

Article

Hadamard Matrices with Cocyclic Core

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Abstract: Since Horadam and de Launey introduced the cocyclic framework on combinatorial designs in the 1990s, it has revealed itself as a powerful technique for looking for (cocyclic) Hadamard matrices. Ten years later, the series of papers by Kotsireas, Koukouvinos and Seberry about Hadamard matrices with one or two circulant cores introduced a different structured approach to the Hadamard conjecture. This paper is built on both strengths, so that Hadamard matrices with cocyclic cores are introduced and studied. They are proved to strictly include usual Hadamard matrices with one and two circulant cores, and therefore provide a wiser uniform approach to a *structured* Hadamard conjecture.

Keywords: Hadamard matrix; circulant matrix; cocyclic matrix; difference set

MSC: 05B20; 05B10



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1. Introduction

Hadamard matrices are square matrices of order n with entries from $\{-1, 1\}$ such that their rows are pairwise orthogonal. They were identified as the extremal solutions of the maximal determinant problem for square matrices with entries from the unit disc [1].

It may be straightforwardly checked that, regardless of the elemental cases $n = 1$ and $n = 2$, as soon as three rows have to be mutually orthogonal, this implies that n is necessarily a multiple of 4. Unexpectedly, the converse assertion (that there exists a Hadamard matrix for every order $n = 4w$), remains one of the most famous open century-old problems in mathematics, the Hadamard Conjecture.

The techniques for constructing Hadamard matrices are usually organized into three types: multiplication theorems, “plug-in” methods and direct constructions [2,3]. Nevertheless, none of them has succeeded to provide a uniform method for constructing these matrices.

The cocyclic approach, as introduced in [4], tries to cast new light on this purpose. A cocyclic matrix (over a group G) consists of a matrix $(\psi(i, j))$ constructed from a binary (2-)cocycle over G , $\psi : G \times G \rightarrow \{-1, 1\}$, so that:

$$\psi(a, b) \psi(a, b \cdot c) \psi(a \cdot b, c) \psi(b, c) = 1, \quad (1)$$

for all $a, b, c \in G$. It is called a 2-coboundary if a map $\phi : G \rightarrow G$ exists such that $\psi(a, b) = \phi(a)\phi(b)\phi(ab)$, for every $a, b \in G$.

Accordingly, checking whether a cocyclic matrix is Hadamard translates into checking if the summation of each row (but the first) is zero [5]. This is not the only advantage of this approach. It endows the matrix with an enriched structure, which is being successfully exploited to look for Hadamard matrices [2,6,7]. Nowadays, this framework is broadening its expected horizons, since more recently even quasigroups [8] and Latin rectangles [9] have come into play.

(Cocyclic) Hadamard matrices are intimately related to other combinatorial objects, such as designs, (almost) difference sets and (almost) perfect sequences. We next briefly recall these notions and relations, as they are introduced in [2,10], since they will be relevant in the exposition of the paper. For further information, the interested reader is referred to [2,10] and the references therein.

A (v, k, λ) -design is a pair $\mathcal{D} = (P, B)$ consisting of a v -set $P = \{p_1, \dots, p_v\}$ of points and a v -set $B = \{B_1, \dots, B_v\}$ of k -blocks, $1 < k < v$, such that each pair of distinct points is contained in exactly λ blocks.

If G is an automorphism group of \mathcal{D} such that for each pair of points p_i, p_j there is a unique $g \in G$ such that $g(p_i) = p_j$, and similarly for blocks, the design \mathcal{D} is called regular with respect to G , and G is called a regular group for \mathcal{D} .

Matricially, \mathcal{D} is easily represented by its $v \times v$ incidence matrix $A_{\mathcal{D}} = (a_{ij})$, rows and columns in correspondence with blocks and points, respectively, such that $a_{ij} = 1$ if and only if $p_j \in B_i$, and 0 otherwise.

It may be straightforwardly checked that a $v \times v$ square matrix A with entries 0,1 is an incidence matrix of a (v, k, λ) -design if and only if $AA^T = (k - \lambda)I + \lambda J$ and $AJ = kJ$, for I the identity matrix and J the all 1s matrix.

In particular, there exists a Hadamard matrix H of order $4w$ if and only if there exists a square $(4w - 1, 2w - 1, w - 1)$ -design \mathcal{D} . Actually, noting $A' = 2A_{\mathcal{D}} - J$, the matrix obtained from $A_{\mathcal{D}}$ by replacing 0s by -1 s, then $H = \begin{pmatrix} 1 & \mathbf{1} \\ \mathbf{1}^T & A' \end{pmatrix}$, is a Hadamard matrix, and vice versa.

Other combinatorial structures intimately related to Hadamard matrices and designs are difference sets.

A (v, k, λ) -difference set in a group (G, \cdot) of order v is a k -subset $D \subset G$ such that the $k(k - 1)$ -list of quotients $d_1 d_2^{-1}$ of distinct elements d_1, d_2 of D contains each non-identity element of G exactly λ times. The order of the difference set is $n = k - \lambda$. The difference set is called cyclic or abelian, if G has the respective property.

Remark 1. Notice that if D is a (v, k, λ) -difference set, then $G \setminus D$ defines a $(v, v - k, v - 2k + \lambda)$ -difference set, the so-called complementary difference set of D .

Let G be a group of v elements, and $D \subset G$ a k -subset, $1 < k < v$. Then D is a (v, k, λ) -difference set in G if and only if $\mathcal{D} = (G, \{gD : g \in G\})$ is a (v, k, λ) -design with regular group G , and vice versa. Therefore, as soon as a $(4w - 1, 2w - 1, w - 1)$ -difference set exists, there exists a Hadamard matrix of order $4w$ in turn.

Now we turn our interest to (almost) perfect sequences, as introduced in [10].

Consider a cyclic binary sequence (a_i) with entries from $\{-1, 1\}$ of period v , such that $a_{i+v} = a_i$, for $i \geq 0$. The autocorrelation of (a_i) for shift t is defined as the sum $c_t(a_i) = \sum_{i=0}^{v-1} a_i a_{i+t}$.

A sequence (a_i) is called perfect when $c_t(a_i) = r$ is constant for $0 < t \leq v - 1$, and equals the smallest possible $\max |c_t(a_i)| \in \{0, 1, 2\}$, since $c_t(a) \equiv v \pmod{4}$.

Remark 2. Notice that a perfect periodic sequence with period v , k positive entries per period and autocorrelation $c_t(a_i) = r, 0 < t \leq v - 1$, is equivalent to a cyclic (v, k, λ) -difference set: it suffices to take $D = \{i : a_i = 1, 0 \leq i \leq v - 1\}$, so that $|D \cap (D + t)| = \lambda$ and $r = v - 4(k - \lambda)$.

Depending on the value $c_t(a_i) = r \in \{0, \pm 1, \pm 2\}$, the existence of perfect sequences (a_i) is summarized in [10]. In particular:

- The case $r = 0$ corresponds to circulant Hadamard matrices of order v , which are conjectured to exist just for $v = 4$.
- The case $r = 1$ corresponds to $(2u^2 + 2u + 1, u^2, \frac{u^2 - u}{2})$ -cyclic difference sets, which are conjectured to exist just for $u = 1, 2$.

- There is no known example for the case $r = 2$, and experimental results suggest that none might exist.
- The case $r = -2$ gives just one difference set, the $(2, 1, 0)$ -difference set.
- The case $r = -1$ gives the so-called Paley–Hadamard difference sets, with parameters $(v, \frac{v-1}{2}, \frac{v-3}{4})$, whose existence implies in turn the existence of Hadamard matrices of order $v + 1$. These include very well known families of difference sets [10].

Remark 3. The case $r = -1$ is of particular interest in this paper, since it includes the so-called Hadamard matrices with one circulant core [11], of the form $\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & A \end{pmatrix}$, for A being a circulant matrix whose first row consists of a perfect sequence (a_i) with $c_t(a_i) = -1, 0 < t \leq 4w - 2$, and $\sum_{i=0}^{4w-2} a_i = -1$.

When relaxing some conditions, the combinatorial structures described above may be generalized, in some sense.

For instance, given a k -subset D of a group (G, \cdot) of order v , it might occur that the number of occurrences of each non identity element in G among the list of quotients $d_1 d_2^{-1}$ of distinct elements $d_1 \neq d_2 \in D$ is either λ or $\lambda + 1$, not just λ . Then D defines a (v, k, λ, s) -almost difference set, where s denotes the amount of elements of $G \setminus \{1\}$ that appears precisely λ times in the list $d_1 d_2^{-1}$, the remaining $v - s - 1$ elements of G appearing exactly $\lambda + 1$ times.

Almost difference sets are related to almost perfect sequences, in the following manner.

A sequence (a_i) is called almost perfect when two possible values r_1, r_2 are allowed for $c_t(a_i)$, for $0 < t \leq v - 1$, with $|r_1| + |r_2| = 4$. As noted in [10], this is equivalent to $D = \{i : a_i = 1, 0 \leq i \leq v - 1\}$ being an $(v, k, k - \lfloor \frac{v}{4} \rfloor - 1, kv - k^2 - (k - 1)\lfloor \frac{v}{4} \rfloor)$ almost difference set, for $k = |D|$.

As claimed in [10], computer experimentation seems to suggest that balanced (i.e., half of the entries are of opposite sign) almost perfect sequences might exist for every $v \equiv 1 \pmod{4}$. As soon as two such balanced sequences (a_i) and (b_i) combine to give $c_t(a_i) + c_t(b_i) = -2$, for $0 < t \leq 2w - 2$, a Hadamard matrix of order $4w$ would exist in turn: this is one of the most prolific techniques for constructing Hadamard matrices of order $4w$, the so-called two circulant cores "plug-in" method [3,12,13].

Actually, the Hadamard matrix would consist in:

$$\begin{pmatrix} -1 & -1 & \mathbf{1} & \mathbf{1} \\ -1 & \mathbf{1} & \mathbf{1} & -\mathbf{1} \\ \mathbf{1} & \mathbf{1} & A & B \\ \mathbf{1} & -1 & B^T & -A^T \end{pmatrix}, \tag{2}$$

for A and B being the circulant matrices based on (a_i) and (b_i) , respectively.

As a matter of fact, among the most prominent "plug-in" methods for constructing Hadamard matrices, precisely the two circulant cores technique and the Goethals–Seidel arrays [14] were the only ones that remain out of the cocyclic framework, as indicated in [15]. Actually, Goethals–Seidel arrays have been recently characterized as (pseudo)cocyclic matrices over quasigroups in [16].

Finally, we now recall the characterization of cocyclic Hadamard matrices in terms of certain difference sets and designs, as introduced in Theorem 2.4 of [17], and which is used in [6] to provide an exhaustive classification of cocyclic Hadamard equivalent matrices of orders less than 40. For the sake of simplicity and readability, we will not detail the definition of these particular combinatorial structures, as they are not needed for the comprehension of the paper.

Theorem 1. [17] *The following statements are equivalent:*

1. *There is a cocyclic Hadamard matrix over G , $|G| = 4w$.*
2. *There is a (normal) relative $(4w, 2, 4w, 2w)$ -difference set in a central extension of $\langle -1 \rangle$ by G , relative to $\langle -1 \rangle$.*
3. *There is a divisible $(4w, 2, 4w, 2w)$ -design, class regular with respect to $\langle -1 \rangle$, and with central extension of $\langle -1 \rangle$ by G as a regular group of automorphisms.*

The aim of this paper is to settle a theoretical background for the study of Hadamard matrices consisting of cocyclic cores, instead of circulant cores. Actually, Hadamard matrices with a cocyclic core will be shown to be placed in some sense halfway from full cocyclic Hadamard matrices and usual Hadamard matrices, as $4w \times 4w$ Hadamard matrices with a cocyclic core will be proven to exist if and only if there exist $(4w - 1, 2w - 1, w - 1)$ -difference sets (Theorem 2), which are already known to imply the existence of full Hadamard matrices, as commented before.

The paper is organized as follows. In Section 2, we characterize the existence of Hadamard matrices with a $(4w - 1) \times (4w - 1)$ cocyclic core, in terms of some perfect sequences and difference sets. Section 3 is devoted to the characterization of Hadamard matrices with a $(4w - 2) \times (4w - 2)$ cocyclic core, in terms of certain almost perfect sequences and almost difference sets.

2. Hadamard Matrices with a $(4w - 1) \times (4w - 1)$ Cocyclic Core

As introduced in Section 1, Hadamard matrices with one circulant core consist of matrices of the type $\begin{pmatrix} 1 & \mathbf{1} \\ \mathbf{1} & A \end{pmatrix}$, for A being a circulant matrix whose first row consists of a perfect binary sequence (a_i) with $c_t(a_i) = -1, 0 < t \leq 4w - 2$, and $\sum_{i=0}^{4w-2} a_i = -1$. They were extensively studied in [11].

In this section we study the conditions under which a cocyclic (instead of a simply circulant) matrix M_ψ (for ψ as in (1)) may be plugged in a structure of the type:

$$\begin{pmatrix} ? & ? \\ \alpha & M_\psi \end{pmatrix}, \tag{3}$$

in order to provide a Hadamard matrix as well, for $\alpha = (\alpha_{g_0}, \dots, \alpha_{g_{4w-2}})$, given an ordering $\{g_0 = 1, \dots, g_{4w-2}\}$ of G .

Notice that the first row of such a structure is not of interest, since every partial Hadamard matrix of size $(4w - z) \times 4w, 1 \leq z \leq 7$, is known to be extendable to a full $4w \times 4w$ Hadamard matrix [18].

Theorem 2. *There exists a Hadamard matrix of order $4w$ with a $(4w - 1) \times (4w - 1)$ cocyclic core if and only if there exists a $(4w - 1, 2w, w)$ -difference set (or, equivalently, a $(4w - 1, 2w - 1, w - 1)$ -difference set).*

The following sequence of lemmas will help in the task of proving this result.

Lemma 1. *The first row (and column) of M_ψ consists of a common single value, either 1 or -1 .*

Proof. This is a common fact of cocyclic matrices, and it may be straightforwardly derived from Equation (1): as soon as one fixes $a = b = 1 \in G$, it follows that $\psi(1, 1) \cdot \psi(1, g_k) = 1$, for every $g_k \in G$.

A similar argument applies to the first column, taking $b = c = 1$ in (1) instead. \square

Remark 4. *Cocyclic matrices M_ψ for which $\psi(1, j) = \psi(j, 1) = 1$ are usually termed normalized.*

The following result is at the very heart of the usual cocyclic test for Hadamard matrices [2,5]. However, we will rather state it and reproduce its proof explicitly, since both of them will be extensively used in the paper.

Lemma 2. For every $g_i, g_j \in G$, the dot product of rows indexed by g_i and g_j in M_ψ consists of:

$$\sum_{g_k \in G} \psi(g_i, g_k)\psi(g_j, g_k) = \psi(g_i g_j^{-1}, g_j) \sum_{g_k \in G} \psi(g_i g_j^{-1}, g_k).$$

Proof. Actually, this is an alternative way to read the usual cocyclic Hadamard test of [5]. From (1), taking $a = g_i g_j^{-1}$, $b = j$ and $c = g_k$, it follows that:

$$\psi(g_i g_j^{-1}, g_j) \cdot \psi(g_i g_j^{-1}, g_j g_k) = \psi(g_i, g_k) \cdot \psi(g_j, g_k).$$

Therefore, the dot product of rows indexed by g_i and g_j in M_ψ consists of:

$$\sum_{g_k \in G} \psi(g_i, g_k)\psi(g_j, g_k) = \sum_{g_k \in G} \psi(g_i g_j^{-1}, g_j)\psi(g_i g_j^{-1}, g_j g_k) = \psi(g_i g_j^{-1}, g_j) \sum_{g_k \in G} \psi(g_i g_j^{-1}, g_k).$$

□

Lemma 3. Assume that the matrix in (3) is Hadamard. Then $\sum_{g_k \in G} \alpha_{g_k} = \pm 1$.

Proof. From Remark 4 we know that $\psi(g_k, 1)$ is constant (for both 1 and -1) for every $g_k \in G$. Therefore, the dot product of columns 1 and 2 of the matrix in (3) consists of:

$$0 = \pm 1 \pm \sum_{g_k \in G} \alpha_{g_k},$$

and the result follows. □

Lemma 4. Assume that the matrix in (3) is Hadamard. Then $\sum_{g_k \in G} \psi(g_i, g_k) = -\psi(g_i, 1) \cdot \alpha_1 \cdot \alpha_{g_i}$, for every $g_i \in G \setminus \{1\}$.

Proof. Taking $g_j = 1 \in G$ in Lemma 2, it follows that the dot product of rows indexed by g_i and 1 in (3) consists of:

$$0 = \alpha_{g_i} \cdot \alpha_1 + \psi(g_i, 1) \sum_{g_k \in G} \psi(g_i, g_k),$$

from which the asserted result derives at once. □

Lemma 5. Assume that the matrix in (3) is Hadamard. Then $\psi(g_i, g_j) = \psi(g_i, 1) \cdot \alpha_1 \cdot \alpha_{g_i} \cdot \alpha_{g_j} \cdot \alpha_{g_i g_j}$, for every $g_i, g_j \in G$.

Proof. From Lemma 2, the dot product of the rows indexed by $g_i g_j$ and g_j in (3) consists of:

$$0 = \alpha_{g_i g_j} \cdot \alpha_{g_j} + \psi(g_i, g_j) \sum_{g_k \in G} \psi(g_i, g_k).$$

Consequently, taking into account Lemma 4,

$$0 = \alpha_{g_i g_j} \cdot \alpha_{g_j} - \psi(g_i, 1) \cdot \alpha_1 \cdot \alpha_{g_i} \psi(g_i, g_j),$$

so that $\psi(g_i, g_j) = \psi(g_i, 1) \cdot \alpha_1 \cdot \alpha_{g_i} \cdot \alpha_{g_j} \cdot \alpha_{g_i g_j}$. □

In the circumstances above, the following results may be straightforwardly derived.

Corollary 1. ψ is completely characterized as a 2-coboundary $\psi(g_i, g_j) = \phi(g_i)\phi(g_j)\phi(g_i g_j)$, for $\phi : G \rightarrow \{-1, 1\}$ defined as $\phi(g_i) = \phi(g_1) \cdot \alpha_1 \cdot \alpha_{g_i}$, $\phi(g_1) = \pm 1$.

Remark 5. Notice that it is irrelevant whether one fixes $\phi(g_1)$ to be either 1 or -1 . This just has to be with the (un)normalized character of the related 2-cocycle, as noted in Remark 4.

Corollary 2. $\sum_{g_j \in G} \alpha_{g_j} \alpha_{g_i g_j} = -1$, for every $g_i \in G$.

Proof. From Lemma 5, $\psi(g_i, g_j) = \alpha_1 \alpha_{g_i} \alpha_{g_j} \alpha_{g_i g_j}$. Adding as g_j runs on G , we obtain:

$$-\alpha_1 \alpha_{g_i} = \sum_{g_j \in G} \psi(g_i, g_j) = \alpha_1 \alpha_{g_i} \sum_{g_j \in G} \alpha_{g_j} \alpha_{g_i g_j},$$

from which the result derives at once. \square

Lemma 6. Assume that (α_{g_k}) and M_ψ are given so that rows from 2 to $4w$ in (3) are pairwise orthogonal. Then this matrix may be completed to a full Hadamard matrix, in essentially one way (up to negation of the row).

Proof. Let $(s_1, s_2, \dots, s_{4w})$ be the unknown entries of the first row of (3).

Lemma 1 guarantees that the first column in M_ψ consists of a common constant, $\psi(1, 1)$. Take $s_2 = \psi(1, 1)$, so that the second column of (3) consists of a common constant.

From the orthogonality law, the dot product of column $i \neq 2$ with the second column has to be zero. Consequently, since the second column is formed by a common constant, the summation of each column $i \neq 2$ of (3) has to be zero, as well. This characterizes uniquely the first row of the matrix (once s_2 has been fixed). \square

We can now prove Theorem 2.

Proof. We first prove the sufficient condition.

Assume that a Hadamard matrix of the form (3) exists, with a cocyclic core consisting of M_ψ .

In these circumstances, notice that (α_{g_k}) defines a *generalized* perfect sequence, in the sense that:

- It is balanced, as Lemma 3 indicates.
- A generalized autocorrelation function for (α_{g_i}) may be defined, so that shift g_t depends on the right action of g_t by means of the group law of G : $c_t(\alpha_{g_k}) = \sum_{g_k \in G} \alpha_{g_k} \alpha_{g_k g_t}$.
- Corollary 2 shows that (α_{g_k}) is perfect.

In these circumstances, the set $D = \{g_k : \alpha_{g_k} = 1\}$ defines a difference set, as noted in Remark 2.

Since the sequence is balanced, it follows that $|D| = 2w$ or $|D| = 2w - 1$. Furthermore, it may be straightforwardly checked that $|D \cap D \cdot g_t| = |D| - w$. No matter what the concrete value of $|D|$ is, a couple of complementary difference sets with parameters $(4w - 1, 2w, w)$ and $(4w - 1, 2w - 1, w - 1)$ are thus defined (see Remark 1).

Conversely, starting from such a difference set, as noted in Remark 2, the process works the other way around, and the proof ends, as soon as the matrix so obtained may be extended to a full Hadamard matrix, attending to Lemma 6. \square

Example 1. Consider the multiplicative cyclic group of order 3, $G = (C_3, \cdot) = \langle a : a^3 = 1 \rangle$, endowed with the natural ordering $\{1, a, a^2\}$.

The subset $D = \{a, a^2\}$ defines a $(3, 2, 1)$ -difference set in G , since $a \cdot a^{-2} = a^2$ and $a^2 \cdot a^{-1} = a$.

Let us consider the sequence $\alpha = (\alpha_g)_{g \in D} = (-1, 1, 1)$, such that $\alpha_g = 1$ if $g \in D$, and $\alpha_g = -1$ otherwise.

By construction, (α_g) defines a perfect sequence. Actually:

- $\alpha_1 \cdot \alpha_{1.a} + \alpha_a \cdot \alpha_{a.a} + \alpha_{a^2} \cdot \alpha_{a^2.a} = -1.$
- $\alpha_1 \cdot \alpha_{1.a^2} + \alpha_a \cdot \alpha_{a.a^2} + \alpha_{a^2} \cdot \alpha_{a^2.a^2} = -1.$

Consider the map $\phi : (C_3, \cdot) \rightarrow \{-1, 1\}$ defined as $\phi(1) = 1, \phi(a) = \alpha_1 \cdot \alpha_a = -1, \phi(a^2) = \alpha_1 \cdot \alpha_{a^2} = -1.$ Let $\psi : C_3 \times C_3 \rightarrow \{-1, 1\}$ be the 2-coboundary defined as $\psi(i, j) = \phi(i)\phi(j)\phi(i \cdot j),$ so that M_ψ reads as:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Plugging this matrix in (3), we get:

$$\left(\begin{array}{c|ccc} ? & ? & ? & ? \\ -1 & 1 & 1 & 1 \\ \hline 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{array} \right).$$

This matrix may be completed to form a full Hadamard matrix, as soon as one takes its first row to be $(-1, 1, -1, -1),$ attending to Lemma 6.

Remark 6. Notice that if one negates both of the third and fourth rows and columns of the Hadamard matrix of Example 1, one obtains the Hadamard equivalent matrix:

$$\left(\begin{array}{c|ccc} -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ \hline -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \end{array} \right),$$

which is indeed a Hadamard matrix with a back-circulant core. This may be transformed into a Hadamard equivalent matrix consisting of a usual circulant core, as soon as rows are permuted accordingly. This is not a casual fact, as the following result asserts.

Proposition 1. Any Hadamard matrix with a circulant core is Hadamard equivalent to a Hadamard matrix with a cocyclic core.

Proof. As noted in Remark 3 (see [11] for details), any $4w \times 4w$ Hadamard matrix H with a circulant core H_C is in one-to-one correspondence with the cyclic $(4w, 2w, w)$ -difference set $D = \{a^i : H_C(1, i - 1) = 1\} \subset (C_{4w-1}, \cdot),$ consisting of the elements that correspond to the positions at which the 1s of the first row of H_C are located.

As illustrated in Example 1, starting from such a cyclic difference set, application of the constructive proof of Theorem 2 provides a Hadamard matrix with a cocyclic core $M_\psi.$ Furthermore, Corollary 1 shows that the map ϕ underlying $M_\psi,$ as defined, provides a cocyclic core, which is Hadamard equivalent to the initial circulant core. To check this, it suffices to negate both the sets of rows and columns indexed by the elements in $D,$ and reorder the rows from bottom to top. \square

A question arises in a natural way: is there any advantage in looking for Hadamard matrices with cocyclic cores instead of Hadamard matrices with circulant cores?

Proposition 2. Hadamard matrices with cocyclic cores strictly include Hadamard matrices with one circulant core.

Proof. Hadamard matrices of order $4w$ with a circulant core are in one-to-one correspondence with cyclic $(4w - 1, 2w - 1, w - 1)$ -difference sets.

However, Hadamard matrices of order $4w$ with a cocyclic core are in one-to-one correspondence with $(4w - 1, 2w - 1, w - 1)$ -difference sets, which are not necessarily cyclic.

Actually, there are orders $4w$ for which no such cyclic difference sets exist, but as yet, general difference sets do exist. \square

Example 2. Consider the case $4w = 28$, for instance.

On the one hand, Theorem 2 in [11] states that there are no 28×28 Hadamard matrices with a circulant core. Consequently, no cyclic $(27, 13, 6)$ -difference set can exist.

On the other hand, $(27, 13, 6)$ -difference sets do exist: as noted in [19],

$$D = \{1, a, a^2, b, ab, b^2, c, ac, bc, ac^2, a^2bc^2, b^2c^2, a^2b^2c^2\}$$

constitutes a $(27, 13, 6)$ -difference set in $G = (C_3 \times C_3 \times C_3, \cdot) = \langle a, b, c : a^3 = b^3 = c^3 = 1 \rangle$, with ordering $a^{i_1}b^{j_1}c^{k_1} < a^{i_2}b^{j_2}c^{k_2}$ if and only if $k_1 < k_2$, or $k_1 = k_2$ and $j_1 < j_2$, or $(j_1, k_1) = (j_2, k_2)$ and $i_1 < i_2$.

Let us consider the sequence $\alpha = (\alpha_g)_{g \in D}$, such that $\alpha_g = 1$ if $g \in D$, and $\alpha_g = -1$ otherwise.

Since $\alpha_1 = 1$, consider the map $\phi : G \rightarrow \{-1, 1\}$ defined as $\phi(g) = \alpha_g$, for $g \in G$. Let $\psi : G \times G \rightarrow \{-1, 1\}$ be the 2-coboundary defined as $\psi(i, j) = \phi(i)\phi(j)\phi(i \cdot j)$. Plugging (α_g) and M_ψ in (3) and completing with regards to Lemma 6 leads to a 28×28 Hadamard matrix with a cocyclic core.

3. Hadamard Matrices with a $(4w - 2) \times (4w - 2)$ Cocyclic Core

Hadamard matrices with two circulant cores [3,12,13] come from a new “plug-in” technique, which extends the underlying idea applied when constructing Hadamard matrices with one circulant core. In this occasion, properly fixing two rows and columns, a full Hadamard matrix may be constructed as soon as two $(2w - 1) \times (2w - 1)$ circulant matrices A and B exist, satisfying the constraint:

$$AA^T + BB^T = 4wI_{2w-1} - 2J_{2w-1},$$

in which case the matrix:

$$\begin{pmatrix} -1 & -1 & \mathbf{1} & \mathbf{1} \\ -1 & \mathbf{1} & \mathbf{1} & -1 \\ \mathbf{1} & \mathbf{1} & A & B \\ \mathbf{1} & -1 & B^T & -A^T \end{pmatrix} \tag{4}$$

would be Hadamard.

As commented in Section 1, this is the case if and only if the first rows of A and B , (a_i) and (b_i) , define two balanced almost perfect sequences with $c_t(a_i) + c_t(b_i) = -2$, for $0 < t \leq w - 1$.

In this section we study the conditions under which a cocyclic matrix M_ψ (for ψ as in (1)) may be plugged in a structure of the type:

$$\begin{pmatrix} ? & ? & ? \\ \alpha & \beta & M_\psi \end{pmatrix}, \tag{5}$$

in order to provide a Hadamard matrix as well, for $\alpha = (\alpha_{g_0}, \dots, \alpha_{g_{4w-3}})$ and $\beta = (\beta_{g_0}, \dots, \beta_{g_{4w-3}})$, given an ordering $\{g_0 = 1, \dots, g_{4w-3}\}$ of G .

As noted in Section 1, as soon as the rows from 3 to $4w$ are pairwise orthogonal, the matrix can be completed to a full Hadamard matrix [18]. Therefore we will focus on the rows from 3 to $4w$ in (5).

In what follows, the notation used in Section 2 still applies, so that the rows from 3 to $4w$ in (5) will be indexed by the corresponding elements in G . In particular, it is remarkable that the row indexed by $1 \in G$ corresponds to the third row in (5).

Furthermore, attending to Hadamard equivalence, up to negation of columns 1 or 2 in (5), without loss of generality we may assume that $\alpha_1 = \beta_1 = \psi(1, 1)$, so that the third row in (5) (that is indexed by $1 \in G$) consists of the constant vector $\psi(1, 1) \cdot \mathbf{1}$.

We now determine a collection of necessary and sufficient conditions for (5) being a Hadamard matrix with a cocyclic core.

Lemma 7. Assume that (5) is a Hadamard matrix with a cocyclic core M_ψ . Then $\sum_{g \in G} \psi(h, g) \in$

$\{-2, 0, 2\}$, for all $h \in G \setminus \{1\}$. Moreover,

- $\sum_{g \in G} \psi(h, g) = 0$ if and only if $(\alpha_h, \beta_h) = (-1, 1) \cdot s_h$, for some (undetermined by now) $s_h \in \{1, -1\}$.
- $0 \neq \sum_{g \in G} \psi(h, g) = \pm 2$ if and only if $(\alpha_h, \beta_h) = (-1, -1) \cdot s_h$, for $s_h = \frac{1}{2} \sum_{g \in G} \psi(h, g)$.

Proof. Let $h \in G \setminus \{1\}$.

Taking into account the orthogonality constraint, it follows that the dot product $\langle \text{row } h, \text{row } 1 \rangle$ of rows indexed by h and $1 \in G$ in (5) must be zero. Therefore, from Lemma 2,

$$0 = \alpha_h \alpha_1 + \beta_h \beta_1 + \psi(h, 1) \cdot \sum_{g \in G} \psi(h, g).$$

Since $\alpha_1 = \beta_1 = \psi(1, 1) = \psi(h, 1)$, we get $\sum_{g \in G} \psi(h, g) = -\alpha_h - \beta_h$ and the result follows. \square

Corollary 3. Assume that (5) is a Hadamard matrix with a cocyclic core M_ψ , such that $\sum_{g \in G} \psi(h, g) \in \{-2, 2\}$, for all $h \in G \setminus \{1\}$. Then ψ is a 2-coboundary and $w = 1$.

Proof. For every $g, h \in G$, attending to Lemma 2, the dot product of rows indexed by gh and h is:

$$0 = 2s_{gh}s_h + \psi(g, h)s_g,$$

so that $\psi(g, h) = -s_g s_h s_{gh}$ is actually a 2-coboundary ($\psi(g, h) = \phi(g)\phi(h)\phi(gh)$, for $\phi : G \rightarrow \{1, -1\}$ defined as $\phi(g) = -s_g$).

Furthermore, the summation of the row indexed by g in M_ψ is $2s_g = \sum_{h \in G} \psi(g, h) = -s_g \sum_{h \in G} s_h s_{gh}$, so that $\sum_{h \in G} s_h s_{gh} = -2$ and $(s_g)_{g \in G}$ defines a perfect sequence. It is known (see [10] for instance) that the only difference set corresponding to a perfect sequence of this type is the trivial $(2, 1, 0)$ -difference set, and hence $w = 1$. \square

Remark 7. Consequently, from now on, we assume that there are elements $h \in G$ for which $\sum_{g \in G} \psi(h, g) = 0$. Actually, in these circumstances, notice that ψ becomes a quasi-orthogonal cocycle, as introduced in [20].

Lemma 8. In the circumstances above, the elements in $G \setminus \{1\}$ may be organized into two subsets, $S_0 = \{h \in G \setminus \{1\} : \sum_{g \in G} \psi(h, g) = 0\}$, and $\bar{S}_0 = (G \setminus \{1\}) \setminus S_0$, such that:

1. $h \in S_0 \iff h^{-1} \in S_0$. Accordingly, $h \in \bar{S}_0 \iff h^{-1} \in \bar{S}_0$.
2. For all $g \in S_0$, $g \cdot S_0 \subseteq G \setminus S_0$. Therefore, $|S_0| \leq |G \setminus S_0|$.
3. For all $g \in G \setminus S_0$, $g \cdot (G \setminus S_0) \subseteq G \setminus S_0$. Therefore, $|G \setminus S_0| \leq |S_0|$.
4. For all $g \in S_0$, $g \cdot (G \setminus S_0) \subseteq S_0$ and $(G \setminus S_0) \cdot g \subseteq S_0$.

Proof. We proceed with the demonstration point by point.

1. For every $h \in G \setminus \{1\}$, since $\langle \text{row } 1, \text{row } h \rangle = \langle \text{row } h, \text{row } 1 \rangle$. Therefore $\psi(h^{-1}, h) \sum_{g \in G} \psi(h^{-1}, g) = \psi(h, 1) \sum_{g \in G} \psi(h, g)$, and the result follows.
2. Let $g, h \in S_0$. Then $h^{-1} \in S_0$ as well. Now:

$$0 = \langle \text{row } g, \text{row } h^{-1} \rangle = \alpha_g \alpha_{h^{-1}} + \alpha_g \alpha_{h^{-1}} + \psi(gh, h) \cdot \sum_{k \in G} \psi(gh, k).$$

Therefore $gh \in G \setminus S_0$.

3. Let $g, h \in G \setminus S_0$. Then $h^{-1} \in G \setminus S_0$ as well. Now:

$$0 = \langle \text{row } g, \text{row } h^{-1} \rangle = \alpha_g \alpha_{h^{-1}} + \alpha_g \alpha_{h^{-1}} + \psi(gh, h) \cdot \sum_{k \in G} \psi(gh, k).$$

Therefore $gh \in G \setminus S_0$.

4. Let $g \in S_0$ and $h \in G \setminus S_0$. Then $h^{-1} \in G \setminus S_0$ as well. Now:

$$0 = \langle \text{row } g, \text{row } h^{-1} \rangle = \alpha_g \alpha_{h^{-1}} - \alpha_g \alpha_{h^{-1}} + \psi(gh, h) \cdot \sum_{k \in G} \psi(gh, k).$$

Therefore, $gh \in S_0$.

□

Corollary 4. In the circumstances above, $|S_0| = |G \setminus S_0| = 2w - 1$, $G \setminus S_0$ is a normal subgroup of G and $G / (G \setminus S_0) \simeq C_2$.

Proof. This is a straightforward consequence of Lemma 8. □

Remark 8. In the circumstances above, one may define a new ordering for the elements in G , such that the last $2w - 1$ elements correspond to S_0 . For instance, when any $g \in S_0$ are fixed, such an ordering would be $\{1 = n_1, \dots, n_{2w-1}, g, gn_2, \dots, gn_{2w-1}\}$.

Corollary 5. In the circumstances above, for $h \in \bar{S}_0$ and $g \in G$, $\psi(h, g) = -s_h s_g s_{hg}$. Furthermore, $\sum_{g \in G} s_g s_{hg} = -2$.

Proof. Let $h \in \bar{S}_0$ and $g \in G$. Consider the dot product of rows indexed by hg and g in (5). From Lemma 8, depending on whether:

- $g \in G \setminus S_0$, then $hg \in G \setminus S_0$. Attending to Lemma 7, then:

$$0 = \langle \text{row } hg, \text{row } g \rangle = 2s_{hg} s_g + \psi(h, g) \sum_{k \in G} \psi(h, k) = 2s_{hg} s_g + 2s_h \psi(h, g).$$

- $g \in S_0$, then $hg \in S_0$. Attending to Lemma 7, then:

$$0 = \langle \text{row } hg, \text{row } g \rangle = 2s_{hg} s_g + \psi(h, g) \sum_{k \in G} \psi(h, k) = 2s_{hg} s_g + 2s_h \psi(h, g).$$

Whichever is the case, for $h \in \bar{S}_0$ and $g \in G$, it is satisfied that $\psi(h, g) = -s_h s_g s_{hg}$. Taking a summation as g runs in G , we get $2s_h = \sum_{g \in G} \psi(h, g) = -s_h \sum_{g \in G} s_g s_{hg}$. The result follows immediately. □

We next summarize these results.

Theorem 3. Let (5) be a matrix with a cocyclic core M_ψ over G . Then it is Hadamard if and only if the following conditions are satisfied:

- G contains a normal subgroup N such that $G/N \simeq C_2$.
- There is a binary sequence $(s_g)_{g \in G}$ such that:
 - For every $h \in N$, $\psi(h, g) = -s_h s_g s_{hg}$, and $\sum_{g \in G} s_g s_{hg} = -2$.
 - For every $k \in G \setminus N$, $\psi(k, g) = -s_k s_g s_{kg} \varphi(k, g)$ (for some 2-cocycle $\varphi : G \times G \rightarrow \{1, -1\}$ such that $\varphi(h, g) = 1$ for all $h \in N$), and $\sum_{g \in G} s_g s_{kg} \varphi(g, k) = 0$.

Even though it might seem difficult to meet the hypothesis described in Theorem 3 altogether, we next describe a family of Hadamard matrices with a cocyclic core.

Proposition 3. *Hadamard matrices with two circulant cores are Hadamard equivalent to Hadamard matrices with a cocyclic core over D_{4w-2} .*

Proof. Consider the group $D_{4w-2} = \langle a, b : a^{2w-1} = b^2 = (ab)^2 = 1 \rangle$, with ordering $G = \{1, a, \dots, a^{2w-2}, b, \dots, a^{2w-2}b\}$. Notice that $N = \langle a \rangle$ is a normal subgroup of G , and that every element $k \in G \setminus N$ is of order 2, that is $k^2 = 1$. Consider the 2-cocycle $\varphi : G \times G \rightarrow \{1, -1\}$ defined as $\varphi(h, g) = -1$ if and only if both $h, g \in G \setminus N$, and 1 otherwise.

Attending to the description of cocyclic matrices over D_{4w-2} developed in [21,22], it may be straightforwardly derived that the two-circulant core structure of (2) is Hadamard equivalent to a cocyclic matrix over D_{4w-2} .

More concretely, consider two balanced sequences (a_i) and (b_i) such that $\sum a_i \cdot a_{i+t} + \sum b_i \cdot b_{i+t} = -2$, for $0 < t \leq w - 1$. Take $s_{a^i} = -a_i$ and $s_{a^i b} = -b_i$, for $0 \leq i \leq 2w - 2$.

Now, consider the subset $C \subset G$ of indices at which the positive entries of the sequence (s_i) (i.e., negative entries of $((a_i)|(b_i))$) occur, and select the corresponding subset of elementary coboundaries ∂_c , so that $\psi = \varphi \cdot \prod_{c \in C} \partial_c$. For the sake of convenience, it might be appropriate to recall that these elementary coboundaries are commonly defined as $\partial_g(h, k) = \delta_g(h)\delta_g(k)\delta_g(hk)$, for $\delta_i(j) = -1$ if and only if $i = j$, and 1 otherwise.

Actually, as indicated in [21,22], the process of successively negating the row and column indexed by $c \in C$ in M_ψ , ends in a Hadamard equivalent matrix with two back circulant cores similar to (2). Permuting the rows accordingly, the proper core of the matrix (2) may be obtained.

Now, consider the conditions $\sum a_i \cdot a_{i+t} + \sum b_i \cdot b_{i+t} = -2$, for $0 < t \leq w - 1$, which guarantee the orthogonal constraint of (2) in [12,13]. These are straightforwardly translated to the conditions $\sum_{g \in G} s_g s_{hg} = -2$, for every $h \in N$, of Theorem 3.

Notice that, as defined, the constraints $\sum_{g \in G} s_g s_{kg} \varphi(g, k) = 0$, for every $k \in G \setminus N$, are naturally satisfied. Actually,

$$\sum_{g \in G} s_g s_{kg} \varphi(g, k) = \sum_{h \in N} s_h s_{kh} - \sum_{g \in G \setminus N} s_g s_{kg} = 0,$$

since there is a one-to-one correspondence from N to $kN = G \setminus N$, taking $h \in N$ on $g_h = kh$; so that each term $s_h s_{kh}$ on the left-hand side of the equation cancels with the corresponding term on the right-hand side, as $-s_{g_h} s_{kg_h} = -s_{kh} s_{k^2 h} = -s_{kh} s_h$. \square

Example 3. *For instance, consider the 8×8 Hadamard matrix with two circulant cores of the form (2), which arises from the sequences $(a_i) = (b_i) = (1, 1, -1)$, with $a_0 a_1 + a_1 a_2 + a_2 a_0 = -1$:*

$$H = \left(\begin{array}{cc|ccc|ccc} -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ \hline 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\ \hline 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 \end{array} \right).$$

Consider the dihedral group $D_6 = \langle a, b : a^3 = b^2 = (ab)^2 = 1 \rangle$, with ordering $\{1, a, a^2, b, ab, a^2b\}$, normal subgroup $N = \{1, a, a^2\}$ and the subset $S_0 = \{b, ab, a^2b\}$. Let $C = \{a^2, a^2b\}$ and consider the 2-cocycle $\psi : D_6 \times D_6 \rightarrow \{1, -1\}$ given by $\psi = \phi\phi$, for $\phi = \partial_{a^2}\partial_{a^2b}$ and $\phi(g, h) = -1$ if and only if both $g, h \in S_0$. The related cocyclic matrix is then:

$$M_\psi = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}.$$

Since $\phi = \partial_{a^2}\partial_{a^2b}$, we define $s_1 = -1, s_a = -1, s_{a^2} = 1, s_b = -1, s_{ab} = -1, s_{a^2b} = 1$, which is consistent with Lemma 7 and Proposition 3.

We may therefore extend M_ψ to a full Hadamard matrix of the form (5) with a cocyclic core M_ψ , as soon as one takes $\alpha = (-s_1, -s_a, -s_{a^2}, -s_b, -s_{ab}, -s_{a^2b}) = (1, 1, -1, 1, 1, -1)$ and $\beta = (-s_1, -s_a, -s_{a^2}, s_b, s_{ab}, s_{a^2b}) = (1, 1, -1, -1, -1, 1)$:

$$H_\psi = \left(\begin{array}{cc|ccc|ccc} 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \\ \hline 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 \end{array} \right).$$

Notice that starting from H_ψ and negating both rows and columns 5 (indexed by a^2) and 8 (indexed by a^2b), and then permuting rows (4,5) to be ordered as (5,4), ends with the matrix H above, as claimed in Proposition 3.

Nevertheless, Hadamard matrices with a cocyclic core strictly include those Hadamard matrices with a circulant core. For the sake of completeness, we conclude the paper by describing an example of a Hadamard matrix with a cocyclic core over a group not isomorphic to any dihedral group, which supports this claim.

Example 4. Consider the group $G = C_3 \times C_2 = \langle a, b : a^3 = b^2 = 1 \rangle$. The subgroup $N = \langle a \rangle$ is normal in G .

Take $(s_i)_{i \in G} = (-1, -1, 1, -1, -1, 1)$. It may be readily checked that:

- $\sum_{g \in G} s_g s_{a \cdot g} = s_1 s_a + s_a s_{a^2} + s_{a^2} s_1 + s_b s_{ab} + s_{ab} s_{a^2b} + s_{a^2b} s_b = -2.$
- $\sum_{g \in G} s_g s_{a^2 \cdot g} = s_1 s_{a^2} + s_a s_1 + s_{a^2} s_a + s_b s_{a^2b} + s_{ab} s_b + s_{a^2b} s_{ab} = -2.$
- $\sum_{g \in G} s_g s_{b \cdot g} = s_1 s_b + s_a s_{ab} + s_{a^2} s_{a^2b} - s_b s_1 - s_{ab} s_a - s_{a^2b} s_{a^2} = 0.$
- $\sum_{g \in G} s_g s_{ab \cdot g} = s_1 s_{ab} + s_a s_{a^2b} + s_{a^2} s_b - s_b s_a - s_{ab} s_{a^2} - s_{a^2b} s_1 = 0.$

- $$\sum_{g \in G} s_g s_{a^2 b \cdot g} = s_1 s_{a^2 b} + s_a s_b + s_{a^2} s_{ab} - s_b s_{a^2} - s_{ab} s_1 - s_{a^2 b} s_a = 0.$$

Accordingly, consider the 2-cocycle $\psi = \varphi \cdot \partial_{a^2} \partial_{a^2 b}$, for $\varphi(g, h) = -1$ if and only if both $g, h \in G \setminus N$. By construction, the matrix M_ψ may be extended to a full Hadamard matrix with a cocyclic core,

$$\left(\begin{array}{cc|cccccc} 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{array} \right).$$

4. Conclusions and Further Work

In this paper we have explored the notion of Hadamard matrices with a cocyclic core, inspired by a series of papers [3,11–13] on Hadamard matrices with circulant cores.

Actually, we have described two new “plug-in” techniques for constructing Hadamard matrices, which strictly include those based on circulant cores. They depend on the existence of certain (almost) perfect sequences and (almost) difference sets, which will be our concern in the future.

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