On D_{4t}-Cocyclic Hadamard Matrices

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Abstract: In this paper, we describe some necessary and sufficient conditions for a set of coboundaries to yield a cocyclic Hadamard matrix over the dihedral group D_{4t}. Using this characterization, new classification results for certain cohomology classes of cocycles over D_{4t} are obtained, extending existing exhaustive calculations for cocyclic Hadamard matrices over D_{4t} from order 36 to order 44. We also define some transformations over coboundaries, which preserve orthogonality of D_{4t} -cocycles. These transformations are shown to correspond to Horadam's bundle equivalence operations enriched with duals of cocycles.

Keywords: *Hadamard matrix; cocyclic matrix; Ito's type Q Hadamard matrix; shift equivalence;*

1. INTRODUCTION

A square matrix H of entries ± 1 of order n is said to be *Hadamard* if $HH^T = nI_n$, that is, its rows (equivalently, columns) are pairwise orthogonal. From this condition, it may be easily derived that n must be 1, 2, or a multiple of 4. It is conjectured that Hadamard matrices exist for every order $n = 4t$. Although no proof of this fact is known so far, there is much evidence (such as asymptotic formulas for their existence, as well as many different construction methods) that supports the idea that this conjecture might be true. Actually, uncertainty of their existence remains only for 12 orders up to 2,000, just 3 up to 1,000. Interested readers are referred to [4, 9] and references therein for further information about Hadamard matrices and their applications.

Most of the methods used for constructing Hadamard matrices are based on algebraic properties or block structures. These include Sylvester, Paley, Williamson, Ito, 1,2-circulant cores Hadamard matrices, for instance. Unfortunately, no matter what the construction method is, the search space grows exponentially in the size of the matrices, and new insights would be very appreciated.

Another difficult problem is that of classification. Hadamard matrices may be grouped into *Hadamard equivalence* classes, as soon as two matrices are identified if they differ in row/column permutations and/or row/column negations. Very recently, classification of Hadamard matrices up to order 32 has been fulfilled [17, 18], counting more than 10 millions of inequivalent classes in order 32! A weaker notion of equivalence (Q-equivalence, as termed in [20], which adjoins the *switching operations*), drastically reduces the total number of equivalences classes as the size of the matrices grows, from millions to just a few tens.

Cocyclic Hadamard matrices were introduced in the mid-1990s, derived from the work of de Launey and Horadam concerning the problem of extending two-dimensional combinatorial designs to higher dimensional designs [4, 9–11]. The internal structure of cocyclic matrices could be of help in constructing Hadamard matrices in an easier way. To start with, checking whether a cocyclic matrix is Hadamard is faster than the analogous inspection on usual Hadamard matrices [11]. Furthermore, most classical constructions (including those cited above, excepting the 1,2-circulant cores constructions) have been identified as cocyclic [9].

Recall that a (normalized, binary, two-dimensional) *cocycle* ψ over G is a set map $\psi : G \times G \to \langle -1 \rangle$ satisfying $\psi(1, 1) = 1$ and the cocycle equation:

$$\psi(g, h)\,\psi(gh, k) = \psi(g, hk)\,\psi(h, k), \ \forall\, g, h, k \in G. \tag{1}$$

A cocycle ψ over G is naturally displayed as a *G-cocyclic matrix* $M_\psi = (\psi(g, h))$ over G, once some ordering is fixed on the elements of G, which indexes rows and columns (notice that these orderings might not coincide). When a G-cocyclic matrix M_ψ is Hadamard, the cocycle ψ is said to be *orthogonal*.

The set of cocycles from G to $\langle -1 \rangle$ forms an abelian group $Z^2(G, \mathbb{Z}_2)$ under pointwise multiplication. The simplest cocycles are the coboundaries ∂f, defined for any function $f : G \to \langle -1 \rangle$ by $\partial f(g, h) = f(g)^{-1} f(h)^{-1} f(gh)$. The subgroup of coboundaries, $B^2(G, \mathbb{Z}_2)$, is naturally generated by the set of elementary coboundaries $\partial_i := \partial \delta_i$, where δ_i is the Kronecker delta function of the ith element in G in the given ordering. Cocycles may be grouped into cohomological classes, to form $H^2(G; \mathbb{Z}_2) = Z^2(G, \mathbb{Z}_2)/B^2(G, \mathbb{Z}_2)$, so that $[\psi] = [\psi'] \in H^2(G, \mathbb{Z}_2) \Leftrightarrow \psi' = \psi \prod_{i=1}^{|G|} \partial_i^{r_i}$, for $r_i \in \{0, 1\}$.

In the past 20 years, different characterizations of cocyclic Hadamard matrices have been introduced and exploited. Actually, cocyclic Hadamard matrices, Hadamard groups, normal relative difference sets with parameters $(4t, 2, 4t, 2t)$, and divisible $(4t, 2, 4t, 2t)$-designs are known to be equivalent [4, 5, 7, 9].

Thus we have different descriptions of the same phenomena, all of which have been successfully used for studying cocyclic Hadamard matrices. This circumstance has been exploited in [21] to completely classify cocyclic Hadamard matrices of order less than 40, up to Hadamard equivalence. Notice that equivalent cocycles (from the cohomology point of view) may give rise to *Hadamard inequivalent* cocyclic matrices. Moreover, M_ψ might be Hadamard, whereas $M_{\psi'}$ might not be Hadamard, no matter $[\psi] = [\psi'] \in H^2(G; \mathbb{Z}_2)$. In fact, cocycles in the same cohomological class may be split into independent orbits, by means of the *shift action* [9], which discriminates between orthogonal and nonorthogonal cocycles. Actually, for any $a \in G$, the *shifts* $\psi \cdot a$ and $a \cdot \psi$ of ψ are the cocycles

$(\psi \cdot a)(g,h) = \psi(ag,h)\psi(a,h)^{-1}$ and $(a \cdot \psi)(g,h) = \psi(g,ha)\psi(g,a)^{-1}$. Notice that $[\psi \cdot a] = [a \cdot \psi] = [\psi]$. They are orthogonal if ψ is. For any automorphism $\theta \in \mathrm{Aut}(G)$, the cocycle $\psi \circ (\theta \times \theta)$ is orthogonal if ψ is. The orbit of ψ (termed *bundle*) is given by the combination of both actions:

$$\mathcal{B}(\psi) = \{(\psi \cdot a) \circ (\theta \times \theta),\ (a \cdot \psi) \circ (\theta \times \theta) :\ a \in G,\ \theta \in \mathrm{Aut}(G)\}. \tag{2}$$

Ito introduced and studied Hadamard groups in a series of papers [13–16]. Of special interest for us are *type Q* Hadamard matrices introduced in [12], which are related to *type Q* Hadamard groups $Q_{8t} = \langle a, b :\ a^{4t} = b^4 = 1, a^{2t} = b^2, b^{-1}ab = a^{-1}\rangle$ [15], obtained as the extension of $\langle b^2\rangle \cong \mathbb{Z}_2$ by the dihedral group $D_{4t} = \langle a, b :\ a^{2t} = b^2 = 1, ab = ba^{-1}\rangle$. Assume the ordering $\{1, a, \ldots, a^{2t-1}, b, ab, \ldots, a^{2t-1}b\}$ in D_{4t}. In these circumstances, Flannery identified in [7] that the cocycle ρ gives rise to this central extension, where

$$M_\rho = \begin{pmatrix} BN_{2t} & BN_{2t} \\ BN_{2t}^s & -BN_{2t}^s \end{pmatrix} \tag{3}$$

for BN_{2t} being the back negacyclic matrix of order $2t$, and BN_{2t}^s being the “symmetric” matrix obtained from BN_{2t}, by displaying its rows from bottom to top. Thus Q_{8t} is a Hadamard group if and only if $[\rho] \in H^2(D_{4t}; \mathbb{Z}_2)$ contains an orthogonal cocycle [7].

Ito conjectured in [15] that Q_{8t} is a Hadamard group for every integer t, so type Q Hadamard matrices would exist for every order $4t$ as well. Several works in terms of relative difference sets support this idea [22, 23]. As far as we know, the first undecided case is $t = 47$. Actually, notice that $4 \cdot 47$ is the first undecided order for the existence of a cocyclic Hadamard matrix, no matter the base group G is [9].

The purpose of this paper is studying D_{4t}-cocyclic matrices in a different way, attending to cocycles over D_{4t} directly.

In this paper, we describe some necessary and sufficient conditions for a set $\{\partial_{i_j}\}$ of coboundaries to yield a cocyclic Hadamard matrix $\psi = \rho \prod_{j=1}^{k} \partial_{i_j}$ over the dihedral group D_{4t}, for ρ as described in (3). Using this characterization and performing calculations in a conventional PC, new classification results for this cohomology class $[\rho]$ of cocycles over D_{4t} are obtained, extending existing exhaustive calculations for cocyclic Hadamard matrices over D_{4t} from order 36 to order 44. In light of the work in [6], it could happen that this characterization could enable looking for larger D_{4t}-cocyclic Hadamard matrices, provided high-performance computing techniques were used.

We organize the paper as follows.

The notion of r-path of coboundaries and the characterization of orthogonality in terms of r-paths of coboundaries for cocyclic matrices is described in Section 2.

In Section 3 we prove that D_{4t}-coboundaries $\{\partial_1, \ldots, \partial_{4t}\}$ may be organized in two subsets $\{\partial_1, \ldots, \partial_{2t}\}$ and $\{\partial_{2t+1}, \ldots, \partial_{4t}\}$, which naturally give rise to *ingredients* and *recipes*, in order to characterize any D_{4t}-cocyclic Hadamard matrix. This information will allow us to design an exhaustive search for D_{4t}-cocyclic Hadamard matrices for $t \leq 11$, so that the table in p. 132 in [9] is extended.

In Section 4 we will define four different transformations on D_{4t}-cocyclic Hadamard matrices that are Hadamard preserving (swapping, rotations, specular symmetries, and dilatations), so that the set of D_{4t}-cocyclic Hadamard matrices splits in orbits, which are completely characterized by any of their matrices. These transformations will be shown to correspond to Horadam's bundle equivalence operations together with transposition.

2. r-PATHS AND ORTHOGONALITY ON COCYCLIC MATRICES

The material presented next is drawn mainly from [1] for finite groups $G = \{g_1 = 1, \ldots, g_{4t}\}$ of order $4t$. Fixed a representative cocycle $[\rho] \in H^2(G; \mathbb{Z}_2)$, in order to look for G-cocyclic Hadamard matrices M_ψ, for $[\psi] = [\rho] \in H^2(G; \mathbb{Z}_2)$, it suffices to look for a subset of elementary coboundaries ∂_{i_j} such that $\psi = \rho \prod_{j=1}^{k} \partial_{i_j}$ is orthogonal. Since M_ψ is Hadamard if and only if the summation of each row (but the first) is zero (see the cocyclic Hadamard test in [11]), the coboundaries ∂_{i_j} should be selected in such a way that the number of negative entries at each row of M_ψ (but the first, formed all of 1s) is $2t$.

In [1] a way to count the number of negative entries is described, in terms of *paths* and *intersections* of coboundaries.

Notice that negating the $i_j th$ row of $M_{\partial_{i_j}}$ gives a matrix (termed a *generalized* coboundary matrix $\overline{M}_{\partial_{i_j}}$) with exactly two negative entries in each row, excepting the first one (consisting only of 1s). More concretely, the negative entries in the row $r \neq 1$ of $\overline{M}_{\partial_{i_j}}$ are located at the columns i_j and e, where $g_e = g_r^{-1} g_{i_j}$.

A set $\{\overline{M}_{\partial_{i_j}} : 1 \leq j \leq w\}$ of generalized coboundary matrices (or more briefly, simply the underlying set of coboundaries ∂_{i_j}) defines a *r-walk* if these matrices may be ordered in a sequence $(\overline{M}_{l_1}, \ldots, \overline{M}_{l_w})$ so that for every $1 \leq i \leq w-1$, consecutive matrices $\overline{M}_{l_i}$ and $\overline{M}_{l_{i+1}}$ share a negative entry at the rth row, precisely at the position (r, l_{i+1}). Such a walk is called a *path* if the initial (equivalently, the final) matrix shares a -1 entry with a generalized coboundary matrix, which is not in the walk itself, and a *cycle* otherwise.

Similarly, a position in which M_ρ and $\overline{M}_{\partial_{i_1}} \ldots \overline{M}_{\partial_{i_w}}$ share a common -1 in their rth row is called a *r-intersection*.

With this notation at hand, the following theorem follows.

Theorem 2.1. *[1] A G-cocyclic matrix $M_\psi = M_{\partial_{i_1}} \ldots M_{\partial_{i_w}} M_\rho$ is Hadamard if and only if*

$$2c_r - 2I_r = 2t - \rho_r, \ 2 \leq r \leq 4t, \tag{4}$$

where c_r is the number of maximal r-paths in $\{\overline{M}_{\partial_{i_1}}, \ldots, \overline{M}_{\partial_{i_w}}\}$, ρ_r is the number of -1s in the rth row of M_ρ, and I_r is the number of r-intersections generated by M_ρ and $\overline{M}_{\partial_{i_1}} \ldots \overline{M}_{\partial_{i_w}}$.

Notice that since ρ is a fixed representative cocycle $[\rho] \in H^2(G; \mathbb{Z}_2)$, the right-hand side of (4) is a constant vector $\mathbf{v_{RHS}} = \mathbf{v}_\rho$, and consequently does not depend on the choice of the coboundaries. However, the left-hand side $\mathbf{v_{LHS}}$ of (4) depends on the subset $\{M_{\partial_{i_j}}\}$ of coboundaries used. Actually, it depends on the way in which r-paths and r-intersections are formed, for $2 \leq r \leq 4t$.

It might occur that elementary coboundary matrices can be organized in $k > 1$ subsets $S_1, \ldots, S_k$, so that generalized coboundary matrices placed in different subsets never share negative entries at rows to be checked.

Consequently, the left-hand side $\mathbf{v_{LHS}}$ of (4) would decompose as a summation $\mathbf{v_{LHS}} = \mathbf{v}_1 + \cdots + \mathbf{v}_k$ of k vectors (called *ingredients*) of the form $\mathbf{v}_h = (2c_2^{(h)} - 2I_2^{(h)}, \ldots, 2c_{4t}^{(h)} - 2I_{4t}^{(h)})^T$, for $1 \le h \le k$, coming from the paths and intersections generated by the coboundaries in S_h along the rows $2 \le r \le 4t$. The matrix $(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ consisting of these ingredients is called a *recipe*.

In these circumstances, (4) is satisfied if and only if $\mathbf{v}_1 + \cdots + \mathbf{v}_k = \mathbf{v}_\rho$. If it is the case, the recipe is called a *Hadamard* recipe.

We now apply these notions to the case of D_{4t}-cocyclic matrices.

3. INGREDIENTS, RECIPES, AND DIAGRAMS

In what follows, we will consider D_{4t}-cocyclic matrices $M_\psi = M_{\partial_{i_1}} \ldots M_{\partial_{i_k}} M_\rho$ with $[\psi] = [\rho] \in H^2(D_{4t}; \mathbb{Z}_2)$, ρ as defined in (3). Though these are just a subset among D_{4t}-cocyclic matrices, there is computational evidence that most orthogonal cocycles over D_{4t} are of this type [1, 7].

The elementary coboundary matrices M_{∂_i} consist in negating both the ith row and column of the matrices

$$\begin{pmatrix} B_i & J_{2t} \\ J_{2t} & B_i P \end{pmatrix}, \; 1 \le i \le 2t, \qquad \begin{pmatrix} J_{2t} & B_{i-2t} \\ B_{i-2t} P & J_{2t} \end{pmatrix}, \; 2t+1 \le i \le 4t, \tag{5}$$

where B_i is the back circulant $2t \times 2t$ matrix with first row formed all of 1s excepting the ith-entry, and P is the back circulant $2t \times 2t$ permutation matrix with first row $(1, 0, \ldots, 0)$. Notice that the product used in $B_i P$ is the usual one of matrices, not the Hadamard (pointwise) product. Consequently, the generalized coboundary matrices $\overline{M}_{\partial_i}$ consist in negating the ith column of the matrices in (5).

A basis for coboundaries is given by $\{\partial_2, \ldots, \partial_{4t-2}\}$, since ∂_1, ∂_{4t-1} and ∂_{4t} may be expressed as a product of the remaining coboundaries:

$$\partial_1 = -\prod_{i=2}^{2t} \partial_i, \quad \partial_{4t-1} = \prod_{i=1}^{t} \partial_{2i} \prod_{i=1}^{t-1} \partial_{2t+2i-1}, \quad \partial_{4t} = \prod_{i=1}^{t} \partial_{2i} \prod_{i=1}^{t-1} \partial_{2t+2i}. \tag{6}$$

Every D_{4t}-cocyclic matrix M_ψ may be expressed in eight ways, precisely one of which uses none of M_{∂_1}, $M_{\partial_{4t-1}}$, and $M_{\partial_{4t}}$.

Next we clarify how r-paths may be formed from D_{4t}-coboundaries. Since the cocyclic Hadamard test (4) for D_{4t}-cocyclic matrices M_ψ just concerns rows from 2 to t [1], we only have to attend to r-paths, for $2 \le r \le t$.

Observe that each generalized coboundary matrix $\overline{M}_{\partial_i}$ has exactly two negative entries in each row r, $2 \le r \le t$, which are located at columns:

$$\begin{array}{l} \bullet\ i \text{ and } [i-(r-1)]_{2t} \text{ if } i \le 2t. \\ \bullet\ i \text{ and } 2t + [i-(r-1)]_{2t} \text{ if } i \ge 2t+1. \end{array} \tag{7}$$

Here we use $[m]_n$ instead of $m \bmod n$ for brevity. In addition, for convenience, we assume that $[0]_{2t} = 2t$.

From this, it is apparent that r-paths of D_{4t}-coboundaries consists in sequences of the type $(\ldots, \partial_i, \partial_{[i-(r-1)]_{2t}}, \ldots)$ or $(\ldots, \partial_{2t+i}, \partial_{2t+[i-(r-1)]_{2t}}, \ldots)$. Therefore, given a subset

S of D_{4t}-coboundaries, calculating the vector of r-paths, $2 \leq r \leq t$, that they generate takes $O(t)$ at worst.

Example 1. *For $t = 5$, consider the subset of coboundaries $\{\partial_2, \partial_5, \partial_7, \partial_{13}, \partial_{14}\}$. For convenience of the reader, we next include rows from 2 to 5 of the related generalized coboundary matrices $\overline{M}_{\partial_i}$, for $i = 2, 5, 7, 13, 14$. We use $-$ instead of -1 for brevity, and $\boxed{-}$ in case that this occurrence is shared with another generalized coboundary matrix in the subset.*

- $\overline{M}_{\partial_2}$:

$$\begin{pmatrix} - & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \boxed{-} & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

- $\overline{M}_{\partial_5}$:

$$\begin{pmatrix} 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & - & 1 & \boxed{-} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \boxed{-} & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ - & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

- $\overline{M}_{\partial_7}$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \boxed{-} & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & - & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

- $\overline{M}_{\partial_{13}}$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & \boxed{-} & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & - \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & 1 \end{pmatrix}$$

- $\overline{M}_{\partial_{14}}$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \boxed{-} & - & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - \end{pmatrix}$$

Now it may be easily checked that this subset of coboundaries defines four 2-paths ((∂_2), (∂_5), (∂_7), and ($\partial_{14}, \partial_{13}$)), four 3-paths (($\partial_2$), ($\partial_7, \partial_5$), ($\partial_{13}$), and ($\partial_{14}$)), four 4-paths (($\partial_5, \partial_2$), ($\partial_7$), ($\partial_{13}$), and ($\partial_{14}$)), and five 5-paths (($\partial_2$), ($\partial_5$), ($\partial_7$), ($\partial_{13}$), and ($\partial_{14}$)). This information may be reported as a vector of length 4, $(4, 4, 4, 5)^T$, the ith entry corresponding to the number of $(i+1)$-paths, $1 \leq i \leq 4$.

From (7), it is apparent that ∂_i would never form an r-path with ∂_{2t+j}, for any $1 \leq i, j \leq 2t$. In particular, the set of elementary coboundaries on D_{4t} may be split into two disjoint subsets $S_1 = \{\partial_1, \ldots, \partial_{2t}\}$ and $S_2 = \{\partial_{2t+1}, \ldots, \partial_{4t}\}$, so that r-paths are always formed from coboundaries belonging to the same subset S_m, for $2 \leq r \leq t$.

Due to this property, any D_{4t}-cocycle $\psi = \rho \prod_{j=1}^{k} \partial_{i_j}$ (respectively, D_{4t}-cocyclic matrix M_ψ) can be visualized as a $2 \times 2t$ matrix $D_\psi = (a_{ij})_{1 \leq i \leq 2,\ 1 \leq j \leq 2t}$, such that $a_{ij} = \times$ if $2t(i-1) + j \in \{i_1, \ldots, i_k\}$ and empty $(-)$ elsewhere. This matrix D_ψ is termed *diagram* in [2].

Remark 1. *The definition of diagram has to do with the expression of the matrix M_ψ in terms of the full set of elementary coboundaries, so every D_{4t}-cocyclic matrix has eight different diagrams (depending on the relations* (6)*).*

Example 2. *For instance, for $t = 5$, consider the (orthogonal) $D_{4\cdot 5}$-cocycle $\psi = \rho\partial_2\partial_5\partial_7\partial_{13}\partial_{14}$. A diagram D_ψ for ψ is*

$$\begin{vmatrix} - & \times & - & - & \times & - & \times & - & - & - \\ - & - & \times & \times & - & - & - & - & - & - \end{vmatrix}. \tag{8}$$

Diagrams will play an essential role in defining some operations preserving orthogonality on D_{4t}-cocyclic Hadamard matrices in Section 4.

The adaptation of the cocyclic Hadamard test (4) to D_{4t}-cocyclic matrices gives

$$c_r - I_r = t - (r-1), \ 2 \leq r \leq t. \tag{9}$$

Since r-paths and intersections may be determined independently for coboundaries in subsets S_1 and S_2 (i.e., appearing in rows 1 or 2 of a diagram), given any D_{4t}-cocyclic matrix $M_\psi = M_\rho \prod_{j=1}^{k} M_{\partial_{i_j}}$ one could calculate the column vectors (ingredients) $\mathbf{v}_h = (c_2^{(h)} - I_2^{(h)}, \ldots, c_t^{(h)} - I_t^{(h)})^T$, $1 \leq h \leq 2$, whose entries correspond to the left-hand side of equation (9) applied to the subset S_h. Thus ψ is orthogonal if and only if $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_\rho = (t-1, \ldots,\ 1)^T$. If this is the case, the matrix $(\mathbf{v}_1, \mathbf{v}_2)$ is called a *Hadamard* recipe.

Notice that since r-intersections consist in those negative occurrences that are shared by r-paths and negative entries at the rth row of M_ρ, then the vector of r-intersections, $2 \leq r \leq t$, may be straightforwardly calculated by means of a loop on the set of r-paths, which takes $O(t)$ at worst.

Example 3. *For $t = 5$, $\psi = \rho\partial_2\partial_5\partial_7\partial_{13}\partial_{14}$ is an orthogonal D_{4t}-cocycle.*

The set of coboundaries $\{\partial_2\partial_5\partial_7\partial_{13}\partial_{14}\}$ splits into $S_1 = \{\partial_2\partial_5\partial_7\}$ and $S_2 = \{\partial_{13}\partial_{14}\}$.

In order to see that ψ is an orthogonal $D_{4\cdot 5}$-cocycle, we calculate the ingredients $\mathbf{v}_1 = \mathbf{c}^{(1)} - \mathbf{I}^{(1)}$ and $\mathbf{v}_2 = \mathbf{c}^{(2)} - \mathbf{I}^{(2)}$ related to this cocycle, as the difference of the corresponding vectors consisting of r-paths and r-intersections, $2 \leq r \leq 5$. And check whether $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_\rho = (4, 3, 2, 1)^T$.

Taking into account the calculations in Example 1, we know that the vectors $\mathbf{c}^{(1)}$ and $\mathbf{c}^{(2)}$ of r-paths, for $2 \leq r \leq 5$, generated by these subsets of coboundaries are $\mathbf{c}^{(1)} = (3, 2, 2, 3)^T$ and $\mathbf{c}^{(2)} = (1, 2, 2, 2)^T$. Notice that the vector $(4, 4, 4, 5)^T$ obtained in Example 1 consists in the summation $\mathbf{c}^{(1)} + \mathbf{c}^{(2)}$.

We now compute the vectors $\mathbf{I}^{(h)}$ of r-intersections, for $2 \leq r \leq 5$, $1 \leq h \leq 2$. For convenience of the reader, we next include rows from 2 to 5 of M_ρ and the product

$\overline{M} = \overline{M}_{\partial_2}\overline{M}_{\partial_5}\overline{M}_{\partial_7}\overline{M}_{\partial_{13}}\overline{M}_{\partial_{14}}$. *We use* $-$ *instead of* -1 *for brevity, and* $\boxed{-}$ *in case that this occurrence gives an intersection (i.e., this negative entry appears in both matrices).*

- M_ρ:

$$\begin{pmatrix}
1&1&1&1&1&1&1&1&1&-&1&1&1&1&1&1&1&1&1&-\\
1&1&1&1&1&1&1&1&-&\boxed{-}&1&1&1&1&1&1&1&1&-&-\\
1&1&1&1&1&1&1&-&\boxed{-}&-&1&1&1&1&1&1&1&-&-&\boxed{-}\\
1&1&1&1&1&1&\boxed{-}&\boxed{-}&-&-&1&1&1&1&1&1&-&-&\boxed{-}&\boxed{-}
\end{pmatrix}$$

- $\overline{M}$:

$$\begin{pmatrix}
-&-&1&-&-&-&-&1&1&1&1&-&1&-&1&1&1&1&1&1\\
1&-&-&1&1&1&-&1&1&\boxed{-}&-&-&-&-&1&1&1&1&1&1\\
1&1&1&-&-&1&-&1&\boxed{-}&1&-&1&-&-&1&1&1&1&1&\boxed{-}\\
-&-&-&1&-&1&\boxed{-}&\boxed{-}&1&1&1&1&-&-&1&1&1&1&\boxed{-}&\boxed{-}
\end{pmatrix}$$

Since $\mathbf{I}^{(1)} = (0, 1, 1, 2)^T$ *and* $\mathbf{I}^{(2)} = (0, 0, 1, 2)^T$, *the set of coboundaries* $\{\partial_2, \partial_5, \partial_7\}$ *gives rise to the ingredient* $\mathbf{v}_1 = (3, 2, 2, 3)^T - (0, 1, 1, 2)^T = (3, 1, 1, 1)^T$. *And the set of coboundaries* $\{\partial_{13}, \partial_{14}\}$ *gives rise to the ingredient* $\mathbf{v}_2 = (1, 2, 2, 2)^T - (0, 0, 1, 2)^T = (1, 2, 1, 0)^T$. *Since* $\mathbf{v}_1 + \mathbf{v}_2 = (3, 1, 1, 1)^T + (1, 2, 1, 0)^T = (4, 3, 2, 1)^T = \mathbf{v}_\rho$, *then* $(\mathbf{v}_1\mathbf{v}_2)$ *is a Hadamard recipe,* ψ *an orthogonal* $D_{4\cdot 5}$*-cocycle, and* M_ψ *a* $D_{4\cdot 5}$*-cocyclic Hadamard matrix.*

Remark 2. *Notice that the notion of recipe does not depend on the order of its ingredients. Consequently, the set of ingredients may be calculated attending just to subsets of coboundaries in* S_1.

Now it is straightforward to design an algorithm searching exhaustively for D_{4t}-cocyclic Hadamard matrices M_ψ, $[\psi] = [\rho] \in H^2(D_{4t}; \mathbb{Z}_2)$.

Algorithm 1. Exhaustive search for orthogonal D_{4t}-cocycles ψ.
INPUT: t.

1. Compute the list $\mathcal{V}_t = \{\mathbf{v} :\ \exists S / \mathbf{v}_S = \mathbf{v}\}$ of ingredients, and classify every subset S of coboundaries in $\{\partial_1, \ldots, \partial_{2t}\}$ attending to the ingredient $\mathbf{v}_S$ that it produces, $\mathcal{S}_\mathbf{v} = \{S :\ \mathbf{v}_S = \mathbf{v}\}$.
2. Determine the set $\mathcal{H}_t$ of (unordered) Hadamard recipes $(\mathbf{v}, \mathbf{w})$.
3. Construct all pairs of subsets (S, R) producing a Hadamard recipe $(\mathbf{v}_S, \mathbf{w}_R)$, $\{(S, R) : \forall\, (\mathbf{v}, \mathbf{w}) \in \mathcal{H}_t,\ \forall\, S \in \mathcal{S}_\mathbf{v},\ \forall\, R \in \mathcal{S}_\mathbf{w}\}$.
4. Both $\psi_{S,R} = \rho \prod_{j \in S \cup \{2t+i:\ i \in R\}} \partial_j$ and $\psi_{R,S} = \rho \prod_{j \in R \cup \{2t+i:\ i \in S\}} \partial_j$ are orthogonal D_{4t}-cocycles.

OUTPUT: the full set of orthogonal D_{4t}-cocycles in $[\rho]$.

Remark 3. *The time consuming parts of this algorithm are steps 1 and 2:*

1. In order to construct the exhaustive list $\mathcal{V}_t$ of ingredients $\mathbf{v}$, one needs to construct the ingredients $\mathbf{v}_S$ provided by each possible subset S of D_{4t}-coboundaries. Recall that given a subset of D_{4t}-coboundaries S, calculating the related ingredient $\mathbf{v}_S$ takes $O(t)$, as it was pointed out before. Unfortunately, the number of candidate subsets S is very large, less than 2^{4t-3} but close to this number (in [1] the number of coboundaries

TABLE I. **D_{4t}-cocyclic Hadamard matrices from Algorithm 1.**

t	$\lvert\mathcal{V}_t\rvert$	$\lvert\mathcal{H}_t\rvert$	$\#M_\psi$	$i_j:\ \psi=\rho\prod_{j=1}^{k}\partial_{i_j}$
3	5	1	72	2, 9
4	12	2	512	3, 5, 11
5	34	6	1,400	3, 5, 14, 17
6	96	13	7,488	3, 6, 16, 17, 19, 21
7	317	17	11,368	4, 6, 9, 11, 17, 20, 21
8	1,040	52	69,632	3, 5, 7, 10, 11, 13, 19, 24, 25
9	3,341	75	130,248	2, 6, 11, 13, 20, 23, 26, 27, 30, 31, 32
10	12,398	234	521,600	4, 6, 8, 11, 15, 23, 26, 31, 32, 33, 36
11	41,821	290	619,564	3, 9, 10, 11, 18, 24, 26, 29, 31, 34, 35, 38, 39, 41

TABLE II. **Orbits of D_{4t}-cocyclic Hadamard matrices.**

t	No. of orbits	Matrices/orbits	No. of Hadamard matrices	$32t^2\phi(2t)$
3	1	72×1	72	576
4	1	512×1	512	2,048
5	3	$200\times 1+400\times 1+800\times 1$	1,400	3,200
6	8	$576\times 3+1{,}152\times 5$	7,488	4,608
7	5	$784\times 1+1{,}176\times 1+2{,}352\times 2+4{,}704\times 1$	11,368	9,408
8	13	$4{,}096\times 9+8{,}192\times 4$	69,632	16,384
9	20	$1{,}944\times 1+3{,}888\times 5+7{,}776\times 14$	130,248	15,552
10	59	$1{,}600\times 2+3{,}200\times 4+6{,}400\times 27+12{,}800\times 26$	521,600	25,600
11	34	$9{,}680\times 4+19{,}360\times 30$	619,520	38,720

whose combination yields a D_{4t}-cocyclic Hadamard matrix is proved to run in the range $[t-1, 3t-2]$). Thus constructing the complete list $\mathcal{V}_t$ of ingredients $\mathbf{v}$ may require a long time.

2. Once the complete list $\mathcal{V}_t$ of ingredients is available, step 2 requires to locate two ingredients such that their summation is equal to $(t-1,\dots,1)^T$. If $\mathcal{V}$ consists of $|\mathcal{V}_t|$ ingredients, looking for the full set of Hadamard recipes takes $O(|\mathcal{V}_t|^2)$. This may be impractical for large $|\mathcal{V}_t|$.

Table I shows a report of Algorithm 1 running for $3\le t\le 11$. Column 2 indicates the number $|\mathcal{V}_t|$ of different ingredients $\mathbf{v}$, which are produced by subsets in $S_1=\{\partial_1,\dots,\partial_{2t}\}$. Column 3 indicates the number $|\mathcal{H}_t|$ of different (unordered) Hadamard recipes $(\mathbf{v},\mathbf{w})$, which may be formed from these ingredients. Column 4 gives the number of different D_{4t}-cocyclic Hadamard matrices M_ψ obtained from these recipes. And column 5 includes an explicit subset of indices $\{i_j\}$ such that the D_{4t}-cocycle $\psi=\rho\prod_{j=1}^{k}\partial_{i_j}$ is orthogonal. All calculations have been performed in a 64 bits Intel(R) Core(TM) i3 CPU M330 2.13GHz RAM 4Gb system.

4. OPERATIONS

In Remark 2 it was noted that a recipe does not depend on the order of its ingredients. In other words, this comes to say that if $D_{s(\psi)}$ is the diagram obtained by swapping the rows of the diagram D_ψ, then ψ is orthogonal if and only if $s(\psi)$ is.

This is just one of the multiple orthogonality-preserving geometric operations that may be defined on diagrams D_ψ. The idea is looking for operations on ψ, which do not modify the final recipe. In fact, in light of the operations described in [2], detecting operations on D_{4t}-cocyclic matrices that preserve r-paths, $2 \leq r \leq t$, is not so difficult. Unfortunately, unlike the case of $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrices, for which four Hadamard-preserving operations were defined depending solely on r-paths [2], the situation here is more subtle. Actually, since the Hadamard test (9) depends not only on the number c_r of r-paths, but also on the number I_r of intersections, the difficult question here is finding out those operations that fairly preserve the difference between r-paths and intersections along rows r, $2 \leq r \leq t$.

At the beginning, we were able to detect three geometric operations of this type: *swapping*, *rotations*, and *specular symmetries*. The authors were convinced that a fourth operation might exist, acting as a homothecy on coboundaries, as it was in the case of *dilatations* for $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrices [2]. In fact, we defined dilatations acting on coboundaries, and r-paths were somehow preserved (we obtained a permutation of the original vector of r-paths). However, we could not determine how dilatations should affect ρ in order to properly shift intersections, in addition. In the end, comparison of these operations and bundles (2) as in [3] took us to the right direction. This provided us essential information to find out an explicit formula for dilatations on D_{4t}-cocyclic matrices.

Let $\psi = \rho \prod_{j=1}^{k} \partial_{i_j}$ be a D_{4t}-cocycle. Denote $\mathbf{c_1} = \{i_j : 1 \leq i_j \leq 2t\}$ and $\mathbf{c_2} = \{i_j : 2t+1 \leq i_j \leq 4t\}$ the indices of any subset of coboundaries defining ψ (recall that there are eight different forms to express ψ in terms of coboundaries, see (6)). Let $\mathbf{c_j} + \mathbf{k}$ denote the set obtained by adding k to each element of $\mathbf{c_j}$ modulo $2t$ (here, one must assume $[0]_{2t} = 2t$ in $\mathbf{c_1}$, whereas $[0]_{2t} = 4t$ in $\mathbf{c_2}$).

Definition 4.1. *Let $\{(\mathbf{c_1}, \mathbf{c_2})\}$ be a set of indices related to a D_{4t}-cocycle ψ, and call $\mathcal{C}_i$ the columns of the diagram $D_\psi = (\mathcal{C}_0, \cdots, \mathcal{C}_{2t-1})$.*

1. The *swapping* $s(\{\mathbf{c_1}, \mathbf{c_2}\})$ is the set $\{(\mathbf{c_2} - \mathbf{2t}, \mathbf{c_1} + \mathbf{2t})\}$.
2. For $0 \leq i, j \leq t-1$, the (i, j)-*rotation* $\mathrm{T}_{ij}(\{\mathbf{c_1}, \mathbf{c_2}\})$ is the set obtained from $\{(\mathbf{c_1} + \mathbf{i}, \mathbf{c_2} + \mathbf{j})\}$ by adding the indices $(\{1, \ldots, i\}, \{2t+1, \ldots, 2t+j\})$.
3. For $0 \leq i, j \leq 1$, the (i, j)-*specular symmetry* $\sigma_{ij}(\{\mathbf{c_1}, \mathbf{c_2}\})$ of this set is the set $\{((\mathbf{2t+1})\mathbf{i} + (-\mathbf{1})^{\mathbf{i}}\mathbf{c_1}, (\mathbf{6t+1})\mathbf{j} + (-\mathbf{1})^{\mathbf{j}}\mathbf{c_2})\}$.
4. The j*th dilatation* $\mathrm{V}_j(\{\mathbf{c_1}, \mathbf{c_2}\})$, for $j \in \mathbb{Z}_{2t}^*$, is obtained by modifying the diagram $(\mathcal{C}_0, \ldots, \mathcal{C}_{[(2t-1)j^{-1}]_{2t}})$ (resulting from the homothecy of ratio j applied to D_ψ), in the following way:
   (i) Compute the numbers $I = \{[1-j]_{2t}, [1-2j]_{2t}, \ldots, [1-j^{-1}j]_{2t}\}$.
   (ii) Reorder all the nonzero numbers $i \in I$ as $i_1 < i_2 < \ldots < i_{2s}$, for $2s = j^{-1} - 1 \bmod 2t$.
   (iii) For $1 \leq k \leq s$, change in the diagram the positions corresponding to $\partial_{i_{2k-1}+1}, \ldots, \partial_{i_{2k}}$ and $\partial_{2t+i_{2k-1}+1}, \ldots, \partial_{2t+i_{2k}}$.

In terms of diagrams, s swaps the rows; T_{ij} cyclically shifts the first row i places to the right and/or the second row j places to the right, and adds ∂_1 (if $i = 1$) and/or ∂_{2t+1} (if $j = 1$); σ_{ij}, depending on whether $i = 1$ and/or $j = 1$, displays rows 1 and/or 2 of the diagram in reverse order; and V_j initially permutes columns according to multiplication of column index by the invertible element j, and adds some extra coboundaries to fairly shift the cocycle ρ (that is, r-intersections, $2 \le r \le t$).

For instance, if $D_\psi = \begin{vmatrix} - & \times & - & - & \times & - & \times & - & - & - \\ - & - & \times & \times & - & - & - & - & - & - \end{vmatrix}$ is the diagram in (8), then

- $s(D_\psi) = \begin{vmatrix} - & - & \times & \times & - & - & - & - & - & - \\ - & \times & - & - & \times & - & \times & - & - & - \end{vmatrix}$.
- $\mathrm{T}_{13}(D_\psi) = \begin{vmatrix} \times & - & \times & - & - & \times & - & \times & - & - \\ \times & \times & \times & - & - & \times & \times & - & - & - \end{vmatrix}$.
- $\sigma_{01}(D_\psi) = \begin{vmatrix} - & \times & - & - & \times & - & \times & - & - & - \\ - & - & - & - & - & - & \times & \times & - & - \end{vmatrix}$.
- $\mathrm{V}_3(D_\psi) = \begin{vmatrix} - & - & \times & \times & - & \times & - & - & - & - \\ - & - & \times & - & - & \times & \times & - & \times & \times \end{vmatrix}$.

Clearly the order of the swapping s is 2 and $\langle s \rangle \cong \mathbb{Z}_2$. Each of the elementary rotations T_{10} and T_{01} has order $2t$, and generate a group $\langle T_{10}, T_{01} \rangle \cong \mathbb{Z}_{2t} \times \mathbb{Z}_{2t}$. The specular symmetries each have order 2 and generate a group $\langle \sigma_{10}, \sigma_{01} \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. And the dilatations $\langle \mathrm{V}_j, j \in \mathbb{Z}_{2t}^* \rangle \cong \mathbb{Z}_{2t}^*$. It is straightforward to check that $s\sigma_{10} = \sigma_{01}s$, $\mathrm{T}_{10}s = s\mathrm{T}_{01}$, $\mathrm{T}_{mn}\sigma_{mn} = \sigma_{mn}\mathrm{T}_{mn}^{2t-1}$, $\mathrm{T}_{mn}^{j}\mathrm{V}_j = \mathrm{V}_j\mathrm{T}_{mn}$, and $\mathrm{T}_{mn}^{j-1}\mathrm{V}_j\sigma_{mn} = \sigma_{mn}\mathrm{V}_j$, for $(m, n) \in \{(1, 0), (0, 1), (1, 1)\}$, $j \in \mathbb{Z}_{2t}^*$. These relations give a presentation of the group $Diag(D_{4t})$ of diagrammatic operations on diagrams D_ψ of D_{4t}-cocycles ψ. The most remarkable fact is that every operation in $Diag(D_{4t})$ preserves orthogonal cocycles.

Theorem 4.2. *For every $f \in Diag(D_{4t})$, D_ψ defines an orthogonal cocycle ψ if and only if $f(D_\psi) = D_{f(\psi)}$ does.*

Proof. The swapping s simply swaps the ingredients of a recipe.

Given a diagram D_ψ, it is apparent that rotating independently any of its rows does not affect the way in which r-paths are formed, $2 \le r \le t$. However, intersections may be affected by this rotation, unless a way to rotate M_ρ fairly with coboundary matrices $M_{\partial_{i_j}}$ is found. Notice that the effect of $M_\rho M_{\partial_1}$ is rotating the block of negative entries of M_ρ one column to the right cyclically in the range $[1, 2t]$. Similarly, the effect of $M_\rho M_{\partial_{2t+1}}$ is rotating the block of negative entries of M_ρ one column to the right cyclically in the range $[2t + 1, 4t]$. This way, as defined, the operation T_{ij} rotates simultaneously both r-paths and intersections, and hence preserve ingredients and recipes.

Every specular symmetry σ_{ij} preserves not only r-paths (due to the fact that the relative positions of coboundaries remain unchanged), but also intersections (notice that a generalized coboundary matrix $\overline{M}_{\partial_k}$ shares a negative entry with M_ρ at row r if and only $\overline{M}_{\sigma_{ij}(\partial_k)}$ does, see (7)), and consequently ingredients and recipes as well.

Finally, every dilatation V_j acts in two steps. First, it permutes columns of the diagram according to multiplication of column index by the invertible element j. In particular, r-paths turn into $(r - 1)j$-paths. Second, in order to shift intersections accordingly, some coboundaries are introduced, so that negative entries in M_ρ are properly moved. Thus the initial ingredients are permuted (according to the homothecy of ratio j), and a valid recipe is obtained. $\square$

In fact, the group $Bund(D_{4t})$ of bundles (2) of D_{4t}-cocyclic matrices may be identified as a subgroup of $Diag(D_{4t})$.

Theorem 4.3. *$Diag(D_{4t})$ strictly contains the group of bundles $Bund(D_{4t}) = D_{4t} \rtimes \mathrm{Aut}(D_{4t})$, $f_{a^ib^j,\theta}(\psi) = (\psi \cdot a^ib^j) \circ (\theta \times \theta)$, $f'_{a^ib^j,\theta}(\psi) = (a^ib^j \cdot \psi) \circ (\theta \times \theta)$ such that*

$$\begin{aligned} f_{a^ib^j,\theta}(\psi)(g,h) &= \psi(a^ib^j\theta(g), \theta(h))\psi(a^ib^j, \theta(h)), \\ f'_{a^ib^j,\theta}(\psi)(g,h) &= \psi(\theta(g), \theta(h)a^ib^j)\psi(\theta(g), a^ib^j). \end{aligned} \tag{10}$$

More concretely:

1. *The right shift action on a consists in $f_{a,1} = T_{11}^{-1}$.*
2. *The right shift action on b consists in $f_{b,1} = T_{11}\sigma_{11}s$.*
3. *The left shift action on a consists in $f'_{a,1} = T_{10}^{-1}T_{01}$.*
4. *The left shift action on b consists in $f'_{b,1} = T_{10}^{-1}T_{01}s$.*
5. *The automorphism action $\theta_{a,ab}$ such that $\theta_{a,ab}(a) = a$, $\theta_{a,ab}(b) = ab$, consists in $f_{1,\theta_{a,ab}} = T_{01}$.*
6. *For $j \in \mathbb{Z}_{2t}^*$, the automorphism action $\theta_{a^j,b}$ such that $\theta_{a^j,b}(a) = a^j$, $\theta_{a^j,b}(b) = b$, consists in $f_{1,\theta_{a^j,b}} = V_{j^{-1}}$.*
7. *Consequently, $Bund(D_{4t}) \cong \langle T_{ij}, \sigma_{11}, s, V_j \rangle \subset Diag(D_{4t})$. Hence the single specular symmetries σ_{10} and σ_{01} define genuinely new operations on D_{4t}-cocycles, not included in bundles* (10).

Proof. This may be checked by direct inspection. □

Thus there seems to be a gap between geometric and algebraic operations on D_{4t}-cocycles. This is by no means the case, as it may be straightforwardly checked:

Theorem 4.4. *For every ψ in the circumstances above, the dual cocycle ψ^* (defined as $\psi^*(g,h) = \psi(h^{-1}, g^{-1})$), coincides with $T_{10}\sigma_{10}(\psi)$. Consequently, the group generated by duals and bundles of D_{4t}-cocycles ψ coincides with the group $Diag(D_{4t})$ of geometric transformations on diagrams D_ψ.*

The *total orbit* of a cocyclic Hadamard matrix over D_{4t} is the union of all orbits under the action of swapping, rotations, specular symmetries, and dilatations.

Table A1 shows an exhaustive calculation of D_{4t}-cocyclic Hadamard orbits for t, $3 \leq t \leq 11$, split according to its size. For instance, for $t = 6$, $576 \times 3 + 1,152 \times 5$ means three orbits with 576 matrices each, plus five orbits with 1,152 matrices each, totalizing 7,488 matrices. The maximum theoretical possible size of one orbit (i.e., $|Diag(D_{4t})|$) is displayed in the last column. It is apparent that this theoretical size is never reached up to $t = 11$, though it seems that there always exist orbits as large as a half of this size, for $t \geq 6$. This implies that occasionally the same matrix can be obtained from a given one by means of different sequences of operations in $Diag(D_{4t})$ (this is obvious for $t = 3, 4, 5$).

Table A1 in Appendix includes a list of subsets of indices $\{i_j\}$ for those cocycles $\rho \prod_{j=1}^{k} \partial_{i_j}$ generating each orbit in Table II.

Another question is how do operations interact with recipes. Apart from the swapping and specular symmetry operations (which obviously preserve the recipe, up to swapping its ingredients), we have not found any explicit relation between orbits and recipes. Table III shows the number of recipes and the number of orbits for t, $3 \leq t \leq 11$.

TABLE III. Orbits and recipes of D_{4t}-cocyclic Hadamard matrices.

t	3	4	5	6	7	8	9	10	11
No. of orbits	1	1	3	8	5	13	20	59	34
No. of recipes	1	2	6	13	17	52	75	234	290

We can observe that the number of recipes is significantly larger than the number of orbits in a concrete t, excepting the case $t = 3$. This is not surprising. Actually, many different recipes contribute matrices to a common orbit. For instance, for $t = 5$, consider the orthogonal $D_{4\cdot5}$-cocycle ψ consisting in $\psi = \rho\partial_2\partial_5\partial_7\partial_{13}\partial_{14}$, which defines the recipe $\begin{pmatrix} 3 & 1 \\ 1 & 2 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$. It may be straightforwardly checked that $T^9_{01}(\psi) = \rho\partial_2\partial_5\partial_7\partial_{11}\partial_{14}\partial_{15}\partial_{16}\partial_{17}\partial_{18}\partial_{19}$, which, by means of (6), can be expressed as $\rho\partial_4\partial_5\partial_6\partial_7\partial_8\partial_{10}\partial_{13}\partial_{14}\partial_{16}\partial_{18}$, which defines the recipe $\begin{pmatrix} 1 & 3 \\ 1 & 2 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$.

Nevertheless, it may also occur that orthogonal D_{4t}-cocycles belonging to different orbits define a common recipe. For instance, for $t = 9$, it may be checked that $\rho\partial_4\partial_7\partial_9\partial_{13}\partial_{22}\partial_{25}\partial_{27}\partial_{30}\partial_{31}$ and $\rho\partial_4\partial_7\partial_9\partial_{13}\partial_{21}\partial_{23}\partial_{29}\partial_{30}\partial_{32}\partial_{33}$ define two different orbits of length $3,888$. However, these cocycles define the same recipe, since they share the same first ingredient (which depends only on $\partial_4\partial_7\partial_9\partial_{13}$), and hence the second as well, and consequently the full recipe.

5. CONCLUSIONS

In this paper, we have developed a new description of D_{4t}-cocyclic matrices, in terms of ingredients and recipes. In addition, we have been able to describe four geometric operations (swapping, rotations, specular symmetries, and dilatations), which strictly include the bundles generated by shiftings and automorphisms actions of [9]. Furthermore, we have also proved that the groups generated by the algebraic and geometric operations are isomorphic, if bundles are enriched with duals of cocycles.

We have performed an exhaustive search for D_{4t}-cocyclic Hadamard matrices M_ψ (classified up to orbits of these geometric operations) for $t \leq 11$, in the case of $\psi \in [\rho]$ as described in (3). As far as we know, the cases $t = 10$ and $t = 11$ have not been constructed previously.

It is remarkable that these calculations have been performed on a conventional PC. If high-performance computing systems were available, it is conceivable that larger D_{4t}-cocyclic Hadamard matrices might be found, the cases $t = 47, 167$ being of maximal interest, since no cocyclic Hadamard matrix is known of order $4 \cdot 47$, and no Hadamard matrix is known of order $4 \cdot 167$. Actually, in [6] this kind of resources has been used to find some new orders of Hadamard matrices in the following way:

- Given four large (up to 10 millions lines each) text files F_i, $1 \leq i \leq 4$ consisting of vectors $\mathbf{v}$, one has to find four vectors $\mathbf{v}_i \in L_i$, $1 \leq i \leq 4$ such that $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{w}$, for certain constant vector $\mathbf{w}$ previously fixed.
- A new efficient matching algorithm based on hashing techniques is used to solve this problem (actually, for $t = 251, 631$ for usual Hadamard matrices).

In our approach, as soon as a single file F of ingredients is provided, it suffices to find a Hadamard recipe, that is, two ingredients $\mathbf{v}, \mathbf{w} \in F$ such that $\mathbf{v} + \mathbf{w} = (t-1, \ldots, 1)^T$. In light of the work in [6], it seems that this could be certainly feasible for values t significantly greater than 11.

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APPENDIX

TABLE A1. Explicit D_{4t}-cocycles generating orbits(I).

t	$\{i_j : \psi = \rho \prod_{j=1}^{k} \partial_{i_j}\}$
3	{2, 9}
4	{3, 11, 13}
5	{3, 5, 14, 17}, {2, 6, 12, 15, 16}, {5, 12, 14, 17, 18}
6	{3, 6, 16, 17, 19, 21}, {3, 5, 9, 15, 17, 18}, {2, 6, 7, 15, 17, 22}, {2, 6, 7, 9, 15, 20}, {3, 7, 9, 17, 20}, {2, 4, 7, 16, 19, 20}, {4, 6, 9, 16, 19}, {2, 4, 7, 8, 16, 19}
7	{4, 6, 9, 11, 17, 20, 21}, {2, 6, 11, 16, 19, 22, 23, 24}, {2, 6, 10, 18, 21, 23, 24}, {3, 9, 11, 17, 22, 23, 25}, {3, 5, 7, 12, 17, 22, 23}
8	{3, 5, 7, 10, 11, 13, 19, 24, 25}, {3, 5, 9, 10, 11, 13, 19, 24, 27}, {3, 5, 7, 8, 11, 13, 19, 25, 26}, {3, 5, 11, 13, 21, 24, 25, 28}, {2, 4, 7, 11, 12, 13, 20, 23, 26}, {2, 5, 9, 10, 11, 13, 20, 23, 28}, {2, 4, 7, 8, 10, 14, 20, 25, 26}, {3, 6, 9, 11, 13, 19, 27, 28, 29}, {4, 6, 7, 9, 13, 21, 24, 28}, {2, 6, 7, 9, 11, 19, 22, 28, 29}, {3, 5, 11, 13, 19, 22, 23, 24, 27}, {3, 6, 9, 11, 13, 20, 25, 26}, {2, 4, 8, 11, 12, 20, 21, 23, 26}
9	{6, 8, 11, 13, 22, 23, 26, 29, 33}, {2, 6, 11, 13, 20, 23, 26, 27, 30, 31, 32}, {4, 7, 9, 11, 21, 23, 24, 27, 31, 32}, {4, 7, 9, 13, 22, 25, 27, 30, 31}, {4, 7, 9, 13, 21, 23, 29, 30, 32, 33}, {2, 4, 9, 10, 12, 20, 23, 24, 29, 32, 33}, {2, 4, 8, 11, 12, 14, 23, 24, 27, 32}, {2, 4, 8, 11, 12, 14, 21, 23, 24, 28, 29}, {2, 6, 9, 10, 12, 14, 22, 23, 28, 31}, {3, 5, 7, 10, 11, 13, 21, 26, 30, 31}, {2, 6, 8, 11, 12, 13, 20, 23, 24, 27, 32}, {3, 5, 9, 10, 11, 13, 20, 23, 24, 27, 32}, {3, 6, 8, 12, 23, 24, 27, 28, 30, 32}, {3, 8, 10, 16, 23, 24, 27, 28, 30, 32},

(*Continued*)

TABLE A1. Continued

t	$\{i_j : \psi = \rho \prod_{j=1}^{k} \partial_{i_j}\}$
	{2, 6, 9, 10, 11, 15, 20, 23, 25, 26, 32}, {3, 8, 9, 12, 21, 23, 24, 28, 30, 32}, {3, 8, 10, 12, 21, 23, 24, 27, 31, 32}, {4, 6, 9, 12, 21, 22, 26, 28, 30, 31}, {3, 5, 9, 12, 14, 21, 25, 26, 27, 34}, {3, 5, 8, 12, 14, 22, 24, 25, 28, 29}
10	{2, 6, 9, 13, 14, 15, 24, 27, 30, 32, 37}, {3, 6, 10, 12, 13, 14, 16, 24, 25, 27, 32, 37}, {4, 6, 10, 13, 14, 22, 25, 30, 31, 34, 36}, {3, 4, 10, 12, 13, 23, 25, 27, 30, 31, 34, 36}, {2, 7, 8, 11, 15, 25, 27, 30, 31, 34, 36}, {2, 5, 6, 10, 12, 13, 15, 17, 25, 28, 29, 34}, {2, 4, 7, 11, 13, 14, 15, 23, 26, 31, 35, 36}, {5, 8, 10, 14, 24, 25, 28, 31, 32, 34, 36}, {4, 6, 9, 11, 12, 13, 17, 25, 28, 32, 35}, {2, 5, 7, 10, 11, 13, 17, 24, 27, 28, 29, 36}, {2, 6, 8, 9, 11, 13, 14, 17, 25, 30, 34, 35}, {4, 5, 8, 10, 13, 15, 17, 26, 27, 30, 33}, {6, 8, 10, 15, 23, 25, 28, 31, 32, 35, 36}, {2, 5, 10, 14, 15, 23, 25, 30, 31, 32, 34, 38}, {2, 5, 10, 14, 15, 24, 26, 27, 30, 31, 33, 35}, {2, 5, 6, 10, 12, 14, 15, 16, 22, 25, 32, 35, 36}, {2, 7, 8, 12, 15, 22, 24, 28, 31, 34, 35, 36}, {3, 5, 12, 13, 16, 22, 24, 28, 31, 34, 35, 36}, {2, 7, 10, 11, 16, 22, 25, 26, 32, 33, 35, 37}, {2, 6, 9, 10, 13, 15, 23, 25, 28, 35, 36, 37}, {2, 5, 6, 9, 11, 13, 15, 16, 23, 24, 29, 31, 32}, {3, 4, 9, 11, 14, 24, 26, 28, 29, 32, 33, 35}, {2, 4, 6, 9, 10, 13, 15, 18, 23, 24, 30, 31, 32}, {3, 4, 10, 11, 13, 23, 25, 28, 29, 32, 34, 36}, {3, 5, 8, 11, 12, 24, 25, 28, 30, 32, 34, 35}, {3, 6, 8, 9, 13, 23, 27, 29, 31, 34, 35, 36}, {3, 5, 9, 11, 14, 23, 24, 27, 32, 34, 35, 36}, {2, 5, 8, 12, 13, 15, 17, 23, 27, 28, 29, 34}, {3, 6, 7, 9, 11, 13, 16, 24, 26, 27, 31, 32}, {3, 7, 9, 13, 14, 16, 23, 29, 33, 34, 36}, {4, 7, 9, 10, 17, 24, 27, 31, 33, 35, 36}, {2, 4, 8, 9, 11, 16, 17, 23, 30, 33, 34, 36}, {2, 5, 12, 13, 15, 17, 22, 26, 27, 32, 33, 36}, {3, 7, 9, 13, 14, 16, 24, 27, 29, 30, 37}, {3, 5, 9, 11, 14, 17, 23, 27, 32, 33, 34}, {2, 5, 8, 12, 13, 15, 17, 24, 26, 27, 31, 32}, {3, 5, 8, 9, 13, 15, 23, 26, 28, 29, 30, 37}, {3, 5, 7, 11, 12, 15, 24, 29, 32, 34, 35}, {3, 5, 8, 11, 12, 23, 27, 29, 31, 34, 35, 36}, {3, 5, 7, 9, 10, 13, 24, 25, 29, 32, 34, 35}, {2, 6, 10, 11, 14, 16, 25, 26, 28, 31, 35}, {3, 5, 10, 13, 14, 24, 26, 29, 30, 33, 35}, {2, 7, 10, 14, 15, 24, 26, 29, 30, 33, 35}, {3, 7, 8, 10, 14, 16, 22, 25, 28, 29, 30, 34}, {2, 4, 9, 12, 13, 15, 23, 27, 30, 35, 36, 37}, {2, 4, 9, 12, 13, 15, 23, 25, 26, 30, 34, 35}, {2, 5, 11, 12, 16, 22, 26, 28, 32, 33, 35, 36}, {2, 4, 10, 11, 15, 24, 27, 28, 31, 33, 34, 36}, {3, 5, 8, 10, 12, 16, 25, 26, 29, 32, 33}, {2, 5, 7, 11, 12, 15, 17, 23, 24, 31, 32, 34}, {5, 10, 13, 14, 22, 24, 26, 29, 30, 33, 35, 36}, {5, 8, 12, 13, 22, 24, 26, 29, 30, 33, 35, 36}, {5, 8, 11, 15, 24, 26, 28, 31, 32, 33, 36}, {2, 5, 7, 9, 11, 12, 15, 23, 27, 28, 33, 34}, {2, 4, 7, 8, 12, 14, 18, 23, 26, 31, 32, 33}, {2, 4, 6, 9, 12, 16, 17, 23, 26, 31, 32, 33}, {2, 6, 7, 8, 12, 14, 17, 23, 31, 32, 34, 37}, {2, 6, 7, 8, 12, 14, 17, 23, 27, 28, 30, 33}, {4, 6, 8, 11, 15, 23, 26, 31, 32, 33, 36}

(*Continued*)

TABLE A1. Continued

t	$\{i_j : \psi = \rho \prod_{j=1}^{k} \partial_{i_j}\}$
11	{2, 4, 7, 10, 11, 13, 15, 26, 27, 28, 33, 36, 37, 40}, {4, 8, 11, 13, 15, 25, 26, 29, 31, 32, 37, 39, 40},
	{2, 5, 6, 11, 13, 15, 26, 27, 31, 34, 35, 37, 38, 40}, {3, 5, 7, 12, 13, 15, 25, 29, 33, 34, 36, 39, 40},
	{2, 4, 7, 10, 12, 16, 17, 19, 28, 29, 32, 33, 38}, {2, 5, 8, 10, 12, 13, 17, 19, 27, 28, 32, 35, 36},
	{5, 12, 15, 16, 17, 24, 26, 28, 31, 34, 36, 37, 40, 41}, {2, 4, 7, 10, 14, 17, 19, 25, 29, 30, 31, 32, 36, 37},
	{2, 5, 10, 14, 16, 18, 25, 28, 31, 32, 36, 37, 38}, {3, 5, 7, 11, 13, 14, 16, 26, 27, 31, 32, 35, 38},
	{4, 6, 11, 12, 14, 25, 28, 31, 32, 36, 37, 39, 41}, {2, 4, 8, 9, 12, 14, 15, 16, 18, 25, 30, 33, 38, 39},
	{3, 5, 7, 11, 16, 17, 19, 25, 30, 35, 36, 38, 39}, {3, 7, 9, 14, 15, 17, 19, 25, 26, 32, 35, 37, 38},
	{2, 4, 6, 9, 14, 15, 16, 19, 25, 28, 31, 32, 33, 40}, {2, 4, 6, 9, 14, 17, 18, 19, 25, 28, 33, 34, 35, 42},
	{4, 6, 8, 12, 13, 15, 18, 25, 28, 31, 32, 33, 41}, {2, 7, 8, 10, 12, 14, 17, 18, 26, 29, 32, 33, 38},
	{2, 6, 8, 10, 11, 13, 17, 18, 27, 30, 31, 34, 40}, {6, 7, 10, 13, 15, 26, 27, 31, 33, 35, 38, 41},
	{4, 5, 8, 10, 13, 15, 17, 26, 27, 30, 33, 34, 36}, {4, 6, 9, 12, 14, 16, 17, 18, 25, 29, 34, 35, 38},
	{3, 6, 8, 10, 12, 15, 16, 25, 32, 34, 35, 40, 41}, {2, 5, 9, 14, 15, 17, 19, 24, 27, 28, 35, 38, 39, 40},
	{2, 4, 7, 11, 16, 17, 19, 26, 29, 34, 35, 36, 39}, {3, 5, 7, 8, 11, 17, 19, 26, 29, 34, 35, 36, 39},
	{3, 8, 9, 11, 13, 15, 20, 26, 27, 30, 37, 39, 40}, {2, 4, 6, 9, 12, 15, 16, 17, 27, 31, 32, 36, 39},
	{3, 6, 11, 13, 15, 17, 18, 25, 29, 30, 31, 35, 38}, {3, 5, 12, 13, 14, 17, 25, 28, 30, 34, 38, 40, 41},
	{3, 8, 9, 10, 12, 17, 25, 28, 30, 34, 38, 40, 41}, {3, 8, 13, 15, 16, 17, 24, 27, 30, 34, 35, 39, 41},
	{2, 5, 11, 13, 15, 16, 19, 26, 27, 31, 36, 38, 39}, {2, 4, 8, 13, 15, 16, 25, 27, 30, 31, 33, 34, 40, 41}