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Flexible Log-Linear Birnbaum–Saunders Model

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Abstract: Rieck and Nedelman (1991) introduced the sinh-normal distribution. This model was built as a transformation of a N(0,1) distribution. In this paper, a generalization based on a flexible skew normal distribution is introduced. In this way, a more general model is obtained that can describe a range of asymmetric, unimodal and bimodal situations. The paper is divided into two parts. First, the properties of this new model, called flexible sinh-normal distribution, are obtained. In the second part, the flexible sinh-normal distribution is related to flexible Birnbaum–Saunders, introduced by Martínez-Flórez et al. (2019), to propose a log-linear model for lifetime data. Applications to real datasets are included to illustrate our findings.

1. Introduction

In this paper, a generalization of the sinh-normal distribution introduced by Rieck and Nedelman [1] is proposed. Recall that the sinh-normal model is defined as a transformation, $h(\cdot)$, of a standard normal distribution

$$Y = h(Z) \qquad \text{where } Z \sim N(0,1). \tag{1}$$

Rieck and Nedelman [1] showed that the probability density function (pdf) of Y is symmetric and can be unimodal or bimodal. The most relevant application of (1) is that the sinh-normal distribution can be used to propose a log-linear model for lifetime data distributed as a Birnbaum–Saunders (see [1] for details). This point is of crucial importance since the Birnbaum–Saunders distribution is used in a variety of situations, such as to model lifetime, economics and environmental data. In these applications, departures of the BS model are often found. In recent years, some improvements have been introduced to deal with this problem. In this sense, we highlight the flexible Birnbaum–Saunders model, proposed by Matínez-Flórez et al. [2], where two parameters, $\delta \in \mathbb{R}$, $\lambda > 0$, are added to the usual Birnbaum–Saunders model, in such a way that λ controls asymmetry (skewness) and δ is a shape parameter related to unimodality/bimodality of our proposal.

Throughout this paper, the next points are addressed.

First, details about the precedents, in which our proposal is based on, are given in Section 2.

Second, the flexible sinh-normal distribution is defined as result of applying (1) to Z distributed as a flexible skew-normal distribution, $Z \sim FSN(\delta, \lambda)$ with $\delta \in \mathbb{R}$ and $\lambda > 0$. Recall that the $FSN(\delta, \lambda)$ model was introduced by Gómez et al. [3], as a generalization of Azzalini skew-normal model [4], and it can deal with unimodal and bimodal situations.

The properties of the flexible sinh-normal distribution are given in Section 3. These are explicit expressions for the pdf and cumulative distribution function (cdf), shape and modes of the distribution, location and scale changes, approximations, moments and quantiles, among others. All of them are based on the fact that the flexible sinh-normal is obtained as a given function of $Z \sim FSN(\delta, \lambda)$. The obtained properties are compared to those in the Rieck and Nedelman model. In addition, we discuss in which sense our proposal is more general than others previously introduced in literature. The relationship between the flexible sinh-normal distribution and the flexible Birnbaum–Saunders via the exponential (or neperian logarithm), which allows us to build a regression model for lifetime data, is also proven there. Section 4 is devoted to maximum likelihood estimation for the parameters in the flexible sinh-normal distribution.

Third, the flexible Log-linear Birnbaum–Saunders regression model is studied in Section 5. This can be used for non-negative lifetime variables following the flexible Birnbaum–Saunders model given in [2]. The model is built, particular cases of interest are cited, scores and maximum likelihood equations are given.

Applications to real datasets are included in Section 6. In Section 6.1, a positively skewed environmental dataset is considered. There, it is proven that the flexible sinh-normal model provides a better fit than other precedent skew models. In Section 6.2, a symmetric bimodal dataset is considered. Only a few models in the literature exhibit both properties. They are considered along with a mixture of normal distributions. It is shown that the flexible sinh-normal submodel with skewness parameter equal to zero provides a better fit than the other ones. Section 6.3 deals with failure time data in a fatigue test, where the values of stress are considered as known covariate. It is shown that the flexible log-linear Birnbaum–Saunders provides a superior fit to the log-Birnbaum–Saunders [1] and the log-skew-Birnbaum–Saunders model. A simulation study is presented in Section 7. A final discussion is given as Section 8.

2. Materials and Methods

For completeness, details about the most relevant precedents in which our proposal is based are introduced next.

Flexible skew-normal distribution.

The flexible skew-normal (FSN) model was obtained and studied in detail by [5]. They proved that the FSN model can be bimodal for certain values of δ. A random variable (rv) Z follows a FSN distribution, $Z \sim FSN(\delta, \lambda)$, if its pdf is given by

$$f(z; \delta, \lambda) = c_\delta \phi(|z| + \delta)\Phi(\lambda z), z \in \mathbb{R}, \lambda, \delta \in \mathbb{R}, \tag{2}$$

where ϕ and Φ are the pdf and cdf of the $N(0,1)$ distribution, respectively, and $c_\delta^{-1} = 1 - \Phi(\delta)$.

Particular cases of interest in the flexible skew-normal are the following:

- Taking $\delta = 0$ in (2), the skew-normal distribution proposed by Azzalini [4] is obtained.
- If $\lambda = 0$, then (2) reduces to the flexible normal distribution introduced by Martínez-Flórez et al. [6].
- If $\delta = \lambda = 0$, then $Z \sim N(0,1)$.

Sinh-normal or log-Birnbaum–Saunders distribution.

Rieck and Nedelman [1] developed the sinh-normal (SHN) distribution, which is given as the following transformation of a standard normal distribution

$$Y = arcsinh\left(\frac{\gamma}{2}Z\right)\eta + \xi \qquad \text{where} \ Z \sim N(0,1), \tag{3}$$

$\gamma > 0$ is a shape parameter, $\xi \in \mathbb{R}$ is a location parameter and $\eta > 0$ is a scale parameter. (3) is denoted as $Y \sim SHN(\gamma, \xi, \eta)$.

Proposition 1 (Properties of sinh-normal distribution). *Let $Y \sim SHN(\gamma, \xi, \eta)$. Then,*

1. *Y is symmetric about the location parameter ξ.*
2. *The pdf of Y is unimodal for $\gamma \leq 2$ and bimodal for $\gamma > 2$.*
3. *The mean and variance of Y are $E[Y] = \xi$ and $Var[Y] = \eta^2 v(\gamma)$, where $v(\gamma)$ is the variance when the scale parameter is equal to one ($\eta = 1$). There is no closed-expression for $v(\gamma)$, but Rieck and Nedelman [1] provided asymptotic approximations for small and large values of γ.*
4. *Let $Y_\gamma \sim SHN(\gamma, \xi, \eta)$. Then, $U_\gamma = \frac{2(Y_\gamma - \xi)}{\gamma \eta}$ converges in law to a $N(0,1)$ distribution as $\gamma \longrightarrow 0^+$.*

The next lemma considers the particular case of a sinh-normal distribution in which the scale parameter is equal to 2, $\eta = 2$. This model is related to the Birnbaum–Saunders (BS) distribution. This result is the basis of the use of this distribution as a regression model, as shown in Section 5.

Lemma 1.

1. *If $Y \sim SHN(\gamma, \xi, 2)$, then $T = \exp(Y) \sim BS(\gamma, \exp(\xi))$.*
2. *Reciprocally, if $T \sim BS(\alpha, \beta)$, then $Y = \log T$ follows a sinh-normal distribution with shape parameter α, location parameter $\xi = \log \beta$ and scale parameter $\eta = 2$, that is $Y \sim SHN(\alpha, \log \beta, 2)$.*

Proof. It can be seen in [1] (Theorem 1.1). □

Due to the result given in Lemma 1, the sinh-normal distribution is also named log-Birnbaum–Saunders distribution.

3. Results

Flexible sinh-normal distribution.

This model is obtained by applying the transformation introduced in (3) to $Z \sim FSN(\delta, \lambda)$. The new model is called the flexible sinh-normal (FSHN) distribution. It is denoted as $Y \sim FSHN(\gamma, \xi, \eta, \delta, \lambda)$. By applying the change of variable formula, the pdf of Y, f_Y, is

$$f_Y(y) = f_Z(a_{\gamma,\xi,\eta}(y))a'_{\gamma,\xi,\eta}(y) \tag{4}$$

where f_Z denotes the pdf of $Z \sim FSN(\delta, \lambda)$ given in (2), $a_{\gamma,\xi,\eta}(y)$ and $a'_{\gamma,\xi,\eta}(y) = \frac{\partial}{\partial y} a_{\gamma,\xi,\eta}(y)$ are

$$a_{\gamma,\xi,\eta}(y) = \frac{2}{\gamma} sinh\left(\frac{y-\xi}{\eta}\right), \qquad a'_{\gamma,\xi,\eta}(y) = \frac{2}{\eta\gamma} cosh\left(\frac{y-\xi}{\eta}\right). \tag{5}$$

Explicitly, (4) can be written as

$$f_Y(y) = c_\delta \frac{2}{\eta\gamma} cosh\left(\frac{y-\xi}{\eta}\right) \phi\left(\left|\frac{2}{\gamma} sinh\left(\frac{y-\xi}{\eta}\right)\right| + \delta\right) \Phi\left(\lambda\frac{2}{\gamma} sinh\left(\frac{y-\xi}{\eta}\right)\right), \tag{6}$$

with $c_\delta = (1 - \Phi(\delta))^{-1}$.

The following are particular cases of interest in the *FSHN* model:

- If $\delta = 0$, then the FSHN model reduces to the skew sinh-normal distribution studied by Leiva et al. [7].
- If $\lambda = 0$, then a symmetric model denoted by $FSHN_{\lambda=0}(\gamma, \xi, \eta, \delta)$ is obtained, which allows us to model symmetric bimodal data.
- If $\delta = \lambda = 0$, then the FSHN model reduces to the sinh-normal distribution introduced by Rieck and Nedelman [1].

Next, some results dealing with the inverse transformation to that introduced in (3) are given.

Proposition 2. *Let* $Y \sim FSHN(\gamma, \xi, \eta, \delta, \lambda)$. *Then,*

$$a_{\gamma,\xi,\eta}(Y) = \frac{2}{\gamma}\sinh\left(\frac{Y-\xi}{\eta}\right) \sim FSN(\delta, \lambda).$$

Proof. It is obtained by considering $Z = a_{\gamma,\xi,\eta}(Y)$ and applying the inverse transformation technique to

$$Y = h(Z) = arcsinh\left(\frac{\gamma}{2}Z\right)\eta + \xi, \quad \text{with } Z \sim FSN(\delta, \lambda).$$

□

The following are particular cases of interest in Proposition 2:

1. If $\lambda = 0$, then $Z = a_{\gamma,\xi,\eta}(Y)$ follows the *flexible* − *normal*(δ) model studied by Castillo et al. [8].
2. If $\delta = 0$, then $Z = a_{\gamma,\xi,\eta}(Y) \sim Skew - Normal(\lambda)$.
3. If $\lambda = \delta = 0$, then $Z = a_{\gamma,\xi,\eta}(Y) \sim N(0, 1)$.

Next, an explicit expression for the cumulative distribution function (cdf) of a FSHN distribution is given.

Theorem 1. *Let* $Y \sim FSHN(\gamma, \xi, \eta, \delta, \lambda)$. *Then, the cdf of* Y *is*

$$F_Y(y) = \begin{cases} c_\delta \Phi_{BN_\lambda}\left(\frac{\lambda\delta}{\sqrt{1+\lambda^2}}, a(y) - \delta\right), & \text{if } y < \xi \\[2mm] c_\delta\left[\Phi_{BN_\lambda}\left(\frac{\lambda\delta}{\sqrt{1+\lambda^2}}, -\delta\right) + \Phi_{BN_\lambda}\left(-\frac{\lambda\delta}{\sqrt{1+\lambda^2}}, a(y) + \delta\right) - \Phi_{BN_\lambda}\left(-\frac{\lambda\delta}{\sqrt{1+\lambda^2}}, \delta\right)\right], \\[2mm] \quad \text{if } y \geq \xi, \end{cases} \tag{7}$$

where $a(y) = a_{\gamma,\xi,\eta}(y)$ *is defined in* (5), $\Phi_{BN_\lambda}(\cdot, \cdot)$ *denotes the cdf of a bivariate normal distribution, with mean vector* $\boldsymbol{\mu}' = (0, 0)$ *and covariance matrix*

$$\Omega_\lambda = \begin{pmatrix} 1 & \rho_\lambda \\ \rho_\lambda & 1 \end{pmatrix} \quad \text{where } \rho_\lambda = -\frac{\lambda}{\sqrt{1+\lambda^2}}. \tag{8}$$

Proof. Since $Y \sim FSHN(\gamma, \xi, \eta, \delta, \lambda)$

$$Y = arcsinh\left(\frac{\gamma}{2}Z\right)\eta + \xi, \quad \text{with } Z \sim FSN(\delta, \lambda).$$

Taking into account that $arcsinh(x) = \log\left(x + \sqrt{x^2 + 1}\right)$ is a monotonically increasing function of x, we have the following relationship between the cdf's de Y and Z:

$$F_Y(y) = F_Z(a_{\gamma,\xi,\eta}(y)) \tag{9}$$

with $a_{\gamma,\xi,\eta}(y) = \frac{2}{\gamma}\sinh\left(\frac{y-\xi}{\eta}\right)$.

By applying the expression for the cdf of $Z \sim FSN(\delta, \lambda)$ given in [2] we have that, for $a_{\gamma,\xi,\eta}(y) < 0$, or, equivalently, for $y < \xi$

$$F_Z(a_{\gamma,\xi,\eta}(y)) = c_\delta \Phi_{BN_\lambda}\left(\frac{\lambda\delta}{\sqrt{1 + \lambda^2}}, a(y) - \delta\right),$$

and for $a_{\gamma,\xi,\eta}(y) \geq 0$, or, equivalently, for $y \geq \xi$

$$F_Z(a_{\gamma,\xi,\eta}(y)) =$$

$$c_\delta\left[\Phi_{BN_\lambda}\left(\frac{\lambda\delta}{\sqrt{1 + \lambda^2}}, -\delta\right) + \Phi_{BN_\lambda}\left(-\frac{\lambda\delta}{\sqrt{1 + \lambda^2}}, a(y) + \delta\right) - \Phi_{BN_\lambda}\left(-\frac{\lambda\delta}{\sqrt{1 + \lambda^2}}, \delta\right)\right]$$

where $\Phi_{BN_\lambda}(\cdot, \cdot)$ denotes the cdf of a bivariate normal distribution, with mean vector $\boldsymbol{\mu}' = (0,0)$ and covariance matrix Ω_λ given in (8). Thus, the proposed result holds $\qquad \square$

Next, some particular cases of interest for the cdf in the FSHN model are discussed.

Corollary 1. *Let $Y \sim FSHN(\gamma, \xi, \eta, \delta, \lambda)$.*

1. *If $\lambda = 0$, then the cdf of Y reduces to*

$$F_Y(y) = \begin{cases} \frac{c_\delta}{2} \Phi(a(y) - \delta), & \text{if } y < \xi \\[2mm] \frac{c_\delta}{2} \{\Phi(a(y) + \delta) + 1 - 2\Phi(\delta)\}, & \text{if } y \geq \xi . \end{cases} \qquad (10)$$

2. *If $\delta = 0$, then the cdf of Y reduces to $F_Y(y) = 2\Phi_{BN_\lambda}(0, a(y))$, for $y \in \mathbb{R}$.*

Proof. 1.   If $\lambda = 0$, then $\rho_\lambda = 0$, where ρ_λ is given in (8). Recall that, in the bivariate normal distribution, uncorrelation implies independence, therefore

$$\Phi_{BN_{\lambda=0}}(x_1, x_2) = \Phi(x_1) \, \Phi(x_2), \qquad \forall (x_1, x_2) .$$

Taking into account that $\Phi(0) = 1/2$ and $\Phi(-\delta) = 1 - \Phi(\delta)$, we have that (7) reduces to (10).
2.   If $\delta = 0$, then we have the cdf of the skew-sinh-normal distribution.
  $\square$

The result in Corollary 1 for $\lambda = 0$ corresponds to the model studied by Olmos et al. [9], whereas the result for $\delta = 0$ is a particular case of models studied by Vilca-Labra and Leiva-Sanchez [10].

Shape of $f_Y(\cdot)$.

Proposition 3. *Let $Y \sim FSHN(\gamma, \xi, \eta, \delta, \lambda)$. The pdf given in (16) can be bimodal. The modes would be given as the solution of the following non-linear equations*

1. *$y_1^* < \xi$ solution of*

$$\delta + \lambda \frac{\phi(\lambda s(y))}{\Phi(\lambda s(y))} + \frac{s(y)}{c^2(y)} = s(y) , \qquad (11)$$

2. *$y_2^* > \xi$ solution of*

$$-\delta + \lambda \frac{\phi(\lambda s(y))}{\Phi(\lambda s(y))} + \frac{s(y)}{c^2(y)} = s(y) , \qquad (12)$$

with $s(y) = \frac{2}{\gamma} sinh\left(\frac{y-\xi}{\eta}\right)$ and $c(y) = \frac{2}{\gamma} cosh\left(\frac{y-\xi}{\eta}\right)$.

Proof. To obtain the maximums of f_Y, we take the natural logarithm of f_Y, except additive terms not depending of y. Thus, let us consider

$$g(y) = \ln(c(y)) + \ln \phi(|s(y)| + \delta) + \ln \Phi(\lambda s(y))$$

where $s(y) = \frac{2}{\gamma} sinh\left(\frac{y-\xi}{\eta}\right)$ and $c(y) = \frac{2}{\gamma} cosh\left(\frac{y-\xi}{\eta}\right)$. Taking the first derivative of $g(y)$ with respect to y, $g'(y) = 0$, after straightforward calculations, (11) and (12) are obtained. $\square$

Proposition 4. *Let $Y \sim FSHN(\gamma, \xi, \eta, \delta, \lambda)$ with $\lambda = 0$. Then, the pdf of Y is symmetrical with respect to ξ and can be bimodal for $\delta < 0$.*

Proof. Symmetry follows from the fact that $cosh(\cdot)$ and $\phi(|z| + \delta)$ are even functions. Bimodality for $\delta < 0$ follows from Proposition 3. $\square$

Proposition 5. *Let $Y \sim FSHN(\gamma, \xi, \eta, \delta, \lambda)$. Then, the pdf of Y is non-differentiable at $y = \xi$.*

Proof. This result follows from the fact that, if $y = \xi$, then $s(\xi) = \frac{2}{\gamma} sinh(0) = 0$ and the non-differentiability of the absolute value function at zero. $\square$

Some illustrative plots for the pdfs in the FSHN model are given in Figures 1–3. In Figures 1 and 2, we mainly focus on the study of the shape parameters, δ and γ, so in every figure a positive or negative value of δ is fixed, and a range of possible values is considered for γ (less and greater than two), since our aim is to compare with the sinh-normal distribution, which is strongly unimodal for $\gamma \leq 2$ and bimodal for $\gamma > 2$ [1]. In this way, the effect of δ and/or γ can be appreciated. Moreover, positive and negative values of λ are considered to assess the effect of the skewness parameter. In Figure 3, $\lambda = 0$ is fixed and therefore the symmetric submodel is plotted. Additional details are given next for every figure.
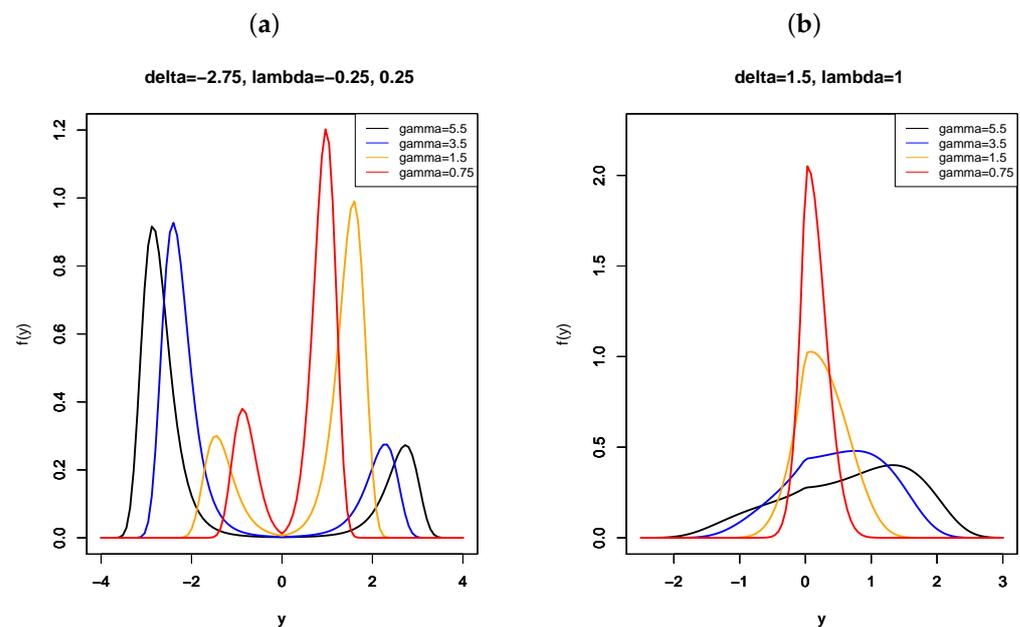

Figure 1. FSHN pdfs: (**a**) *FSHN* (5.5, 0, 1, −2.75, −0.25) (black), *FSHN* (3.5, 0, 1, −2.75, −0.25) (blue), *FSHN* (1.5, 0, 1, −2.75, 0.25) (orange) and *FSHN* (0.75, 0, 1, −2.75, 0.25) (red); and (**b**) *FSHN* (5.5, 0, 1, 1.5,1) (black), *FSHN* (3.5, 0, 1, 1.5,1) (blue), *FSHN* (1.5, 0, 1, 1.5,1) (orange) and *FSHN* (0.75, 0, 1, 1.5,1) (red).

In Figure 1, without loss of generality for our purposes, the location and scale parameters are taken as $\xi = 0$ and $\eta = 1$. We intend to highlight the effect of δ on the bimodality of FSHN model. In Figure 1a, small values are considered for the skewness parameter, $\lambda = -0.25$ and 0.25, and a clearly negative value of $\delta = -2.75$. All pdfs are bimodal. Note that the parameter $\gamma \in \{0.75, 1.5, 3.5, 5.5\}$ and even for values of $\gamma \leq 2$ the pdf is bimodal. These plots suggest that a negative value of δ induces bimodality and the effect of γ is attenuated. It is also possible to appreciate the effect of the sign of skewness parameter λ in Figure 1a; note that black and blue plots correspond to negative λ, whereas orange and red ones correspond to positive λ. On the other hand, Figure 1b, a positive value of $\delta = 1.5$ is considered ($\lambda = 1$). We highlight that the distribution is unimodal even for values of $\gamma > 2$.

In Figure 2, a small value of δ is fixed ($\delta = 0.2$ in Figure 2a and $\delta = -0.2$ in Figure 2b); the plots suggest that, in this case, the bimodality depends on the value of γ. However, we must also be conscious that, in these plots $\lambda = 0.4$, for large values of λ unimodal distributions are nearly always obtained for any values of γ and δ.

(a)

delta=0.2, lambda=0.4

(b)

delta=−0.2, lambda=0.4
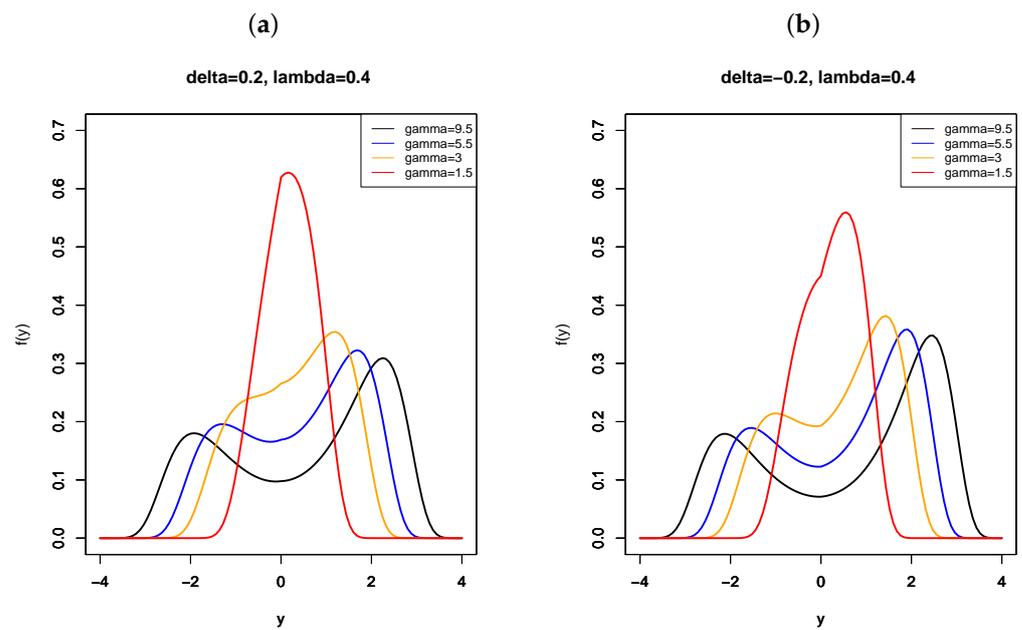
Figure 2. FSHN pdfs: (**a**) *FSHN* (9.5, 0, 1, 0.2,0.4) (black), *FSHN* (5.5, 0, 1, 0.2,0.4) (blue), *FSHN* (3, 0, 1, 0.2,0.4) (orange) and *FSHN* (1.5, 0, 1, 0.2,0.4) (red); and In (**b**) *FSHN* (9.5, 0, 1, −0.2,0.4) (black), *FSHN* (5.5, 0, 1, −0.2,0.4) (blue), *FSHN* (3, 0, 1, −0.2,0.4) (orange) and *FSHN* (1.5, 0, 1, −0.2,0.4) (red).

In Figure 3, the skewness parameter $\lambda = 0$ is fixed, therefore symmetric pdfs are always obtained. In Figure 3a, a negative value of $\delta = -0.75$ is fixed and all plots are bimodal (even for values of the parameter $\gamma < 2$ in which the sinh-normal distribution is unimodal). In Figure 3b, $\delta = 0.75$ is fixed and unimodal symmetric distributions are obtained.

(a)

lambda=0, delta=−0.75

(b)

lambda=0, delta=0.75
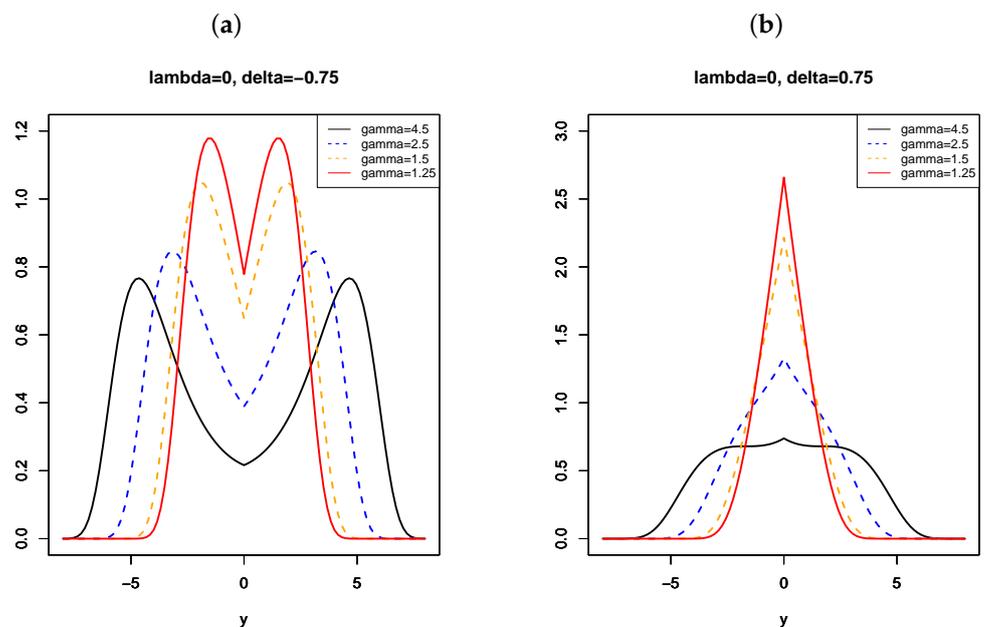
Figure 3. FSHN pdfs: (**a**) *FSHN* (4.5, 0, 2.5, −0.75,0) (black), *FSHN* (2.5, 0, 2.5, −0.75, 0) (blue), *FSHN* (1.5, 0, 2.5, −0.75, 0) (orange) and *FSHN* (1.25, 0, 2.5, −0.75, 0) (red); and (**b**) *FSHN* (4.5, 0, 2.5, 0.75,0) (black), *FSHN* (2.5, 0, 2.5, 0.75, 0) (blue), *FSHN* (1.5, 0, 2.5, 0.75, 0) (orange) and *FSHN* (1.25, 0, 2.5, 0.75, 0) (red).

Next, we show that the FSHN model is closed under location and scale changes.

Proposition 6. *Let* $Y \sim FSHN(\gamma, \xi, \eta, \delta, \lambda)$. *Then,*

1. $aY \sim FSHN(\gamma, a\xi, a\eta, \delta, \lambda), \ a > 0$.
2. $Y + b \sim FSHN(\gamma, \xi + b, \eta, \delta, \lambda), \ \forall b \in \mathbb{R}$.
3. $-Y \sim FSHN(\gamma, -\xi, \eta, \delta, -\lambda)$.

Proof. 1. Let $Z = aY$ with $a > 0$. The result proposed follows from the fact that $f_Z(z) = \frac{1}{a} f_Y\left(\frac{z}{a}\right)$.
2. Let $Z = Y + b$. Then, $f_Z(z) = f_Y(z - b)$.
3. Let $Z = -Y$. Then, $f_Z(z) = f_Y(-z)$. Taking into account that $cosh(\cdot)$ is an even function, $cosh(-x) = cosh(x)$

$$\frac{2}{\gamma} cosh\left(\frac{-z - \xi}{\eta}\right) = \frac{2}{\gamma} cosh\left(\frac{z + \xi}{\eta}\right) = \frac{2}{\gamma} cosh\left(\frac{z - (-\xi)}{\eta}\right),$$

and, since $sinh(\cdot)$ is an odd function, $sinh(-x) = -sinh(x)$

$$\frac{2}{\gamma} sinh\left(\frac{-z - \xi}{\eta}\right) = -\frac{2}{\gamma} sinh\left(\frac{z + \xi}{\eta}\right) = -\frac{2}{\gamma} sinh\left(\frac{z - (-\xi)}{\eta}\right),$$

and the result proposed is obtained. $\square$

Corollary 2. *Let* $Y \sim FSHN(\gamma, \xi, \eta, \delta, \lambda)$. *Then,*

$$\frac{Y - \xi}{\eta} \sim FSHN(\gamma, 0, 1, \delta, \lambda).$$

Next, the limit behavior of FSHN model when γ approaches to 0^+ is studied.

Proposition 7. *Let* $Y \sim FSHN(\gamma, \xi, \eta, \delta, \lambda)$. *Then,* $U_\gamma = \frac{2(Y - \xi)}{\gamma \eta}$ *converges in distribution to the flexible skew-normal* $FSN(\delta, \lambda)$ *distribution as* $\gamma \longrightarrow 0^+$.

Proof. Let $U_\gamma = \frac{2(Y - \xi)}{\gamma \eta}$. By applying Proposition 6, $U_\gamma = \frac{2(Y - \xi)}{\gamma \eta} \sim FSHN(\gamma, 0, \frac{2}{\gamma}, \delta, \lambda)$. Therefore, its pdf is

$$f_{U_\gamma}(u) = c_\delta cosh\left(\frac{\gamma u}{2}\right) \phi\left(\left|\frac{2}{\gamma} sinh\left(\frac{\gamma u}{2}\right)\right| + \delta\right) \Phi\left(\lambda \frac{2}{\gamma} sinh\left(\frac{\gamma u}{2}\right)\right), \quad u \in \mathbb{R}. \quad (13)$$

Considering that $\lim_{\gamma \to 0^+} cosh\left(\frac{\gamma u}{2}\right) = 1$, by L'Hôpital–Bernoulli rule,

$$\lim_{\gamma \to 0^+} \frac{2}{\gamma} sinh\left(\frac{\gamma u}{2}\right) = u \lim_{\gamma \to 0^+} cosh\left(\frac{\gamma u}{2}\right) = u.$$

Since $\phi(\cdot)$ and $\Phi(\cdot)$ are continuous differentiable functions,

$$f_{U_\gamma}(u) \longrightarrow c_\delta \phi(|u| + \delta)\, \Phi(\lambda u), \quad u \in \mathbb{R} \quad \text{as } \gamma \to 0^+,$$

which is the pdf of a flexible skew-normal(δ, λ) distribution. That is, the result proposed. $\square$

Remark 1. *Given $Y \sim FSHN(\gamma, \xi, \eta, \delta, \lambda)$ with γ close to zero, Proposition 7 allows us to approximate the distribution of Y, properly normalized, by a flexible skew-normal(δ, λ) distribution. In addition, note that this result is analogous to the property of the sinh-normal distribution given in Proposition 1.*

Next, it is shown that the *pth* quantile of Y can be obtained from the *pth* quantile of $Z \sim FSN(\delta, \lambda)$.

Theorem 2. *Let $Y \sim FSHN(\gamma, \xi, \eta, \delta, \lambda)$. Then, the pth quantile of Y, y_p, with $0 < p < 1$ is given by*

$$y_p = \xi + \eta \; arcsinh\left(\frac{\gamma}{2} z_p\right) \tag{14}$$

where z_p denotes the pth quantile of $Z \sim FSN(\delta, \lambda)$.

Proof. It follows from (3) and the fact that the arcsinh function is monotonically increasing. □

Submodel with $\lambda = 0$.

For $\lambda = 0$, y_p can be obtained from the quantiles of the $N(0, 1)$ distribution.

Corollary 3. *If $\lambda = 0$, then*

$$y_p = \begin{cases} \xi + \eta \; arcsinh\left\{\frac{\gamma}{2}\left[\delta + \Phi^{-1}(2p/c_\delta)\right]\right\}, & \text{if } p < 0.5 \\[2mm] \xi + \eta \; arcsinh\left\{\frac{\gamma}{2}\left[-\delta + \Phi^{-1}\left(\Phi(\delta) + \frac{(2p-1)}{c_\delta}\right)\right]\right\}, & \text{if } p \geq 0.5. \end{cases} \tag{15}$$

As it can be seen in [1], recall that the log-Birnbaum–Saunders regression model is based on the relationship between the Birnbaum–Saunders and the sinh-normal distribution. In our proposal, a regression model is introduced next that is based on the relationship between the flexible Birnbaum–Saunders, studied in [2] and the FSHN distribution introduced in this paper. First, we recall the expression of the pdf for the flexible Birnbaum–Saunders.

Flexible Birnbaum–Saunders.

Based on the flexible skew-normal model defined in (2), Martínez-Flórez et al. [2] proposed the flexible Birnbaum–Saunders (FBS) distribution, $T \sim FBS(\alpha, \beta, \delta, \lambda)$, whose pdf is given by

$$f(t; \alpha, \beta, \delta, \lambda) = \frac{t^{-3/2}(t + \beta)}{2\alpha\beta^{1/2}(1 - \Phi(\delta))} \phi(|a_t| + \delta)\Phi(\lambda a_t), \qquad t > 0, \tag{16}$$

with a_t

$$a_t = a_t(\alpha, \beta) = \frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right), \tag{17}$$

$\alpha > 0$, $\beta > 0$, $\delta \in \mathbb{R}$, $\lambda \in \mathbb{R}$ and $\phi(\cdot)$ and $\Phi(\cdot)$ the pdf and cdf of a $N(0, 1)$, respectively.

In (16), δ and λ are shape parameters. We highlight that λ is a parameter that controls asymmetry (or skewness) and δ is a shape parameter related to bimodality of FBS model. Specifically, it can be seen in [2] that (16) can be bimodal for some combinations of δ and λ parameters.

The following are particular cases of interest in (16):

- If $\lambda = 0$, then the model introduced by Olmos et al. [9] is obtained.
- If $\delta = 0$, then the skew-Birnbaum–Saunders is obtained.
- If $\lambda = \delta = 0$, then the Birnbaum–Saunders is obtained.

Theorem 3. *Let $Y \sim FSHN(\gamma, \xi, 2, \delta, \lambda)$. Then, $T = exp(Y) \sim FBS(\gamma, exp(\xi), \delta, \lambda)$.*

Proof. This result is obtained by applying the transformation technique for random variables and taking into account (16). $\square$

In practice, it is usual to consider non-negative rv's following a $FBS(\alpha, \beta, \delta, \lambda)$. Then, its neperian logarithm will follow a FSHN distribution, as established in next corollary.

Corollary 4. *Let* $T \sim FBS(\alpha, \beta, \delta, \lambda)$. *Then,*

$$Y = log(T) \sim FSHN(\alpha, \xi = log(\beta), \eta = 2, \delta, \lambda) . \tag{18}$$

If T depends on a known set of covariates then it is of interest to build a generalized linear model related to T [11]. This issue is studied in Section 5.

Moments

Let $Y \sim FSHN(\gamma, \xi, \eta, \delta, \lambda)$. Recall that

$$Y = \xi + \eta arcsinh\left(\frac{\gamma}{2}Z\right) \qquad \text{where} \quad Z \sim FSN(\delta, \lambda).$$

Thus, the moments of Y can be expressed in terms of the moments of $arcsinh\left(\frac{\gamma}{2}Z\right)$ with respect to $Z \sim FSN(\delta, \lambda)$. To simplify the notation, let us introduce

$$c_j(\gamma, \delta, \lambda) = E_{FSHN}\left[\left\{arcsinh\left(\frac{\gamma}{2}Z\right)\right\}^j\right] \qquad j = 1, 2, \ldots,$$

$$V(\gamma, \delta, \lambda) = Var_{FSHN}\left[arcsinh\left(\frac{\gamma}{2}Z\right)\right] .$$

In particular, we have that

$$E[Y] = \xi + \eta c_1(\gamma, \delta, \lambda), \tag{19}$$

$$Var[Y] = \eta^2 V(\gamma, \delta, \lambda) . \tag{20}$$

4. Inference Based on Maximum Likelihood Estimation

Let $Y \sim FSHN(\gamma, \xi, \eta, \delta, \lambda)$. Let us denote the vector of unknown parameters by $\boldsymbol{\theta} = (\gamma, \xi, \eta, \delta, \lambda)^T$. Given $Y_1, \ldots, Y_n$ a random sample of $Y \sim FSHN(\gamma, \xi, \eta, \delta, \lambda)$, let us consider the log-likelihood function for the parameter vector $\boldsymbol{\theta} = (\gamma, \xi, \eta, \delta, \lambda)$, which is given

$$\ell(\boldsymbol{\theta}) = n \log c_\delta - n \log \eta + \sum_{i=1}^{n} \log z_{i,1} + \sum_{i=1}^{n} \log \phi(|z_{i,2}| + \delta) + \sum_{i=1}^{n} \log \Phi(\lambda z_{i,2}) + c, \tag{21}$$

where c denotes a constant independent of $\boldsymbol{\theta}$ and

$$z_{i,1} = \frac{2}{\gamma} cosh\left(\frac{y_i - \xi}{\eta}\right), \quad z_{i,2} = \frac{2}{\gamma} sinh\left(\frac{y_i - \xi}{\eta}\right) .$$

To maximize $l(\boldsymbol{\theta})$ in $\boldsymbol{\theta}$, consider the score functions, denoted as $U(\gamma)$, $U(\xi)$, $U(\eta)$, $U(\delta)$ and $U(\lambda)$, and given as the first derivatives of $l(\boldsymbol{\theta})$ with respect to γ, ξ, η, δ and λ, respectively. Maximum likelihood estimates for the parameters are given as the solutions to $U(\gamma) = 0$, $U(\xi) = 0$, $U(\eta) = 0$, $U(\delta) = 0$ and $U(\lambda) = 0$, which are equivalent to the following equations

$$\sum_{i=1}^{n} z_{i,2}^2 + \delta \sum_{i=1}^{n} |z_{i,2}| - \lambda \sum_{i=1}^{n} z_{i,2} R_{i,2} = n \tag{22}$$

$$\sum_{i=1}^{n} \left(z_{i,1} z_{i,2} - \frac{z_{i,2}}{z_{i,1}}\right) + \delta \sum_{i=1}^{n} sgn(z_{i,2}) z_{i,1} - \lambda \sum_{i=1}^{n} z_{i,1} R_{i,2} = 0 \tag{23}$$

$$\sum_{i=1}^{n}\left(\frac{y_i - \xi}{\eta}\right)\left(z_{i,1}z_{i,2} - \frac{z_{i,2}}{z_{i,1}}\right) + \delta \sum_{i=1}^{n}\left(\frac{y_i - \xi}{\eta}\right)sgn(z_{i,2})z_{i,1} - \lambda \sum_{i=1}^{n}\left(\frac{y_i - \xi}{\eta}\right)z_{i,1}R_{i,2} = n \quad (24)$$

$$\frac{\phi(\delta)}{1 - \Phi(\delta)} - \delta = \frac{\sum_{i=1}^{n}|z_{i,2}|}{n} \quad (25)$$

$$\sum_{i=1}^{n}z_{i,2}R_{i,2} = 0, \quad (26)$$

where $sgn(\cdot)$ denotes the *sign* function and

$$R_{i,2} = \frac{\phi(\lambda z_{i,2})}{\Phi(\lambda z_{i,2})}, \quad i = 1, \ldots, n.$$

Iterative methods must be used to solve these equations [12]. Details about the previous maximum likelihood estimators (MLEs) are postponed to Section 5, where the MLEs for the parameters in the regression model based on the FSHN distribution are studied.

5. The Flexible Log-Birnbaum–Saunders Regression Model

In this section the flexible log-Birnbaum–Saunders regression model is introduced. Let $T_1, \ldots, T_n$ be independent positive continuous random variables such as

$$T_i \sim FBS(\gamma_i, \tau_i, \delta_i, \lambda_i). \quad (27)$$

Let us assume that the distribution of T_i, proposed in (27), depends on a set of known explanatory variables, $X_{i,1}, \ldots, X_{i,p-1}$, satisfying for $i = 1, \ldots, n$

1. $\tau_i = exp(x_i^T \boldsymbol{\beta})$ where $x_i = (1, x_{i,1}, \ldots, x_{i,p-1})^T$ and $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_{p-1})^T$ is a p-dimensional vector of unknown parameters.
2. The shape and skewness parameters in (27) do not involve x_i, that is, $\gamma_i = \gamma$, $\delta_i = \delta$ and $\lambda_i = \lambda$.

Let us consider $Y_i = log(T_i)$. By applying Corollary 4, $Y_i \sim FSHN(\gamma, x_i^T \boldsymbol{\beta}, 2, \delta, \lambda)$. Thus, Y_i can be written as a linear model

$$Y_i = x_i^T \boldsymbol{\beta} + \epsilon_i \quad (28)$$

where the error term $\epsilon_i \sim FSHN(\gamma, 0, 2, \delta, \lambda)$ (Proposition 6 is applied) and $\epsilon_i's$ are independent. As for the expectation, variance of ϵ_i and covariance of ϵ_i and ϵ_j ($i \neq j$), taking into account (19), we have for $i = 1, \ldots, n$

$$\begin{aligned}
\mathbb{E}[\epsilon_i] &= 2c_1(\gamma, \delta, \lambda) \\
Var[\epsilon_i] &= 4V(\gamma, \delta, \lambda) \\
Cov(\epsilon_i, \epsilon_j) &= 0, \quad i \neq j.
\end{aligned}$$

Taking

$$\beta_0^* = \beta_0 + 2c_1(\gamma, \delta, \lambda),$$

we can write

$$E[Y_i] = x_i^T \boldsymbol{\beta}^* \quad \text{with } \boldsymbol{\beta}^* = (\beta_0^*, \boldsymbol{\beta}_1^T)^T \quad (29)$$

where the initial vector of unknown parameters has been partitioned into $\boldsymbol{\beta} = (\beta_0, \boldsymbol{\beta_1}^T)^T$ with $\boldsymbol{\beta_1} = (\beta_1 \ldots, \beta_{p-1})^T$.

By applying the ordinary least squares approach, an unbiased linear estimator of $\boldsymbol{\beta}^*$ is

$$\widehat{\boldsymbol{\beta}}^* = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y},$$

where $\mathbf{y} = (y_1, \ldots, y_n)^T$ and

$$\mathbf{X} = \begin{pmatrix} 1 & x_{1,1} & x_{1,2} & \cdots & x_{1,p-1} \\ 1 & x_{2,1} & x_{2,2} & \cdots & x_{2,p-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_{n,1} & x_{n,2} & \cdots & x_{n,p-1} \end{pmatrix}.$$

The covariance matrix of $\widehat{\boldsymbol{\beta}}^*$ is given by

$$Cov(\widehat{\boldsymbol{\beta}}^*) = 4V(\gamma, \delta, \lambda)(\mathbf{X}^T\mathbf{X})^{-1}.$$

The model previously introduced is named flexible log-Birnbaum–Saunders regression model and is denoted by $FLBS(\gamma, x_i^T\boldsymbol{\beta}, 2, \delta, \lambda)$.

Particular cases of interest.

1. If $\delta = \lambda = 0$, then we have the log-Birnbaum–Saunders regression model proposed by Rieck and Nedelman [1], which is based on the sinh-normal distribution [1].
2. If $\delta = 0$, then (28) reduces to the skewed log-Birnbaum–Saunders regression model studied by Lemonte [13], which is based on the skewed sinh-normal distribution introduced in Leiva et al. [7].
3. If $\lambda = 0$, then (28) reduces to the submodel flexible log-Birnbaum–Saunders, denoted by $FLBS_{\lambda=0}(\gamma, x_i^T\boldsymbol{\beta}, 2, \delta)$, which is based on the $FSHN_{\lambda=0}(\gamma, 0, 2, \delta)$ distribution introduced in this paper.

Likelihood Equations in the Regression Model

Let us consider the log-likelihood function for the parameter vector $\boldsymbol{\theta} = (\gamma, \boldsymbol{\beta}^T, \delta, \lambda)^T$, which is given

$$\ell(\boldsymbol{\theta}) = n\log c_\delta + \sum_{i=1}^n \log \xi_{i,1} + \sum_{i=1}^n \log \phi(|\xi_{i,2}| + \delta) + \sum_{i=1}^n \log \Phi(\lambda\xi_{i,2}) + c, \tag{30}$$

where c denotes a constant independent of $\boldsymbol{\theta}$ and

$$\xi_{i,1} = \frac{2}{\gamma}\cosh\left(\frac{y_i - x_i^T\boldsymbol{\beta}}{2}\right), \quad \xi_{i,2} = \frac{2}{\gamma}\sinh\left(\frac{y_i - x_i^T\boldsymbol{\beta}}{2}\right).$$

To maximize $l(\boldsymbol{\theta})$ in $\boldsymbol{\theta}$, consider the score functions, given as the first derivatives of $l(\boldsymbol{\theta})$ with respect to γ, β_j (with $j = 0, \ldots, p-1$), δ and λ, respectively. They are denoted as $U(\gamma)$, $U(\beta_j)$ with $j = 0, 1, \ldots, p-1$, $U(\delta)$ and $U(\lambda)$,

$$U(\gamma) = -\frac{n}{\gamma} + \frac{1}{\gamma}\sum_{i=1}^n \xi_{i,2}^2 + \frac{\delta}{\gamma}\sum_{i=1}^n sgn(\xi_{i,2})\xi_{i,2} - \frac{\lambda}{\gamma}\sum_{i=1}^n \xi_{i,2}w_{i,2} \tag{31}$$

$$U(\beta_j) = \frac{1}{2}\sum_{i=1}^n x_{i,j}\left(\xi_{i,1}\xi_{i,2} - \frac{\xi_{i,2}}{\xi_{i,1}}\right) + \frac{\delta}{2}\sum_{i=1}^n x_{i,j}sgn(\xi_{i,2})\xi_{i,1} - \frac{\lambda}{2}\sum_{i=1}^n x_{i,j}\xi_{i,1}, w_{i,2} \tag{32}$$

with $j = 0, 1, \ldots, p-1$.

$$U(\delta) = n\frac{\phi(\delta)}{1 - \Phi(\delta)} - \sum_{i=1}^n sgn(\xi_{i,2})\xi_{i,2} - n\delta, \tag{33}$$

$$U(\lambda) = \sum_{i=1}^n \xi_{i,2}w_{i,2}, \tag{34}$$

where $sgn(\cdot)$ denotes the *sign* function and

$$w_{i,2} = \frac{\phi(\lambda\xi_{i,2})}{\Phi(\lambda\xi_{i,2})}, \quad i = 1, \ldots, n.$$

Maximum likelihood estimators for the parameters are given as the solutions to $U(\gamma) = 0$, $U(\beta_j) = 0$, $j = 0, \ldots, p-1$, $U(\delta) = 0$ and $U(\lambda) = 0$, which requires numerical techniques.

6. Applications

6.1. Illustration 1

Dataset 1 consists of $n = 116$ observations of daily ozone mean concentration in the atmosphere (in ppb= $ppm \times 1000$) in New York taken from May to September 1973 [14]. The average concentration of pollutants in the air is of interest in epidemiological studies due to its serious adverse effects on the human health. It is usually assumed that these data are independent, and therefore they do not require a cyclic trend analysis, see for instance the works of Gokhale and Khare [15], Nadarajah [16] and Leiva et al. [7], where this dataset is also studied.

Descriptive summaries. The sample mean, variance, skewness and kurtosis coefficients are: $\bar{y} = 42.1293$, $s^2 = 1088.2010$, $\sqrt{b_1} = 1.2098$ and $b_2 = 1.1122$. From these summaries and Figure 4, we can conclude that this dataset is positively (or right) skewed. Thus, sinh-normal, skew sinh-normal, Birnbaum–Saunders, extended Birnbaum–Saunders [7] and FSHN distributions can be considered as models for this dataset. Their performance is compared throughout the Akaike information criterion, $AIC = -2\hat{l}(\cdot) + 2k$ and Bayesian information criterion, $BIC = -2\hat{l}(\cdot) + log(n)k$, where $\hat{l}(\cdot)$ denotes the log-likelihood function evaluated at the MLEs of parameters and k is the number of parameters in the model. From the results in Table 1, the FSHN distribution provides the best fit to these data since its AIC and BIC are the smallest [17,18].

Table 1. Estimates of parameters and their standard errors in parentheses for sinh-Normal (SHN), skew-SHN, BS, extended BS and FSHN models.

Estimator	SHN	Skew-SHN	BS	Extended BS	FSHN
$\hat{\gamma}$	0.437	0.729	0.982	3.381	3.021
	(0.170)	(0.205)	(0.064)	(0.766)	(1.349)
$\hat{\xi}$	43.445	4.199		1.113	7.043
	(3.218)	(1.484)		(0.448)	(1.868)
$\hat{\eta}$	155.008	147.882	28.021	2.037	88.949
	(56.170)	(35.000)	(2.264)	(0.206)	(15.259)
$\hat{\delta}$				-0.778	2.845
				(0.205)	(1.422)
$\hat{\lambda}$		21.035		2.905	26.549
		(9.841)		(0.914)	(10.4072)
AIC	1147.49	1096.41	1102.19	1091.29	1091.23
BIC	1155.75	1107.43	1107.70	1105.06	1105.00

On the other hand, since the sinh-normal, skew sinh-normal and FSHN are nested models, they can be compared by using the likelihood ratio test, [19].

First, let us consider the test

$$H_0 : (\delta, \lambda) = (0,0) \quad \text{vs} \quad H_1 : (\delta, \lambda) \neq (0,0),$$

which is equivalent to testing the sinh-normal (SHN) versus a FSHN distribution. The associated likelihood ratio statistic is

$$\Lambda_1 = \frac{L_{SHN}(\widehat{\gamma}, \widehat{\widehat{\xi}}, \widehat{\eta})}{L_{FSHN}(\widehat{\gamma}, \widehat{\widehat{\xi}}, \widehat{\eta}, \widehat{\delta}, \widehat{\lambda})},$$

which is asymptotically distributed as a chi-square variable with two degrees of freedom (df). After substituting the estimated values of the parameters, we obtain $-2\log(\Lambda_1) = 60.261$, which is greater than the 5% chi-square critical value with 2 df, which is 5.99. Therefore, the FSHN model is preferred to the SHN model for this dataset.

(a)   **(b)**
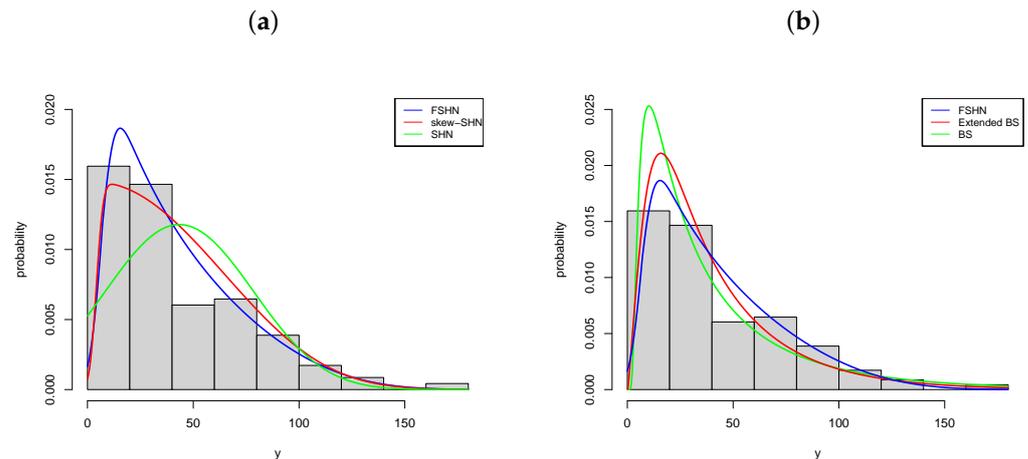
Figure 4. Estimated pdfs: (**a**) FSHN (blue), skew-SHN (red) and SHN (green); and (**b**) FSHN (blue), extended BS (red) and BS (green).

Second, we compare the FSHN model to the skew sinh-normal (skew-SHN) model. Let us now consider the following likelihood ratio statistic

$$\Lambda_2 = \frac{L_{Skew-SHN}(\widehat{\gamma}, \widehat{\widehat{\xi}}, \widehat{\eta}, \widehat{\lambda})}{L_{FSHN}(\widehat{\gamma}, \widehat{\widehat{\xi}}, \widehat{\eta}, \widehat{\delta}, \widehat{\lambda})},$$

which is asymptotically distributed as a chi-square variable with $df = 1$, χ_1^2.

After substituting the estimated values, we obtain $-2\log(\Lambda_2) = 7.182$, which is greater than the 5% chi-square critical value with 1 df, which is 3.84. Therefore, FSHN is preferred to skew-SHN model for this dataset.

Figure 4 presents the histogram and the fitted pdfs. Note that FSHN and extended BS provide a good fit to this dataset.

6.2. Illustration 2

Dataset 2 consists of $n = 500$ observations of fetus weight (in grams) before birth obtained by ultrasound technique. Descriptive summaries are given in Table 2. The data are available at http://www.mat.uda.cl/hsalinas/cursos/2011/R/weight.rar (accessed on 17 May 2021). The variables *b.weigt* (fetal weight in grams) and *a.weigt* (birth weight in grams) can be found there. We study the b.weight variable.

Table 2. Descriptive summaries for b.weight variable.

n	$\bar{y}$	s^2	**Median**	$\sqrt{b_1}$
500	3210.356	695,710.6	3175	0.0712

In Table 2, note that the mean and median are similar and the sample asymmetry coefficient $\sqrt{b_1}$ is close to zero. Therefore, the distribution is fairly symmetrical. This fact can be seen in the histogram plotted in Figure 5, where we can also appreciate the bimodality of this dataset. Bimodality is due to the difference in weight, at the gestation stage, between males and females. Therefore, a symmetric bimodal model must be considered. In the literature, there are few models which exhibit both properties [20]. We propose the bimodal normal [21], the two-piece skew normal [22], the sinh-normal with $\gamma > 2$ and the submodel FSHN with $\lambda = 0$. In addition, we consider a mixture of two normal distributions whose pdf is

$$f(y) = \frac{\alpha}{\sigma_1} \phi \left(\frac{y - \mu_1}{\sigma_1} \right) + \frac{1 - \alpha}{\sigma_2} \phi \left(\frac{y - \mu_2}{\sigma_2} \right) \qquad y \in \mathbb{R} , \tag{35}$$

with $0 < \alpha < 1$, $\mu_i \in \mathbb{R}$ and $\sigma_i > 0$, $i = 1, 2$.

(**a**)   (**b**)
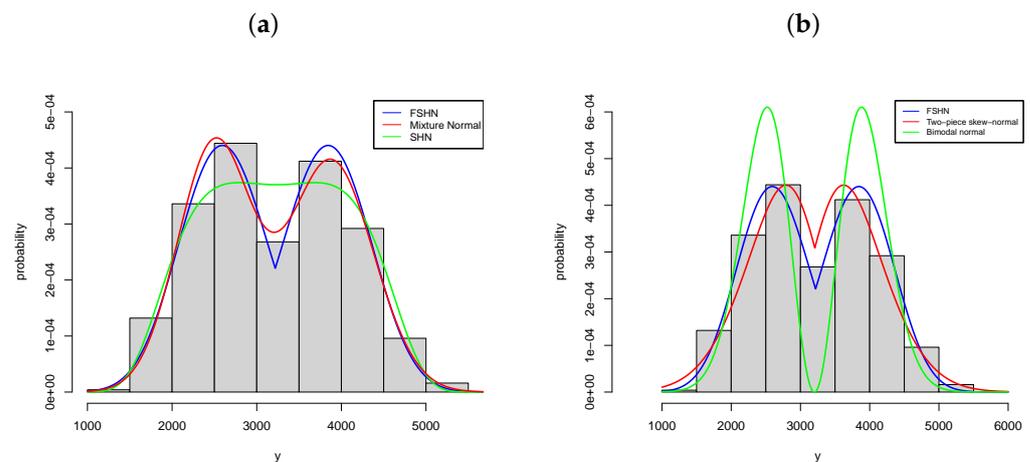
Figure 5. b.weight histogram and fitted bimodal models: (**a**) FSHN (blue), mixture normal (red) and SHN (green); and (**b**) FSHN (blue), two-piece skew normal (red) and bimodal normal (green).

The estimated parameters in these models, along with their standard errors in parentheses, are given in Table 3. In this table, the following abbreviations are used: SHN (sinh-normal), TN (two-piece skew normal [22]), BN (bimodal normal [21]), MN (mixture of two normals given in (35)), $FSHN_{\lambda=0}$ (FSHN with $\lambda = 0$). From AIC and BIC in Table 3, we can conclude that the mixture of two normals and the $FSHN_{\lambda=0}$ model provide the best fit to this dataset. However, it is of interest to point out that the use of mixtures is a quite controversial issue in statistics, mainly due to non-identifiability problems (see, e.g., [23]).

Table 3. Estimates of parameters, (standard errors) in models fitted in Dataset 2.

Estimator	SHN	TN	BN	MN	$FSHN_{\lambda=0}$
$\widehat{\gamma}\|\widehat{\mu}_1$	2.216			2501.859	0.355
	(0.296)			(51.012)	(0.141)
$\widehat{\xi}\|\widehat{\mu}_2$	3237.546	3207.422	3201.666	3880.479	3218.617
	(29.560)	(26.083)	(5.620)	(59.509)	(22.821)
$\widehat{\eta}\|\widehat{\sigma}_1$	972.770	772.688	481.088	436.380	2976.591
	(82.977)	(25.760)	(8.782)	(33.756)	(1043.259)
$\widehat{\delta}\|\widehat{\sigma}_2$				497.050	−1.156
				(40.223)	(0.133)
$\widehat{\alpha}$		1.771		0.4860	
		(0.539)		(0.040)	
AIC	8103.25	8109.67	8238.39	8096.99	8089.38
BIC	8111.51	8122.31	8246.82	8118.06	8112.45

Since SHN and $FSHN_{\lambda=0}$ are nested models, we can test

$$H_0 : \delta = 0 \qquad \text{versus} \qquad H_1 : \delta \neq 0,$$

which is equivalent to compare SHN versus $FSHN_{\lambda=0}$. Taking into account that the Fisher information matrix is non-singular at $(\delta, \lambda) = (0,0)$, we can consider the likelihood ratio statistic

$$\Lambda_1 = \frac{L_{SHN}(\widehat{\gamma}, \widehat{\xi}, \widehat{\eta})}{L_{FSHN}(\widehat{\gamma}, \widehat{\xi}, \widehat{\eta}, \widehat{\delta})}.$$

It is obtained that $-2\log(\Lambda_1) = 15.868$, which is greater than the $\chi^2_{1,0.05} = 3.84$. Thus, H_0 is rejected, leading to the conclusion that the proposed $FSHN_{\lambda=0}$ model fits better than the SHN considered in this case.

In Figure 5, the histogram of bebe weights is plotted along with the proposed bimodal models. There, the good fit provided by our proposal can be checked.

6.3. Illustration 3

Dataset 3 consists of 40 independent observations, which correspond to the failure time, T, for hardened steel specimens in a rolling contact fatigue test. The observations were taken at each of four values of contact stress, which is the covariate x in our proposal. This dataset can be seen in the work of Chan et al. [24]. Let $Y_i = \log T_i$ and consider the regression model

$$Y_i = \beta_0 + \beta_1 \log x_i + \epsilon_i, \quad i = 1, \ldots, 40. \tag{36}$$

We fit the log-Birnbaum–Saunders, the log-skew-Birnbaum–Saunders proposed by Lemonte et al. [13] and the log-flexible Birnbaum–Saunders introduced in this paper to this dataset. The estimated parameters along with the AIC and BIC are given in Table 4. According to AIC and BIC, log-flexible Birnbaum–Saunders provides a better fit than the other ones, so this model would be preferred.

Table 4. MLEs in LBS, LSBS and LFBS models.

Parameters	LBS	LSBS	LFBS
γ	1.279	2.011	1.988
	(0.143)	(0.313)	(0.527)
β_0	0.097	-0.961	-2.053
	(0.170)	(0.166)	(0.530)
β_0^*		0.165	-0.392
β_1	-14.116	-13.870	-13.812
	(1.571)	(1.602)	(1.248)
δ			-1.565
			(0.430)
λ		-0.932	2.073
		(0.174)	(0.910)
AIC	129.23	125.36	121.885
BIC	134.30	132.11	130.32

7. Simulation

Next, a simulation study is presented to illustrate the performance of our results. Two features are studied: (1) the global performance of MLEs of parameters if the sample size increases; and (2) the effect of varying every shape parameter, γ, δ and λ, on the performance of the rest of estimators under consideration. Both issues are studied in the use of the flexible sinh-normal distribution as a regression model.

7.1. Simulation I

The flexible sinh-normal distribution as a regression model with one covariate is considered, that is

$$Y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \qquad i = 1, \ldots, n,$$

where $\epsilon_i \sim FSHN(\gamma, 0, 2, \delta, \lambda)$ or, equivalently, $Y_i \sim FSHN(\gamma, \beta_0 + \beta_1 x_i, 2, \delta, \lambda)$ independent for $i = 1, \ldots, n$. The vector of unknown parameters is $(\gamma, \beta_0, \beta_1, \delta, \lambda)$.

As sample sizes, we consider $n \in \{25, 50, 75, 100, 500\}$, $m = 5000$ simulations and values of the parameters $(\gamma, \beta_0, \beta_1, \delta, \lambda) = (1, 1.75, 1.25, -1.5, 2)$. As x, covariate values of a $U(0,1)$ distribution are considered. Values of the FSHN were generated from values of the FSN distribution, which were obtained by using the stochastic representation of FSN model given in [5]. The results are listed in Tables 5–7. As for measures of performance, the mean of estimates, their standard errors (sd) in parentheses, the relative bias and the square root of the mean squared error (MSE) are given.

Table 5. Simulations for $\widehat{\gamma}$.

n	$\widehat{\gamma}$ (sd)	bias $(\widehat{\gamma})$	$\sqrt{MSE}$
25	0.9968 (0.2878)	0.2026	0.3833
50	1.0674 (0.2338)	0.1461	0.2966
75	1.0972 (0.2221)	0.1222	0.2696
100	1.1235 (0.2196)	0.1012	0.2534
500	1.0960 (0.2114)	0.0432	0.2181

Table 6. Simulations for $\widehat{\beta}_0$, $\widehat{\beta}_1$.

n	$\widehat{\beta}_0$ (sd)	bias $(\widehat{\beta}_0)$	$\sqrt{MSE}$	$\widehat{\beta}_1$ (sd)	bias $(\widehat{\beta}_1)$	$\sqrt{MSE}$
25	1.8205 (0.4539)	0.0403	0.4593	1.2866 (0.5022)	0.0293	0.5034
50	1.7290 (0.4054)	0.0120	0.4059	1.2681 (0.3424)	0.0145	0.3428
75	1.6813 (0.4027)	0.0393	0.4085	1.2621 (0.2702)	0.0097	0.2704
100	1.6348 (0.4074)	0.0659	0.4234	1.2617 (0.2373)	0.0094	0.2376
500	1.7141 (0.4475)	0.0205	0.5058	1.2514 (0.1027)	0.0011	0.1027

Table 7. Simulations for $\widehat{\delta}$ and $\widehat{\lambda}$.

n	$\widehat{\delta}$ (sd)	bias $(\widehat{\delta})$	$\sqrt{MSE}$	$\widehat{\lambda}$ (sd)	bias $(\widehat{\lambda})$	$\sqrt{MSE}$
25	-1.6190 (0.4530)	0.0793	0.4683	1.8495 (2.4948)	0.0753	2.4991
50	-1.6021 (0.3325)	0.0680	0.3478	2.0366 (1.0632)	0.0183	1.0638
75	-1.6048 (0.3013)	0.0699	0.3190	2.1029 (1.0358)	0.0515	1.0408
100	-1.6239 (0.2934)	0.0826	0.3185	2.1537 (1.0253)	0.0769	1.0366
500	-1.6025 (0.2981)	0.1083	0.3395	2.1683 (1.3550)	0.2342	1.4335

From the results in Tables 5–7, we can conclude that the MLEs are biased; in general, the standard error, relative bias and the squared root of MSE decrease if the sample size n increases. We highlight that, in this simulation, $\widehat{\beta}_1$ exhibits a good behavior, whereas the greater variability corresponds to $\widehat{\lambda}$.

Simulations were carried out by using the optim function of software R [25] by applying the Nelder–Mead method.

7.2. Simulation II

Next, a sensitivity study about the effect of varying the shape parameters (and the sample size) on the estimates of other parameters is presented.

Summaries varying the γ parameter are given in Table 8. Similarly, results varying the parameters δ and λ are given in Tables 9 and 10, respectively.

Table 8. Empirical sd, relative bias and $\sqrt{MSE}$ for the $FSHN(\gamma, 1.75, 1.25, -1.5, 2)$ model.

		$\hat{\gamma}$			$\hat{\beta}_0$			$\hat{\beta}_1$			$\hat{\delta}$			$\hat{\lambda}$		
γ	n	sd	RB	$\sqrt{MSE}$	sd	RB	$\sqrt{MSE}$	sd	RB	$\sqrt{MSE}$	sd	RB	$\sqrt{MSE}$	sd	RB	$\sqrt{MSE}$
	25	0.1223	0.0841	0.1241	0.1108	0.0214	0.1169	0.1600	0.0012	0.1600	0.7379	0.0040	0.7378	1.7371	0.0332	1.7382
	50	0.0362	0.048	0.0381	0.0851	0.0124	0.0878	0.1122	0.0003	0.1122	0.3840	0.0005	0.3839	0.9076	0.0463	0.9123
	75	0.0285	0.0338	0.0297	0.0732	0.0091	0.0749	0.0913	0.0019	0.0914	0.3288	0.0023	0.3288	0.8327	0.0698	0.8443
0.25	100	0.0246	0.0246	0.0254	0.0647	0.0078	0.0661	0.0766	0.0000	0.0766	0.3012	0.0093	0.3015	0.7395	0.0607	0.7494
	200	0.0174	0.0126	0.0177	0.0508	0.0037	0.0512	0.0548	0.0005	0.0548	0.2465	0.0028	0.2465	0.4737	0.0365	0.4793
	25	0.1801	0.0819	0.1903	0.3016	0.0561	0.3171	0.4195	0.0164	0.4199	0.5691	0.0069	0.5692	1.0871	0.0487	1.0914
	50	0.1285	0.0437	0.1326	0.2413	0.0327	0.2479	0.2896	0.0012	0.2895	0.4314	0.0075	0.4315	1.1060	0.0696	1.1146
	75	0.0989	0.0284	0.1012	0.2055	0.0208	0.2087	0.2326	0.0077	0.2327	0.3555	0.0071	0.3557	1.2154	0.0833	1.2266
0.75	100	0.0853	0.0221	0.0869	0.1829	0.0129	0.1843	0.1987	0.0023	0.1987	0.3147	0.0017	0.3147	0.8197	0.0753	0.8334
	200	0.0596	0.0100	0.0600	0.1444	0.0062	0.1448	0.1401	0.0013	0.1401	0.2387	0.0007	0.2387	0.5718	0.0478	0.5797
	25	0.3304	0.1037	0.3549	0.4336	0.1015	0.4685	0.5576	0.0212	0.5582	0.5984	0.0069	0.5985	3.4611	0.0179	3.4610
	50	0.2370	0.0551	0.2468	0.3606	0.0534	0.3725	0.3872	0.0046	0.3872	0.4080	0.0031	0.4079	1.0082	0.0505	1.0131
	75	0.1951	0.0388	0.2010	0.3106	0.0353	0.3166	0.3065	0.0065	0.3066	0.3286	0.0030	0.3286	0.9589	0.0750	0.9705
1.25	100	0.1703	0.0279	0.1738	0.2803	0.0255	0.2839	0.2596	0.0011	0.2596	0.2942	0.0010	0.2942	1.0168	0.0834	1.0303
	200	0.1278	0.0138	0.1290	0.2219	0.0110	0.2228	0.1858	0.0027	0.1858	0.2198	0.0025	0.2198	0.6109	0.0472	0.6181

Table 9. Empirical sd, relative bias and $\sqrt{MSE}$ for the $FSHN(1, 1.75, 1.25, \delta, 2)$ model.

		$\hat{\gamma}$			$\hat{\beta}_0$			$\hat{\beta}_1$			$\hat{\delta}$			$\hat{\lambda}$		
δ	n	sd	RB	$\sqrt{MSE}$	sd	RB	$\sqrt{MSE}$	sd	RB	$\sqrt{MSE}$	sd	RB	$\sqrt{MSE}$	sd	RB	$\sqrt{MSE}$
	25	0.7149	0.0835	0.7197	0.3610	0.0488	0.3709	0.5022	0.0080	0.5023	1.2311	0.1081	1.2358	5.1219	0.0603	5.1228
	50	0.5221	0.0493	0.5244	0.2857	0.0213	0.2881	0.3528	0.0022	0.3528	0.7337	0.0750	0.7374	1.3591	0.0676	1.3657
	75	0.1679	0.0347	0.1714	0.2515	0.0071	0.2518	0.2918	0.0073	0.2919	0.4105	0.0572	0.4144	0.9445	0.0657	0.9535
-1	100	0.1450	0.0270	0.1475	0.2311	0.0101	0.2317	0.2482	0.0003	0.2482	0.3644	0.0357	0.3661	0.8753	0.0660	0.8851
	200	0.1009	0.0138	0.1018	0.1842	0.0041	0.1843	0.1683	0.0016	0.1683	0.2750	0.0186	0.2756	0.6144	0.0355	0.6185
	25	0.9140	0.0934	0.9187	0.3478	0.0137	0.3486	0.4973	0.0149	0.4976	1.388	0.5287	1.4128	6.2775	0.0923	6.2795
	50	0.3620	0.0535	0.3659	0.2714	0.0018	0.2714	0.3426	0.0003	0.3426	0.7485	0.3018	0.7635	1.3659	0.0951	1.3790
	75	0.2473	0.0403	0.2506	0.2482	0.0012	0.2482	0.2754	0.0056	0.2755	0.5598	0.2215	0.5706	1.0731	0.0816	1.0853
-0.5	100	0.1735	0.0348	0.1770	0.2217	0.0041	0.2217	0.2415	0.0005	0.2415	0.4555	0.1987	0.4662	1.6011	0.0805	1.6090
	200	0.1195	0.0216	0.1214	0.1850	0.0025	0.1850	0.1643	0.0005	0.1643	0.3406	0.1183	0.3456	0.7319	0.0442	0.7371
	25	1.3881	0.0596	1.3892	0.3364	0.0093	0.3368	0.4892	0.0094	0.4893	2.0955	1.0817	2.1126	7.9980	0.2026	8.0075
	50	0.6369	0.0448	0.6384	0.2713	0.0036	0.2713	0.3298	0.0109	0.3301	1.1466	0.6245	1.1570	2.6481	0.1318	2.6609
	75	0.2728	0.0372	0.2753	0.2455	0.0026	0.2455	0.2654	0.0004	0.2653	0.6709	0.4761	0.6813	1.4155	0.1159	1.4342
-0.25	100	0.2067	0.0266	0.2084	0.2292	0.0074	0.2296	0.2272	0.0006	0.2272	0.5568	0.3704	0.5644	1.2191	0.1269	1.2451
	200	0.1518	0.0164	0.1527	0.1963	0.0050	0.1965	0.1603	0.0002	0.1603	0.4422	0.2235	0.4456	0.9483	0.0807	0.9619
	25	2.0897	0.1139	2.0925	0.3216	0.0211	0.3237	0.4466	0.0009	0.4466	3.4203	0.4546	3.4218	11.1958	0.479	11.2356
	50	0.8816	0.0242	0.8818	0.2650	0.0007	0.2650	0.3036	0.0053	0.3037	1.6859	0.0111	1.6857	2.8473	0.2296	2.8838
	75	0.5832	0.0087	0.5832	0.2383	0.0067	0.2386	0.2372	0.0038	0.2372	1.1711	0.1668	1.1717	2.1403	0.2225	2.1859
0.25	100	0.3810	0.0155	0.3813	0.2211	0.0100	0.2218	0.2083	0.0008	0.2083	0.9596	0.0649	0.9597	1.4984	0.2060	1.5538
	200	0.1888	0.0036	0.1888	0.1847	0.0116	0.1858	0.1459	0.0002	0.1459	0.5933	0.1700	0.5947	0.9973	0.1412	1.0364
	25	1.9152	0.1240	1.9191	0.2973	0.0191	0.2991	0.4197	0.0051	0.4197	3.4184	0.4688	3.4261	15.9834	0.5792	16.0237
	50	1.0250	0.0667	1.0271	0.2435	0.0022	0.2435	0.2802	0.0050	0.2802	2.0445	0.2053	2.0469	3.6007	0.3098	3.6533
	75	0.7143	0.0455	0.7157	0.2228	0.0076	0.2232	0.2284	0.0015	0.2284	1.4870	0.0934	1.4876	2.7262	0.2506	2.7716
0.5	100	0.3324	0.0137	0.3326	0.2076	0.0102	0.2084	0.1904	0.0029	0.1904	0.9070	0.0385	0.9071	1.3857	0.1926	1.4381
	200	0.2089	0.0110	0.2091	0.1698	0.0124	0.1711	0.1359	0.0002	0.1359	0.6743	0.0530	0.6747	0.9694	0.1442	1.0113
	25	1.9723	0.2052	1.9828	0.2573	0.0170	0.2590	0.3659	0.0037	0.3659	3.8193	0.5343	3.8562	18.1767	0.7324	18.2338
	50	1.3344	0.1345	1.3410	0.2116	0.0025	0.2116	0.2548	0.0034	0.2548	2.8532	0.3136	2.8701	4.7951	0.3444	4.8439
	75	0.8267	0.0764	0.8302	0.1906	0.0063	0.1909	0.2034	0.0013	0.2034	1.9203	0.1296	1.9245	2.4815	0.2837	2.5453
1.0	100	0.7316	0.0690	0.7347	0.1743	0.0106	0.1753	0.1711	0.0014	0.1711	1.7036	0.1055	1.7067	2.0484	0.2545	2.1105
	200	0.4334	0.0374	0.4350	0.1350	0.0105	0.1362	0.1184	0.0000	0.1183	1.1408	0.0382	1.1413	1.1312	0.1508	1.1706

Table 10. Empirical sd, relative bias and $\sqrt{MSE}$ for the $FSHN(1, 1.75, 1.25, -1.5, \lambda)$ model.

| | | $\hat{\gamma}$ | | | $\hat{\beta}_0$ | | | $\hat{\beta}_1$ | | | $\hat{\delta}$ | | | $\hat{\lambda}$ | | |
λ	n	sd	RB	$\sqrt{MSE}$	sd	RB	$\sqrt{MSE}$	sd	RB	$\sqrt{MSE}$	sd	RB	$\sqrt{MSE}$	sd	RB	$\sqrt{MSE}$
	25	0.2404	0.1088	0.2639	0.3883	0.0407	0.3948	0.5086	0.0004	0.5085	0.5320	0.1424	0.5732	0.5807	0.0151	0.5809
	50	0.1763	0.0508	0.1835	0.3073	0.0110	0.3079	0.3521	0.0068	0.3522	0.3709	0.0838	0.3916	0.4488	0.0282	0.4496
	75	0.1510	0.0325	0.1544	0.2648	0.0011	0.2648	0.2838	0.0019	0.2838	0.2999	0.0606	0.3134	0.3266	0.0193	0.3272
-1	100	0.1343	0.0284	0.1372	0.2301	0.0053	0.2303	0.2445	0.0029	0.2445	0.2604	0.0468	0.2696	0.2926	0.0096	0.2927
	200	0.0979	0.0099	0.0983	0.1628	0.0013	0.1628	0.1693	0.0020	0.1693	0.1828	0.0194	0.1851	0.1893	0.0066	0.1894
	25	0.2206	0.0935	0.2396	0.3742	0.0102	0.3746	0.5369	0.0040	0.5369	0.4774	0.1842	0.5515	0.3220	0.0099	0.3220
	50	0.1729	0.0363	0.1766	0.2690	0.0003	0.2690	0.3778	0.0045	0.3778	0.3421	0.0819	0.3635	0.2121	0.0134	0.2122
	75	0.1446	0.0258	0.1468	0.2160	0.0008	0.2160	0.2956	0.0060	0.2957	0.2785	0.0565	0.2911	0.1656	0.0009	0.1655
-0.5	100	0.1235	0.0212	0.1253	0.1817	0.0016	0.1817	0.2532	0.0014	0.2532	0.2390	0.0438	0.2479	0.1413	0.0004	0.1413
	200	0.0870	0.0102	0.0876	0.1277	0.0008	0.1277	0.1754	0.0028	0.1754	0.1702	0.0224	0.1734	0.0984	0.0008	0.0984
	25	0.2086	0.0831	0.2245	0.3642	0.0052	0.3643	0.5740	0.0008	0.5739	0.4812	0.1713	0.5454	0.2217	0.0212	0.2217
	50	0.1527	0.0389	0.1576	0.2436	0.0003	0.2436	0.3869	0.0011	0.3869	0.3409	0.0785	0.3606	0.1442	0.0008	0.1442
	75	0.1245	0.0247	0.1269	0.1979	0.0008	0.1979	0.3083	0.0039	0.3083	0.2841	0.0510	0.2942	0.1142	0.0040	0.1142
-0.25	100	0.1068	0.0194	0.1086	0.1714	0.0002	0.1714	0.2675	0.0016	0.2675	0.2410	0.0398	0.2482	0.0962	0.0064	0.0962
	200	0.0774	0.0097	0.0780	0.1168	0.0008	0.1168	0.1813	0.0002	0.1813	0.1729	0.0183	0.1750	0.0688	0.0051	0.0688
	25	0.2099	0.0859	0.2267	0.3646	0.0075	0.3648	0.5633	0.0033	0.5633	0.4799	0.1791	0.5499	0.2034	0.0335	0.2035
	50	0.1537	0.0407	0.1590	0.2517	0.0003	0.2517	0.3856	0.0064	0.3857	0.3423	0.0830	0.3642	0.1473	0.0040	0.1473
	75	0.1263	0.0288	0.1296	0.1979	0.0011	0.1979	0.3079	0.0018	0.3079	0.2813	0.0566	0.2938	0.1142	0.0006	0.1142
0.25	100	0.1094	0.0194	0.1111	0.1696	0.0032	0.1697	0.2644	0.0031	0.2644	0.2449	0.0399	0.2521	0.0984	0.0127	0.0984
	200	0.0757	0.0092	0.0762	0.1185	0.0001	0.1185	0.1858	0.0011	0.1858	0.1689	0.0194	0.1714	0.0683	0.0008	0.0683
	25	0.2258	0.0986	0.2463	0.3932	0.0166	0.3942	0.5549	0.0022	0.5549	0.4783	0.1898	0.5566	0.3089	0.0263	0.3091
	50	0.1761	0.0447	0.1817	0.2753	0.0032	0.2754	0.373	0.0040	0.3730	0.3343	0.0893	0.3601	0.2124	0.0042	0.2124
	75	0.1420	0.0317	0.1455	0.2131	0.0031	0.2131	0.2950	0.0021	0.2950	0.2779	0.0630	0.2935	0.1695	0.0016	0.1694
0.5	100	0.1236	0.0216	0.1255	0.1885	0.0037	0.1886	0.2571	0.0060	0.2572	0.2379	0.0434	0.2466	0.1416	0.0017	0.1416
	200	0.088	0.0087	0.0884	0.1294	0.0014	0.1294	0.1765	0.0002	0.1765	0.1695	0.0175	0.1715	0.0969	0.0007	0.0969
	25	0.2453	0.1065	0.2674	0.3931	0.0453	0.401	0.5165	0.0009	0.5165	0.5063	0.1383	0.5471	0.5795	0.0012	0.5795
	50	0.1809	0.0613	0.1910	0.3007	0.0230	0.3034	0.3507	0.0021	0.3507	0.3503	0.0803	0.3704	0.4679	0.0108	0.4680
	75	0.1550	0.0380	0.1595	0.2602	0.0117	0.2609	0.2878	0.0018	0.2877	0.2877	0.0541	0.2989	0.3588	0.0176	0.3592
1.0	100	0.1345	0.0308	0.1380	0.2310	0.0087	0.2315	0.2468	0.0000	0.2467	0.2534	0.0421	0.2611	0.2835	0.0060	0.2835
	200	0.0982	0.0140	0.0992	0.1662	0.0037	0.1663	0.1725	0.0030	0.1726	0.1814	0.0223	0.1845	0.1915	0.0028	0.1915

Comments to Table 8 (varying $\gamma > 0$).

In this table, we consider $\beta_0 = 1.75$, $\beta_1 = 1.25$, $\delta = -1.5$, $\lambda = 2$, $\gamma \in \{0.25, 0.75, 1.25\}$ and sample sizes $n \in \{25, 50, 75, 100, 200\}$. For $\widehat{\beta_0}$ and $\widehat{\beta_1}$, it can be seen that the relative biases are small. Moreover, we can appreciate that the relative bias, standard error and $\sqrt{MSE}$ decrease when the sample size increases, especially for $\widehat{\beta_1}$. $\widehat{\delta}$ and $\widehat{\lambda}$ exhibit greater variability, but their relative bias, sd and $\sqrt{MSE}$ also decrease when the sample size increases. As for $\widehat{\gamma}$, we can point out that this estimator is also well behaved when n increases. Its relative bias for small sample sizes, $n = 25$, is small and the $\sqrt{MSE}$ is a moderate value.

Comments to Table 9 (varying $\delta \in \mathbb{R}$).

In this table, we consider $\gamma = 1$, $\beta_0 = 1.75$, $\beta_1 = 1.25$, $\lambda = 2$, $\delta \in \{-1, -0.5, -0.25, 0.25, 0.5, 1\}$ and $n \in \{25, 50, 75, 100, 200\}$. We can see that, for $\widehat{\beta_0}$ and $\widehat{\beta_1}$, the relative bias is small even for $n = 25$, and the relative bias, sd and $\sqrt{MSE}$ decrease when n increases. Again, $\widehat{\delta}$ and $\widehat{\lambda}$ exhibit greater variability than the other estimators, but the error measures under consideration are satisfactory. The relative bias, sd and $\sqrt{MSE}$ of $\widehat{\gamma}$ decrease when n increases, and their values are moderate even for small sample sizes.

Comments to Table 10 (varying $\lambda \in \mathbb{R}$).

In this case, we consider $\gamma = 1$, $\beta_0 = 1.75$, $\beta_1 = 1.25$, $\delta = -1.5$, $\lambda \in \{-1, -0.5, -0.25, 0.25, 0.5, 1\}$ and $n \in \{25, 50, 75, 100, 200\}$. Again, a similar behavior to the previously explained is obtained.

As final conclusion, we point out that these simulation studies cover a variety of situations as for the shapes of the FSHN distribution (unimodal and bimodal) and suggest that the estimators of the parameters are consistent when the sample size increases [26].

8. Discussion

The BS distribution is an asymmetric model used for survival time data and material lifetime subject to stress as it can be seen in [27]. Due to its practical and theoretical interest, a number of generalizations can be found in literature. We can cite the extensions provided in [28] to the family of elliptical distributions; in [10] based on the elliptical asymmetric distributions; and the extended Birnbaum–Saunders (EBS) distribution introduced in [7].

Other generalizations intend to solve specific deficiencies observed when this model is fitted to a dataset. In this sense, we can cite the epsilon-Birnbaum–Saunders model introduced in [8] to accommodate outliers; the extension based on the slash-elliptical family of distributions given in [3]; and the generalized modified slash Birnbaum–Saunders, which is based on the work in [29] and proposed in [30]. All these extensions are appropriate to fit data with greater or smaller asymmetry (or kurtosis) than that of the usual BS model, but they are not appropriate for fitting bimodal data. This issue is of great interest, as discussed by Olmos et al. [9], Bolfarine et al. [31], Martínez-Flórez et al. [2] and Elal-Olivero et al. [32]. The flexible BS distribution was proposed to model skewness and to fit data with and without bimodality. In this paper, this model is spread out to be used as a regression model in those situations in which regression models based on other generalizations of BS, such as those proposed in [7,13,33–37], do not provide a satisfactory fit.

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Data Availability Statement: Details about data availability are given in Sections 6.1–6.3.

Abbreviations

The following abbreviations are used in this manuscript:

BS	Birnbaum–Saunders
cdf	cumulative distribution function
FBS	Flexible Birnbaum-Saunders
FSHN	Flexible sinh-normal
FSN	Flexible skew Normal
pdf	probability density function
SHN	Sinh-normal

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