] this question was studied for the remaining bounds, (2), (3) and (4), and some lower bounds were described which were attained infinitely often. Today we know that the bounds (1), (2) and (3) are sharp, in the above sense. Nevertheless, it is not known whether the bound (4) is sharp in the same sense, or even if it is achievable beyond $n = 3$. It is conceivable that it is not sharp.

When a $n \times n$ determinant is found that attains the relevant one of the above bounds, it is immediate that the maximal determinant for that order is just the bound itself. Nevertheless when the upper bound is not attained, finding the maximal $n \times n$ determinant can be exceedingly difficult. For $n \leqslant 30$, orders 19, 22, 23, 27 and 29 are unresolved. The interested reader is addressed to [20] and the website [30] for further information on what is known about maximal determinants.

Table 1
Proportion of inequivalent Hadamard matrices (cocyclic/general framework).

n	4	8	12	16	20	24	28	32	36
$\#[CH]$	1	1	1	5	3	16	6	100	35
$\#[H]$	1	1	1	5	3	60	487	$\geqslant 13.7 \times 10^6$	$\geqslant 3 \cdot 10^6$
$\frac{\#[CH]}{\#[H]}$	1	1	1	1	1	$2.67 \cdot 10^{-1}$	$1.23 \cdot 10^{-2}$	$\leqslant 7.29 \cdot 10^{-6}$	$\leqslant 1.16 \cdot 10^{-5}$

Traditionally, matrices meeting the bound (1) are classified attending to Hadamard equivalence, so that two Hadamard matrices are equivalent if and only if one can be converted into the other by a sequence of permutations of rows and columns, and negations of rows and columns. This classification problem translates naturally to the case of the remaining bounds. The classification of $(-1, 1)$-matrices achieving the maximal determinant remains as an unanswered question in general. What is known (see [19,29,21] for details), is that there is only one equivalence class of D-optimal designs for each of the orders up to $n = 15$, except for $n = 11$. And there are 3 equivalence classes for $n = 11$, 5 for $n = 16$, 3 for $n = 17, 18, 20$, 7 for $n = 21$, 60 for $n = 24$, 78 for $n = 25$ and 487 for $n = 28$. For updates on the lower bounds for the number of equivalence classes for other orders, visit these websites [22,30].

In the early 90s, a surprising link between homological algebra and Hadamard matrices [17] led to the study of cocyclic Hadamard matrices [18]. As was introduced before, a Hadamard matrix of order $4t$ is a $(-1, 1)$ square $4t \times 4t$ matrix such that its distinct row (resp. column) vectors are pairwise orthogonal. A Hadamard matrix is said to be *normalized* if it has its first row and column all of 1's (see [19] for more details and constructions methods).

Hadamard matrices of many types are revealed to be (equivalent to) cocyclic matrices [10,19]. Among them, Sylvester Hadamard matrices, Williamson Hadamard matrices, Ito Hadamard matrices and Paley Hadamard matrices. Furthermore, the cocyclic construction is the most uniform construction technique for Hadamard matrices currently known, and cocyclic Hadamard matrices may consequently provide a uniform approach to the famous Hadamard conjecture.

The main advantages of the cocyclic framework concerning Hadamard matrices may be summarized in the following facts:

- The test to decide whether a cocyclic matrix is Hadamard runs in $O(t^2)$ time, better than the $O(t^3)$ algorithm for usual (not necessarily cocyclic) matrices.
- The search space is reduced to the set of cocyclic matrices over a given group (that is, 2^s matrices, provided that a basis for cocycles over G consists of s generators), instead of the whole set of 2^{16t^2} matrices of order $4t$ with entries in $\{-1, 1\}$.

Now an interesting question arises, is it better to look for Hadamard matrices in the general framework or in the cocyclic context instead?

A recent work of Ó Catháin and Röder (see [27] for details) has permitted the calculation of the exact number $\#[CH]$ of inequivalent cocyclic Hadamard matrices, for orders less than 40. This way, a comparison in terms of the total number $\#[H]$ of inequivalent Hadamard matrices is feasible, up to order 36 (see Table 1). Here we have taken into account the work of Kharaghani and Tayfeh-Rezaie in [21], about the number of equivalence classes of Hadamard matrices of order 32.

Notice that a cocyclic Hadamard matrix may be Hadamard equivalent to a matrix which is not cocyclic at all. The cocyclic character is not preserved by Hadamard equivalence, in general.

From Table 1, it seems that whereas t increases the quotient $\frac{\#[CH]}{\#[H]}$ between the number of inequivalent cocyclic Hadamard matrices and the number of inequivalent Hadamard matrices decreases drastically. Nevertheless, this comparison is somehow biassed, since the set of Hadamard matrices is not uniformly distributed among the equivalence classes. If we attend to the summation of the number of Hadamard matrices equivalent to a matrix of $[H]$ and $[CH]$ in Table 1, denoted by $\#H$ and $\#CH$ respectively (the required information may be extracted from [27] and [32]), we obtain Table 2.

Notice that the number $\#H$ of Hadamard matrices of a given order, and the number $\#[H]$ of equivalence classes in which they distribute, are linked by the notion of *mass*. The mass of Hadamard matrices of a given order is defined to be the sum of the reciprocals of the sizes of the automorphism groups over

Table 2
Proportion of Hadamard matrices belonging to equivalence classes $[CH]$ and $[H]$.

n	4	8	12	16	20	24	28
$\#CH$	192	21,504	190,080	10,838,016	16,440	790,224	64,488
$\#H$	192	21,504	190,080	10,838,016	16,440	823,616	74,306
$\frac{\#CH}{\#H}$	1	1	1	1	1	0.9594	0.8678

Table 3
Density of D_{4t}-Hadamard matrices versus that of usual Hadamard matrices.

n	4	8	12	16	20	24	28
$\%H(D_{4t})$	$3.75 \cdot 10^{-1}$	$1.25 \cdot 10^{-1}$	$1.76 \cdot 10^{-2}$	$1.17 \cdot 10^{-2}$	$2.1 \cdot 10^{-3}$	$4.46 \cdot 10^{-4}$	$4.23 \cdot 10^{-5}$
$\%H$	$2.99 \cdot 10^{-3}$	$1.17 \cdot 10^{-15}$	$8.52 \cdot 10^{-39}$	$9.36 \cdot 10^{-71}$	$6.37 \cdot 10^{-117}$	$3.33 \cdot 10^{-168}$	$7.30 \cdot 10^{-232}$

the equivalence classes of Hadamard matrices of this order. This gives another measure of how many distinct Hadamard matrices there are, without regard to equivalence. See [33, A048615,A048616] for details.

In fact, searching for Hadamard matrices, no matter the context (cocyclic or general), is computationally a very hard task, as difficult as looking for a needle in a haystack. Nevertheless, one should compare the sizes of the needle and the haystack to get an objective impression about the difficulty of finding such a needle in such a haystack. Thus what can be said about the proportion of Hadamard matrices in the general framework and in the cocyclic context? Unfortunately, we have no information about the total number of cocyclic matrices, even for small values of t. The work in [27] could shed light on this problem.

Anyway, we can compare the framework of the usual Hadamard matrices with a concrete family of cocyclic matrices. Among them, the most prolific case seems to be dihedral groups D_{4t} (see [19,3] for instance). Since a basis for normalized cocycles over D_{4t} consists of $4t$ cocycles (see [3], for instance), then a full basis for cocycles over D_{4t} consists of $4t + 1$ elements, and hence the size of the search space for D_{4t}-cocyclic Hadamard matrices is 2^{4t+1}. The search space for the usual Hadamard matrices is the complete set of $(-1, 1)$-matrices square matrices of order $4t$, which consists of 2^{16t^2} matrices. The number of D_{4t}-cocyclic Hadamard matrices for small values of t may be calculated progressing from the work in [2]. Now we can compare the density $\%H(D_{4t})$ of D_{4t}-Hadamard matrices among D_{4t}-cocyclic matrices, and the density $\%H$ of usual Hadamard matrices among the set of $(-1, 1)$-matrices of order $4t$.

Undoubtedly, the information in Table 3 is once again biassed, since we should be comparing with the full set of cocyclic matrices of order $4t$. Anyway, there is some evidence that searching for cocyclic Hadamard matrices, and in particular for D_{4t}-cocyclic Hadamard matrices, makes sense.

Despite the fact that cocyclic construction provides a successful approach for Hadamard matrices, and hence for $(-1, 1)$-matrices meeting the bound (1), as far as the authors know this technique has not yet been used to tackle the maximal determinant problem when $n \not\equiv 0 \pmod 4$.

The main purpose of this paper is to show that the cocyclic technique can certainly be extended to handle the maximal determinant problem at least when $n \equiv 2 \pmod 4$. More concretely, we will focus on cocyclic matrices over the dihedral group D_{2t}, with t odd, so that we give:

- A reformulation of the criterion to decide whether or not a D_{2t}-cocyclic matrix has a determinant attaining Ehlich–Wojtas' bound.
- Some algorithms for constructing D_{2t}-cocyclic matrices with large determinants, based on exhaustive and heuristic searches. Unfortunately, although the largest determinants obtained by these methods so far (up to $n = 2t = 38$) meet the optimal bound (3) when $n - 1$ is the sum of two squares, no D_{2t}-cocyclic matrix has been found neither meeting nor improving the already known lower bounds when $n = 22, 34$.

Apart from this introductory section, we organize the paper as follows. The second section is devoted to explain the theoretical results about how to determine D_{2t}-matrices meeting Ehlich and Wojtas' bound (3). The algorithms and some executions are described in the third section. The last section is devoted to conclusions and future work.

2. Main results

From now on, we assume that $n \equiv 2 \pmod 4$. When necessary, we will use $n = 2t$, for some odd integer $t \geqslant 1$.

Our goal in this section is to characterize the form of D_{2t}-cocyclic matrices which might meet Ehlich and Wojtas' bound (3). The first part of the section is devoted to introduce some notations and technical results. Afterwards, the main statements of the paper are described and proved.

As introduced in Section 1, equality in (3) holds if and only if there exists a $(1, -1)$-matrix B of order n, such that

$$BB^T = B^TB = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}, \tag{5}$$

with $L_t = (n-2)I_t + 2J_t$. Moreover, in these circumstances $n-1$ is necessarily the sum of two squares.

Condition (5) implies some combinatorial properties, regarding the number of positive entries of the rows (resp. columns) of B. The rows of any $(-1, 1)$-matrix of size n can be classified as of even or odd type, depending on the parity of the number of 1s that they contain. It is apparent that the inner product of two rows of the same type is congruent to 2 modulo 4, while the inner product of two rows of opposite type is congruent to 0 modulo 4. In these circumstances, the block structure of the matrix in (5) implies that rows from 1 to t of B share a common type, whereas rows from $t+1$ to $2t$ share the opposite type. The same argument translates to the columns of B. This is a main difference with usual Hadamard matrices of order a multiple of 4, in which rows of different type cannot occur.

Notice that this balanced structure of even and odd type rows does not need to be attained anymore when $n-1$ is not the sum of two squares. In particular, record-determinant matrices are known in sizes 22, 34, 70 and 106 for which the number of even type rows is greater than the number of odd type rows (see [30] for details).

How do these conditions translate to the cocyclic framework? In order to answer this question properly, it seems reasonable to give in advance a brief introduction to cocyclic matrices.

Assume throughout that $G = \{g_1 = 1, g_2, \ldots, g_n\}$ is a multiplicative group, not necessarily abelian. Functions $\psi : G \times G \rightarrow \langle -1 \rangle \cong \mathbf{Z}_2$ which satisfy

$$\psi(g_i, g_j)\psi(g_ig_j, g_k) = \psi(g_j, g_k)\psi(g_i, g_jg_k), \quad \forall g_i, g_j, g_k \in G \tag{6}$$

are called (binary) cocycles (over G) [24]. A cocycle is a coboundary $\partial\phi$ if it is derived from a set mapping $\phi : G \rightarrow \langle -1 \rangle$ by $\partial\phi(a, b) = \phi(a)\phi(b)\phi(ab)^{-1}$.

A cocycle ψ is naturally displayed as *a cocyclic matrix (or G-matrix)* M_ψ; that is, the entry in the (i, j)th position of the cocyclic matrix is $\psi(g_i, g_j)$, for all $1 \leqslant i, j \leqslant n$.

A cocycle ψ is *normalized* if $\psi(1, g_j) = \psi(g_i, 1) = 1$ for all $g_i, g_j \in G$. The cocyclic matrix coming from a normalized cocycle is called *normalized* as well. Each unnormalized cocycle ψ determines a normalized one $-\psi$, and vice versa. Therefore, we may reduce, without loss of generality, to the case of normalized cocycles.

The set of cocycles forms an abelian group $Z(G)$ under pointwise multiplication, and the coboundaries form a subgroup $B(G)$. A basis $\mathcal{B}$ for cocycles over G consists of some elementary coboundaries ∂_i and some representative cocycles, so that every cocyclic matrix admits a unique representation as a Hadamard (pointwise) product $M = M_{\partial_{i_1}} \circ \cdots \circ M_{\partial_{i_w}} \circ R$, in terms of some coboundary matrices $M_{\partial_{i_j}}$ and a matrix R formed from representative cocycles.

Recall that every *elementary coboundary* ∂_d is constructed from the characteristic set map $\delta_d : G \rightarrow \{-1, 1\}$ associated with an element $g_d \in G$, so that

$$\partial_d(g_i, g_j) = \delta_d(g_i)\delta_d(g_j)\delta_d(g_ig_j) \quad \text{for} \quad \delta_d(g_i) = \begin{cases} -1 & g_d = g_i, \\ 1 & g_d \neq g_i. \end{cases}$$

Remark 1 [2, Lemma 1]. In particular, for $d \neq 1$, every row $s \notin \{1, d\}$ in M_{∂_d} contains precisely two -1s, which are located at the positions (s, d) and (s, e), for $g_e = g_s^{-1} g_d$. Furthermore, the first row is always formed by 1s, while the dth row is formed all by -1s, excepting the positions $(d, 1)$ and (d, d).

Although the elementary coboundaries generate the set of all coboundaries, they might not be linearly independent (see [3] for details).

At this point, it is worthwhile to notice that every row (resp. column) in M_{∂_d} consists of an even number of 1s (see Remark 1). Consequently, for a cocyclic matrix $M = M_{\partial_{i_1}} \circ \cdots \circ M_{\partial_{i_w}} \circ R$ to be a candidate B for meeting (5), a necessary (in general, not sufficient) condition is that half the rows (resp. columns) of R are of even type, whereas the remaining t rows (resp. columns) are of odd type.

Let $G_r(M)$ (resp. $G_c(M)$) be the Gram matrix of the rows (resp. columns) of M,

$$G_r(M) = MM^T \quad (\text{resp. } G_c(M) = M^T M).$$

The Gram matrices of a cocyclic matrix can be calculated as follows.

Proposition 1. [19, lemma 6.6]
Let M_ψ be a cocyclic matrix,

$$[G_r(M_\psi)]_{ij} = \psi(g_i g_j^{-1}, g_j) \sum_{g \in G} \psi(g_i g_j^{-1}, g), \tag{7}$$

$$[G_c(M_\psi)]_{ij} = \psi(g_i, g_i^{-1} g_j) \sum_{g \in G} \psi(g, g_i^{-1} g_j). \tag{8}$$

If a cocyclic matrix M_ψ is Hadamard, we say that the cocycle involved, ψ, is orthogonal and M_ψ is *a cocyclic Hadamard matrix*. The cocyclic Hadamard test asserts that a normalized cocyclic matrix is Hadamard if and only if every row sum (apart from the first) is zero [18]. In fact, this is a straightforward consequence of Proposition 1.

Analyzing this relation from a new perspective, one could think of normalized cocyclic matrices meeting the bound (1) as normalized cocyclic matrices for which every row sum is zero. Could it be possible that such a relation translates somehow to the case $n \equiv 2 \pmod 4$? We now prove that, in fact, the answer to this question is affirmative.

A natural way to measure if the rows of a normalized cocyclic matrix $M = [m_{ij}]$ are close to sum zero, is to define an *absolute row excess* function RE, such that

$$RE(M) = \sum_{i=2}^{n} \left| \sum_{j=1}^{n} m_{ij} \right|.$$

This is a natural extension of the usual notion of *excess* of a Hadamard matrix, $E(H)$, which consists in the summation of the entries of H.

With this definition at hand, it is evident that a cocyclic matrix M is Hadamard if and only if $RE(M) = 0$. That is, a cocyclic matrix M meets (1) if and only if $RE(M)$ is minimum. This condition may be generalized to the case $n \equiv 2 \pmod 4$.

Proposition 2. *Let M be a normalized cocyclic matrix over G. Then $RE(M) \geqslant 2t - 2$.*

Proof. Let M be a cocyclic matrix over G. Let M have e rows of even type (precisely, those whose summations are congruent to 2 modulo 4), and consequently $2t - e$ rows of odd type (those whose summations are congruent to 0 modulo 4) In these circumstances, in order to prove that $RE(M) \geqslant 2t-2$ it suffices to prove that $e \geqslant t$ (notice that the first row of M is always of even type).

As we commented before, since $n \equiv 2 \pmod 4$, the inner product of two rows of the same type is congruent to 2 modulo 4, while the inner product of two rows of opposite type is congruent to 0 modulo 4. This way, the number of inner products $\equiv 0 \pmod 4$ is $2e(2t-e)$, the total number of ordered pairs of rows of different type. An upper bound of this value is $2e(2t-e) \leqslant 2t^2$, and equality holds if and only if $e=t$ since $2e(2t-e) \leqslant 2t^2 \Leftrightarrow 2t^2-4et+2e^2 \geqslant 0 \Leftrightarrow 2(t-e)^2 \geqslant 0$.

Since each of the $2t-e$ group elements g_r corresponding to rows of odd type can be represented as $g_s g_j^{-1}$, where $g_j = g_r^{-1} g_s$, Proposition 1 implies that row s of the Gram matrix $G_r(M) = MM^T$ contains $2t-e$ elements $\equiv 0 \pmod 4$ for each $1 \leqslant s \leqslant 2t$, and therefore that the Gram matrix G_r contains $2t(2t-e)$ elements $\equiv 0 \pmod 4$. Hence $2t(2t-e) \leqslant 2t^2 \Leftrightarrow 2t-e \leqslant t \Leftrightarrow e \geqslant t$. $\square$

But we may go even further. Having the minimum possible value $2t-2$ is a necessary condition for a cocyclic matrix M to meet the bound (3).

Proposition 3. *If a cocyclic matrix M meets the bound (3), then $RE(M) = 2t-2$.*

Proof. Let M be a cocyclic matrix meeting (3). By means of rows and columns permutations and row negations (no column negations are needed), M can be transformed in a Hadamard equivalent matrix B satisfying (5). From (5), it is evident that $RE(B) = 2t-2$. Since no column negations have been used, $RE(M) = RE(B)$. $\square$

Unfortunately, although having minimum absolute row excess is a necessary and sufficient condition for meeting the bound (1), it is just a necessary (but not sufficient, in general, see Table 5) condition for meeting the bound (3).

From now on, we fix $G = D_{2t}$, the dihedral group with presentation $\langle a, b\colon\ a^t = b^2 = (ab)^2 = 1\rangle$, with ordering $\{1, a, \ldots, a^{t-1}, b, ab, \ldots, a^{t-1}b\}$ and indexed as $\{1, \ldots, 2t\}$ where t is an odd positive integer.

From the results in [2] and [1], it may be proved that a basis for cocycles over D_{2t} consists in $2t-1$ generators, $\mathcal{B} = \{\partial_2, \ldots, \partial_{2t-1}, \beta\}$. Here ∂_i denotes the coboundary associated with the i^{th}-element of the dihedral group D_{2t}, that is $a^{i-1 \pmod t} b^{\lfloor \frac{i-1}{t} \rfloor}$. And β is the representative cocycle in cohomology, i.e. $M_\beta = \begin{pmatrix} J_t & J_t \\ J_t & -J_t \end{pmatrix}$.

Remark 2. Since half the rows of M_β are of even type (those from 1st to $2t$th), it is apparent that a cocyclic matrix M over D_{2t} can attain the bound (3) only if M decomposes as a combination of the form $M_{\partial_{i_1}} \circ \cdots \circ M_{\partial_{i_w}} \circ M_\beta$. If, on the contrary, M_β is not used, then all the rows of the cocyclic matrix would be of even type, and the condition (5) could not be satisfied. Notice that for every D_{2t}-cocyclic matrix of the form $M_{\partial_{i_1}} \circ \cdots \circ M_{\partial_{i_w}} \circ M_\beta$, rows (resp. columns) from 1st to $2t$th are of even type, whereas the remaining rows (resp. columns) are of odd type.

The following technical result will be used throughout the paper.

Lemma 4. *Let M be a cocyclic matrix over D_{2t}.*

- *MM^T has the form $\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$, for some symmetric square matrices X and Y of order t, if and only if it admits a decomposition of the form $M = M_{\partial_{i_1}} \circ \cdots \circ M_{\partial_{i_w}} \circ M_\beta$.*
- *If it is the case, in addition, then $MM^T = M^T M$.*

Proof. The argument described in [2, Proposition 11] may be adapted to the case of dihedral groups D_{2t}, so that the summation of any row s, $t+1 \leqslant s \leqslant 2t$, is 0.

Actually, consider a matrix $N = M_{\partial_{i_1}} \circ \cdots \circ M_{\partial_{i_w}}$. Attending to the presentation of D_{2t}, it may be readily checked that $(a^k b)^{-1} = a^k b$. In these circumstances, Remark 1 implies that the (necessarily even) number $2f_s$ of -1s located at row s, $t+1 \leqslant s \leqslant 2t$, are distributed in such a way that precisely

f_s of them occur through columns 1 to t, whereas the remaining f_s occur through columns $t+1$ to $2t$. Furthermore, fixed a row s, $t+1 \leqslant s \leqslant 2t$, any two coboundary matrices M_{∂_i} and M_{∂_j} either share their two -1s entries at row s, or do not share any of them at row s. Consequently, attending to the form of M_β, the summation of row s, $t+1 \leqslant s \leqslant 2t$, of any cocyclic matrix $M_{\partial_{i_1}} \circ \cdots \circ M_{\partial_{i_w}} \circ M_\beta$ is zero.

Now the first part of the Lemma becomes apparent, from Proposition 1 and Remark 2.

The proof of the second part of this lemma follows from the study of the distribution of -1 by rows and by columns in the elementary coboundary (see [2,4]). This study leads to the notion of called (row) n-paths in [2], analogously the notion for columns can be defined. The distribution of -1 by rows and columns in $M = [m_{i,j}]$ can be found by means of n-path. As a consequence, we have the following properties of M:

1. If $t+1 \leqslant j \leqslant 2t$, then jth column sum is zero and $m_{j,j} = 1$.
2. Assume $1 \leqslant i \leqslant t$. Then, any sequence of coboundaries making up a i-path for rows, also makes up a i-path for columns. (Unfortunately, this situation does not hold when $t+1 \leqslant i \leqslant 2t$.) Hence:
   - The ith row sum is equal to the ith column sum.
   - If $1 \leqslant j \leqslant t$ then the inner product of rows ith and jth is equal to the inner product of columns ith and jth.

Let us distinguish three cases:

1. If $1 \leqslant i \leqslant t$ and $t+1 \leqslant j \leqslant 2t$ (or vice versa) then by (8), we have:
$$[M^TM]_{i,j} = \pm \sum_{l=1}^{2t} m_{l,k}$$
   with $t+1 \leqslant k \leqslant 2t$. Using the properties above, we have that this column sum is zero.
2. Assuming $1 \leqslant i, j, \leqslant t$, and taking into account the last property stated above. It follows that
$$[M^TM]_{i,j} = [MM^T]_{i,j}.$$
3. Let us show that $[M^TM]_{i,j} = [MM^T]_{i,j}$ when $t+1 \leqslant i, j \leqslant 2t$.
   Firstly, let us observe that $i = j$ the result is trivial. For the remaindering of the proof, we suppose that $i \neq j$. Using (7) and (8), we have:
$$[MM^T]_{i,j} = m_{k,j} \sum_{l=1}^{2t} m_{k,l},$$
   and
$$[M^TM]_{i,j} = m_{i,k} \sum_{l=1}^{2t} m_{l,k}.$$
   where $2 \leqslant k \leqslant t$ since $g_i^{-1} = g_i$ and $g_i g_j^{-1} = g_i^{-1} g_j = g_k$. In this situation, the kth row sum is equal to the kth column sum. Now, using that the entries of M satisfying (6) and $m_{i,i} = 1$, it follows that $m_{k,j} = m_{i,k}$, and this concludes the proof. □

In this paper, not only do we pursue a characterization of the D_{2t}-cocyclic matrices whose determinant is equal to $(4t-2)(2t-2)^{t-1}$, but we will also develop some methods for finding them.

If M is a D_{2t}-cocyclic matrix whose determinant is equal to $(4t-2)(2t-2)^{t-1}$ then $G_r(M)$ is equivalent to

$$C = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix},$$

with $L = (2t-2)I_t + 2J_t$.

Starting from C one may construct $\mathcal{S}$ the full set of equivalent matrices $U_T C U_T$, such that U_T is the negation of the diagonal entries with indices in $T \subset \{1, \ldots, 2t\} = Q$ of the identity matrix I_{2t}.

$$\mathcal{S} = \{U_T C U_T : \ T \subset Q\}. \tag{9}$$

This list has cardinality 2^{2t-1} since $U_T C U_T = U_{Q\setminus T} C U_{Q\setminus T}$. It is a remarkable fact that $\mathcal{S}$ constitutes the complete list of *candidate Gram matrices* (i.e., symmetric, have diagonal elements equal to $2t$ and positive definite and determinant equal to $(4t-2)^2(2t-2)^{2t-2}$ in our cocyclic context. In the general framework, simultaneous permutation of rows and the corresponding columns in the candidate Gram matrices is also allowed. Because of the assumed ordering of the group elements, and the relation of elements of the form $a^j b$ with rows of odd type, arbitrary permutation is not allowed in our context.

Given a candidate Gram matrix, not only would we like to determine whether it admits a decomposition $U_T C U_T = MM^T$, M being a D_{2t}-cocyclic matrix, but we also aim to compute such decomposition whenever possible. The next result will play an essential role in the design of the algorithm solving this problem.

Theorem 5. *Let M be a normalized cocyclic matrix over D_{2t} and $G_r(M) = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$ be the Gram matrix of M. If X and Y are square matrices of order t, then the entries of $M = [m_{i,j}]$ are given by the formulas:*

1. $1 \leqslant i, j \leqslant 2t$

$$m_{1,j} = 1 = m_{i,1} \quad \textit{(M normalized)}$$

2. $2 \leqslant i \leqslant t$

 2.1. $2 \leqslant j \leqslant t$

$$m_{i,j} = \frac{x_{[j+i-1],j}}{x_{i,1}}, \quad \textit{where} \quad [n] = 1 + (n-1) \bmod t$$

 2.2. $t+1 \leqslant j \leqslant 2t$

$$m_{i,j} = \frac{y_{[j+i-1-t],j-t}}{x_{i,1}}$$

3. $i = t+1$ *(By lemma 4, $MM^T = M^T M$. Hence we have uniqueness of the entries and these are the values given below.)*

 3.1. $2 \leqslant j \leqslant t$

$$m_{t+1,j} = \frac{y_{1,t+2-j}}{x_{1,j}}$$

 3.2. $j = t+1$

$$m_{t+1,t+1} = -\sum_{j=1}^{t} m_{t+1,j} \Bigg/ \left(1 + \sum_{j=t+2}^{2t} m_{j-t,t+1}\, m_{t+1,j-t}\, m_{2t+2-j,t+1}\right).$$

 3.3. $t+2 \leqslant j \leqslant 2t$

$$m_{t+1,j} = m_{j-t,t+1}\, m_{t+1,j-t}\, m_{2t+2-j,t+1}\, m_{t+1,t+1}$$

4. $t+2 \leqslant i \leqslant 2t$

 4.1. $2 \leqslant j \leqslant t$

$$m_{i,j} = m_{i-t,2t-j+2}\, m_{i-t,t+1}\, m_{t+1,j}$$

4.2. $j = t+1$

$$m_{i,t+1} = m_{i-t,t+1} \cdot m_{t+1,t+1}$$

4.3. $t+2 \leqslant j \leqslant 2t$

$$m_{i,j} = m_{i-t,2t-j+2}\, m_{i-t,t+1}\, m_{t+1,j}$$

Proof. The statement of the theorem follows by direct inspection. □

In what follows, we rewritten the criterion to decide whether or not the determinant of a $2t \times 2t$ $(-1, 1)$-matrix attains the Ehlich–Wojtas' bound in the cocyclic framework.

Theorem 6. *Let $M = [m_{i,j}]$ be a cocyclic matrix over D_{2t}, then*

$$\det M \leqslant (4t-2)(2t-2)^{t-1}. \tag{10}$$

Moreover, the equality in (10) holds if and only if

- *Each row of M from $(t+1)$th to $2t$th has row sum zero.*
- *The block matrix $\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$ is a candidate a Gram matrix, where $X = [x_{i,j}]$ and $Y = [y_{i,j}]$ are symmetric square matrices of order t with entries:*

$$x_{i,j} = \begin{cases} m_{j-i+1,i} \sum\limits_{k=1}^{2t} m_{j-i+1,k} & i<j \\ 2t & i=j \\ x_{j,i} & i>j \end{cases}$$

and

$$y_{i,j} = \begin{cases} m_{j-i+1,t+i} \ \ \sum_{k=1}^{2t} m_{j-i+1,k} & i<j \\ 2t & i=j \\ y_{j,i} & i>j \end{cases}$$

Proof. The inequality (10) is just Ehlich–Wojtas' bound. Using the identity (7) for computing MM^T, we obtain

$$MM^T = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \Longleftrightarrow \text{the } i\text{th row sum is 0, for all } i \text{ with } \ t+1 \leqslant i \leqslant 2t.$$

Furthermore, if MM^T is a candidate Gram matrix then $\det M = (4t-2)(2t-2)^{t-1}$. □

Remark 3. Two further necessary conditions for equality in (10) to hold are:

- $2t - 1 = \alpha^2 + \beta^2$, where α and β are integers.
- Each row from 2nd to tth has row sum either 2 or -2.

3. Explicit calculations

All the calculations of this section have been worked out in MATHEMATICA 4ZX.0, running on a *Pentium IV 2.400 Mhz DIMM DDR266 512 MB*.

Table 4
Maximum determinant of D_{2t}-matrices.

t	R	#	#[ ]	M_ψ
3	1	6	1	3
5	1	25	1	35
7	1	196	1	151
9	1	972	1	1611
11	0.900972	9680		47271

We have performed three different searches:

- An exhaustive search running over the full set of D_{2t}-matrices.
- An exhaustive search running on the full set of candidate Gram matrices (Algorithm 2).
- A heuristic search, in terms of a genetic algorithm, where the population are D_{2t}-matrices and the fitness function depends on the ratio between the determinant value of an individual and the corresponding bound.

Next these approaches are explained in detail.

3.1. Exhaustive search

We have performed an exhaustive search looking for the set of D_{2t}-matrices with maximum determinant, for $3 \leqslant t \leqslant 11$ odd.

Recall that every cocycle ψ over D_{2t} is expressed with regards to the basis $\mathcal{B} = \{\partial_2, \ldots, \partial_{2t-1}, \beta\}$. Here ∂_i denotes the coboundary associated to the i^{th}-element of the dihedral group D_{2t}, $a^{i-1 \pmod t} b^{\lfloor \frac{i-1}{t} \rfloor}$. And β is the representative cocycle in cohomology, i.e. $M_\beta = \begin{pmatrix} J_t & J_t \\ J_t & -J_t \end{pmatrix}$.

Due to obvious size limitations, in Table 4 we prefer to include the ratio $R = \frac{det(M)}{(4t-2)(2t-2)^{t-1}}$ (which is called *efficiency of the design* in [31]) instead of the value $det(M)$ of the determinant itself. The second column of the table shows the total number # of D_{2t}-matrices found which meet the maximal determinant value, whereas the third column informs about the number #[ ] of different Hadamard equivalence classes in which these D-optimal designs are organized. We also include an explicit D_{2t}-matrix M_ψ meeting the corresponding maximum determinant value, in terms of the coordinates of ψ with regards to $\mathcal{B}$. For brevity, a binary vector of coordinates $(f_1, \ldots, f_{2t-1})_{\mathcal{B}}$ will simply be denoted as its decimal number representation.

It is known that there is only one equivalence class for $t = 3, 5, 7$ (see [30]) and three equivalence classes for $t = 9$ (see [8]). The one that corresponds to optimal D_{2t}-cocyclic matrices for $t = 9$ is the first as listed in [30]. We did not check the number of equivalence classes for $t = 11$, since these matrices do not attain the maximal determinant value already known (see [30]).

Notice that the optimal D_{2t}-cocyclic matrices enumerated in Table 4 are not the only ones with minimum absolute row excess. Table 5 shows, for each $3 \leqslant t \leqslant 11$ odd, the number # of D_{2t}-matrices with minimum absolute row excess, and how they are distributed with regards to their ratio $R = \frac{det(M)}{(4t-2)(2t-2)^{t-1}}$, as well as the required computing time.

3.2. Exhaustive search revisited

The search methods usually employed in the literature [14,26,6,28] for finding $(-1, 1)$-matrices with large determinant are based on two steps. Firstly, in generating a set of candidate Gram matrices having determinant greater than or equal to the square of a known lower bound on the maximum. Secondly, in attempting to decompose each candidate as the product of a $(-1, 1)$-matrix and its transpose. Bearing this in mind, in what follows, we have designed another algorithm (Algorithm 2) searching exhaustively for D_{2t}-matrices with maximum determinant, in case that $2t - 1 = \alpha^2 + \beta^2$.

Table 5
Determinants of D_{2t}-matrices with minimum absolute row excess.

t	#	$\#_i$ / R_i								Time
3	15	9 0.75	6 1							0.016"
5	175	50 0.5625	100 0.868056	25 1						0.484"
7	1568	196 0.432204	294 0.79867	882 0.886884	196 1					17.456"
9	13122	972 0.350616	972 0.72928	1944 0.756409	1944 0.819419	4374 0.853128	972 0.895782	972 0.900735	972 1	9'24.8"
11	82764	2904 0.294845	13310 0.72806	8470 0.774513	24200 0.803574	12100 0.844079	12100 0.876751	9680 0.900972		4^h31'6.3"

Instead of working with the set of D_{2t}-matrices, we prefer to construct the cocyclic matrices related to candidate Gram matrices.

Let us recall that $\mathcal{S}$ (see (9)) constitutes the complete list of candidate Gram matrices with determinant equal to $(4t-2)^2(2t-2)^{2t-2}$. For each of these Gram matrices U_TCU_T in $\mathcal{S}$, one may reconstruct the uniquely determined cocyclic matrix M, which is the candidate to satisfy the relation $MM^T = U_TCU_T$. Notice that this relation will fail only if no cocyclic matrix N exists such that $NN^T = U_TCU_T$, since if equality holds then $N = M$ necessarily (see Theorem 5).

Given $A \in \mathcal{S}$, we now outline a method to determine whether it admits a decomposition $A = MM^T$ where M is a cocyclic $(-1, 1)$-matrix over D_{2t}.

Algorithm 1. Cocyclic test to decompose a candidate Gram matrix.

Input: a candidate Gram matrix A.
Output: a cocyclic matrix M in the case that A admits to be decomposed as $A = MM^T$.

Step 1. Calculate M using the formulas given Theorem 5 and assuming $G_r(M) = A$.
Step 2. Calculate MM^T.
Step 3. If $A = MM^T$ then A admits the decomposition. Otherwise, such decomposition does not exist for A.

Verification: By construction, M is a cocyclic matrix over D_{2t}, the entry $m_{t+1,t+1} = -1$ and every row sum from the $t+1$th to $2t$th is zero, but MM^T might be different from A. Theorem 5 guarantees the uniqueness of M.

Algorithm 2. Search for cocyclic matrices with determinant equal to $(4t-2)(2t-2)^{t-1}$.
Input: an integer n such that $n \equiv 2 \bmod 4$ and $n-1$ is the sum of two squares.
Output: a cocyclic matrix M with determinant equal to $(4t-2)(2t-2)^{t-1}$, in the case that such matrix exists.
$\Omega \leftarrow \emptyset$
$\mathcal{S} \leftarrow$ The complete list of candidate Gram matrices
while $\mathcal{S}$ is not empty {

1. Choose a matrix A in $\mathcal{S}$.
2. $\mathcal{S} \leftarrow \mathcal{S} \setminus \{A\}$.
3. Check whether A admits to be decomposed as $A = MM^T$ for a cocycle M. If not, go to 1; otherwise $\Omega \leftarrow M$.
4. End while.

}
Ω

Table 6
Exhaustive search from Gram matrices.

t	3	5	7	9	13	15
R	4	1,6	1,8,9,11	1,2,10,12,15	1,2,5,14,15,16,18,21,23	1,2,4,11,16,17,18,20,23,27

Remark 4. These constraints

$$\sum_{i=1}^{t} a_{[i+j],i} + a_{t+[i+j],t+i} = a_{1+j,1}^2 = 4, \quad \forall j = 1, \ldots, \frac{t-3}{2},$$

on the entries of a candidate Gram matrix $A = [a_{ij}]$ are a necessary condition in order Algorithm 1 to have a successful output. Concretely, these constraints guarantee that every row sum from 2nd to tth is either 2 or -2. Obviously, they reduce the size of the search in Algorithm 2.

In Table 6 we show, for each $3 \leqslant t \leqslant 15$ odd, an example of subset $T \subset \{1, \ldots, 2t\}$ such that the cocycle associated to the Gram matrix $U_T C U_T$ has maximum determinant (with regards to Table 4).

3.3. Heuristic search

Genetic algorithms (more briefly, GAs in the sequel) are appropriate for searching through large spaces, where exhaustive methods cannot be employed.

The father of the original Genetic Algorithm was John Holland who invented it in the early 1970's [16]. We next include a brief introduction to the subject. The interested reader is referred to [25] for more extensive background on GAs.

The aim of GAs is to mimic the principle of evolution in order to find an optimum solution for solving a given optimization problem. More concretely, starting from an initial "population" of potential solutions to the problem (traditionally termed *chromosomes*), some transformations are applied (may be just to some individuals or even to the whole population), as images of the "mutation" and "crossover" mechanisms in natural evolution. Mutation consists of modifying a "gene" of a chromosome. Crossover interchanges the information of some genes of two chromosomes.

Only some of these individuals will move on to the next generation (the more fit individuals, according to the optimization problem, in terms of the measure of an "evaluation function"). Here "generation" is synonymous to iteration. The mutation and crossover transformations are applied from generation to generation, and individuals go on striving for survival. After some number of iterations, the evaluation function is expected to measure an optimum solution, which solves the given problem. Although no bounds are known on the number of iterations which are needed to produce the fittest individual, it is a remarkable fact that GAs usually converge to an optimum solution significantly faster than exhaustive methods do. Indeed, GAs need not to explore the whole space.

A genetic algorithm for finding D_{2t}-matrices with maximum determinant may be designed as follows.

The population consists of the whole space of D_{2t}-matrices, $M_\psi = (\psi(g_i, g_j))$, ψ being a cocycle over D_{2t}. Each of the individuals f of the population (i.e. potential solutions to the problem) is identified to a binary $(2t-1)$-tuple $(f_1, \ldots, f_{2t-1})_{\mathcal{B}}$, the coordinates of the cocycle ψ with regards to the basis $\mathcal{B}$. This way, the coordinates f_k are the genes of the individual $\psi = (f_1, \ldots, f_{2t-1})_{\mathcal{B}}$.

The initial population P_0 is formed by t^2 binary $(2t-1)$-tuples randomly generated.

The population is expected to evolve generation through generation until an optimum individual (i.e. a D_{2t}-matrix with maximum determinant) is located. We now describe how to form a new generation P_{i+1} from an old P_i:

1. Firstly, we must evaluate the fitness of every individual (i.e. cocycle f) of P_i. This function measuring the adaptation of an individual f is calculated as the ratio $R = \dfrac{det(M_\psi)}{(4t-2)(2t-2)^{t-1}}$.
2. Once the evaluation is finished, the crossover comes into play. All individuals are paired at random, so that crossover combines the features of two parent chromosomes to form two similar

Table 7
GA results.

t	*iter.*	*time*	R	M_ψ
3	0	$0.016''$	1	25
5	1	$0.093''$	1	63
7	1	$0.328''$	1	4997
9	1	$1.092''$	1	12881
11	2	$1.52''$	0.900972	1385361
13	5	$33.587''$	1	13649489
15	11	$2'31.976''$	1	277029099
17	4	$11.23''$	0.908563	2982042693
19	14	$3'05.24''$	1	15847631679

offspring by swapping corresponding segments of the parents. Each time, the break point n is chosen at random, so that two couples of different parents are swapped with possibly different break points.

3. Next we apply the mutation operator. Mutation arbitrarily alters only one gene of a selected individual (i.e. only one coordinate of the corresponding $(2t-1)$-tuple, swapping 0 to 1 or 1 to 0, as it is the case), by a random change with a probability equal to the mutation rate (for instance, 1%).
4. Now individuals strive for survival: a selection scheme, biased towards fitter individuals (according to their ratio), selects the next generation. In the case that an optimum individual exists (with ratio 1), the algorithm stops. Otherwise the population P_{i+1} is constructed from a selection of t^2 of the fittest individuals.

The process goes on from generation to generation until an optimum is reached.

We have included a table showing some executions of the genetic algorithm (Table 7), for $3 \leqslant t \leqslant 19$ odd, including the number of iterations, the time required in the calculations, the best ratio obtained so far, as well as a D_{2t}-matrix meeting this ratio (expressed as the decimal representation of the binary tuple of its coordinates with respect to $\mathcal{B}$).

Notice that the matrices listed for $t = 3, 5, 7, 9$ necessarily define the correspondent unique equivalence class listed in Table 4. The matrices obtained for $t = 13,\ 17,\ 19$ define equivalent classes which are different from those listed in [30]. We had no oportunity to check whether these matrices are equivalent to those described in [29].

4. Conclusions and further work

Firstly, not only have we characterized cocyclic matrices over D_{2t} with maximal determinant but we also indicated how to study the (cocyclic) decomposability of candidate Gram matrices with determinant equal to $(4t-2)^2(2t-2)^{2(t-1)}$. We point out that Algorithm 1 also works for other types of candidate Gram matrices A which satisfy that $A = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$ where X and Y are matrices of order t. In particular, the not optimal matrices listed in Table 5 which have minimum absolute row excess, provide such Gram matrices, which differ from that of (5) just in some signs. For instance, for $t = 3$, the matrix $M = M_{\partial_3} \circ M_{\partial_5} \circ M_\beta$ satisfies $RE(M) = 4$, $det(M) = 128 < 160$, and has Gram matrix

$$G_r(M) = \begin{pmatrix} 6 & -2 & 2 & 0 & 0 & 0 \\ -2 & 6 & 2 & 0 & 0 & 0 \\ 2 & 2 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 2 & -2 \\ 0 & 0 & 0 & 2 & 6 & 2 \\ 0 & 0 & 0 & -2 & 2 & 6 \end{pmatrix}.$$

Secondly, algorithms for constructing cocyclic matrices with large determinants based on exhaustive and heuristic searches have been presented. We observe that the size of the search space in the cocyclic framework is much smaller than in the general one. Thus exhaustive search is feasible for greater orders here. Unfortunately, the determinants obtained by these methods have not yet improved the known lower bounds on the maximum possible value when $n-1$ is not the sum of two squares. For instance, the maximal determinant of D_{22}-matrices is smaller than $2^{23} \cdot 5^{11}$, which is the maximum determinant value known so far for $n = 22$ (it was reported by Dowdeswell, Neubauer, Solomon and Tumer, see [30]).

Our next goals are:

1. Specify which cocyclic Hadamard matrices of order $4t$ have a $2t \times 2t$ matrix of largest determinant embedded. A cocyclic Hadamard matrix H over D_{4t} has always embedded a cocyclic matrix over D_{2t}, M. This matrix M is obtained by eliminating from H the rows and columns indexed with an even number. M satisfies that it is cocyclic and the row sum from $t+1$th to $2t$th is zero. Therefore, M might be a good candidate to have large determinant.
2. Design a GA with a fitness function dependent on the absolute row excess RE of a matrix, in such a way that we will say that an individual is better adapted than another if its absolute row excess is smaller.
3. Study the spectrum of the determinant function for cocyclic matrices over D_{2t}.
4. Obtain analytical formulas for the determinant of a given cocyclic matrix over D_{2t} and for its minors.
5. Study the maximal determinant problem for cocyclic matrices over other families of groups. We have in mind the group $\mathbb{Z}_t \times \mathbb{Z}_2$, since $\mathbb{Z}_t \times \mathbb{Z}_2^2$ is another prolific group providing cocyclic Hadamard matrices [2]. In fact, it may be checked that a basis for cocycles over $\mathbb{Z}_t \times \mathbb{Z}_2$ is $\mathcal{B} = \left\{ \partial_2, \ldots, \partial_{2t-1}, M_\beta = J_t \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right\}$, so that M_β has t rows of even type and t rows of odd type, and matrices of the form $M_{\partial_{i_1}} \circ \cdots \circ M_{\partial_{i_w}} \circ M_\beta$ could also attain the optimal bound (3).

Acknowledgements

We would like to express our gratitude to the referees for their many valuable suggestions, which have significantly improved the readability and comprehensiveness of the paper. In particular, we are in debt with them for the discussions about equivalence classes along the paper (including the tables in Section 1 about proportions and densities of usual and cocyclic Hadamard matrices), as well as the notion of parity type on rows/columns, which has helped to clarify many of the proofs included in Section 2. A special mention should be given to Proposition 2, which was initially stated just for D_{2t}-matrices. Proposition 2 and its proof, as they are presented now, were kindly provided by one of the referees.

This work has been partially supported by the research projects FQM-016 and P07-FQM-02980 from JJAA and MTM2008-06578 from MICINN (Spain) and FEDER (European Union). The authors would also like to thank Kristeen Cheng for her reading of this article.

References

[1] V. Álvarez, J.A. Armario, M.D. Frau, P. Real, A Mathematica Notebook for Computing the Homology of Iterated Products of Groups, in: A. Iglesias, N. Takayama (Eds.), ICMS 2006, LNCS, vol. 4151, Springer Verlag, Heidelberg, 2006, pp. 47–57.
[2] V. Álvarez, J.A. Armario, M.D. Frau, P. Real, A system of equations for describing cocyclic Hadamard matrices, J. Comb. Des. 16 (4) (2008) 276–290.
[3] V. Álvarez, J.A. Armario, M.D. Frau, P. Real, The homological reduction method for computing cocyclic Hadamard matrices, J. Symb. Comput. 44 (2009) 558–570.
[4] J.A. Armario, On an inequivalence criterion for cocyclic Hadamard matrices, Cryptogr. Commun. 2 (2010) 247–259.
[5] G. Barba, Intorno al teorema di Hadamard sui determinanti a volre massimo, Giornale di Matematiche di Battaglini 71 (1933) 70–86.

[6] T. Chadjipantelis, S. Kounias, C. Moyssiadis, The maximum determinant of 21×21 $(1, -1)$-matrices and D-optimal designs, J. Stat. Plann. Inference 16 (1987) 167–178.
[7] J.H.E. Cohn, On determinants with elements ± 1, II, Bull. London Math. Soc. 21 (1989) 36–42.
[8] J.H.E. Cohn, On the number of D-optimal designs, J. Combin. Theory Ser. A 66 (1994) 214–225.
[9] J.H.E. Cohn, Almost D-optimal designs, Util. Math. 57 (2000) 121–128.
[10] W. de Launey, K.J. Horadam, A weak difference set construction for higher dimensional designs, Des. Codes Cryptogr. 3 (1993) 75–87.
[11] H. Ehlich, Determiantenabschätzungen für binäre Matrizen, Math. Z. 83 (1964) 123–132.
[12] H. Ehlich, Determiantenabschätzung für binäre Matrizen mit $n \equiv 3 mod 4$), Math. Z. 84 (1964) 438–447.
[13] R.J. Fletcher, C. Koukouvinos, J. Seberry, New skew-Hadamard matrices of order $4 \cdot 59$ and new D-optimal designs of order $2 \cdot 59$, Discrete Matehmatics 286 (2004) 252–253.
[14] Z. Galil, J. Kiefer, D-optimum weighing designs, Ann. Stat. 8 (1980) 1293–1306.
[15] J. Hadamard, Résolution d'une question relative aux déterminants, Bull. Sci. Math. 2 (17) (1893) 240–246.
[16] J.H. Holland, Adaptation in Natural and Artificial Systems, University of Michigan Press, Ann Arbor, 1975.
[17] K.J. Horadam, W. de Launey, Cocyclic development of designs, J. Algebraic Combin. 2 (3) (1993) 267–290 (Erratum: J. Algebraic Combin. 3 (1) (1994) 129).
[18] K.J. Horadam, W. de Launey, Generation of Cocyclic Hadamard matrices, Computational Algebra and Number Theory, Math. Appl., Kluwer Acad. Publ., Dordrecht, 1995, pp. 279–290.
[19] K.J. Horadam, Hadamard Matrices and Their Applications, Princeton University Press, Princeton, NJ, 2007.
[20] H. Kharaghani, W.P. Orrick, D-optimal designs, in: C. Colbourn, J. Dinitz (Eds.), The CRC Handbook of Combinatorial Designs, Taylor and Francis, Boca Raton, 2006
[21] H. Kharaghani, B. Tayfeh-Rezaie, On the classification of Hadamard matrices of order 32, J. Comb. Des. 18 (2010) 328–336.
[22] C. Koukouvinos, <http://math.ntua.gr/ ckoukouv/>.
[23] C. Koukouvinos, M. Mitroulli, J. Seberry, Bounds on the maximum determinant for $(-1, 1)$ matrices, Bull. ICA 29 (2000) 39–48.
[24] S. MacLane, Homology, Classics in Mathematics, Springer-Verlang, Berlin, 1995.. (Reprint of the 1975 edition).
[25] Z. Michalewicz, Genetic Algorithms + Data Structures = Evolution Programs, Springer-Verlag, 1992.
[26] C. Moyssiadis, S. Kounias, The exact D-optimal first order saturated design with 17 observations, J. Statist. Plann. Inference 7 (1982) 13–27.
[27] P. Ó Catháin, M. Röder, The Cocyclic Hadamard matrices of order less than 40, Des. Codes Cryptogr. 58 (1) (2011) 73–88.
[28] W.P. Orrick, The maximal $\{-1, 1\}$-determinant of order 15, Metrika 62 (2005) 195–219.
[29] W.P. Orrick, On the enumeration of some D-optimal designs, J. Statist. Plann. Inference 138 (2008) 286–293.
[30] W. Orrick, B. Solomon, The Hadamard Maximal Determinant Problem (website), <http://www.indiana.edu/maxdet/>.
[31] J. Seberry, T. Xia, C. Koukouvinos, M. Mitrouli, The maximal determinant and subdeterminants of ± 1 matrices, Linear Algebra Appl. 373 (2003) 297–310.
[32] N.J.A. Sloane, A Library of Hadamard Matrices (website), <http://www2.research.att.com/ njas/hadamard/>.
[33] N.J.A. Sloane, The On-line Encyclopedia of Integer Sequences (website), <http://oeis.org/>.
[34] H. Tamura, D-optimal designs and group divisible designs, J. Comb. Des. 14 (2006) 451–462.
[35] W. Wojtas, On Hadamard's inequallity for the determinants of order non-divisible by 4, Colloq. Math. 12 (1964) 73–83.