Backstepping stabilization of an underactuated 3×3 linear hyperbolic system of fluid flow equations

Florent Di Meglio, Rafael Vazquez, Miroslav Krstic, Nicolas Petit

Abstract— **We investigate the boundary stabilization of a particular subset of 3×3 linear hyperbolic systems with varying coefficients on a bounded domain. The system is underactuated since only one of the three hyperbolic PDEs is actuated at the boundary. The setup considered in the paper occurs in control of multiphase flows on oil production systems. We use a backstepping approach to design a full-state feedback law yielding exponential stability of the origin.**

I. Introduction

We consider the problem of stabilizing a certain type of 3×3 linear hyperbolic systems of transport equations with spatially varying coefficients. We consider the case where two uncontrolled PDE states have strictly positive transport speeds, whereas the controlled PDE state has a strictly negative speed. In addition, we assume that one of the uncontrolled states is a Riemann invariant, i.e. it satisfies a pure transport equation with varying speed.

The stability of hyperbolic systems has been previously investigated. In [4] using a Lyapunov approach, the authors consider the stabilization by static output feedback of 2×2 linear (with constant and varying coefficients) and quasilinear hyperbolic systems. Similarly, sufficient conditions for the stability of 2×2 linear systems with varying coefficients and $n \times n$ quasilinear systems are derived in [1] and [5], respectively. However, when these conditions are not satisfied, more advanced feedback laws are needed.

The stability of these systems is also investigated in [11], linking the Lyapunov and frequency domain approaches, as well as in [16], following a Lyapunov approach. Besides, in [10], sufficient conditions for the exact controllability of quasilinear hyperbolic systems are given. Interestingly, they do not apply to the system considered in this paper. A sufficient condition is that the number of controlled states is larger than the number of uncontrolled ones, which is not the case here.

In [14], an output feedback law is introduced, which allows stabilization of 2×2 linear heterodirectional[1] hyperbolic systems. A backstepping observer-controller structure is derived, yielding exponential stability in the $\mathcal{L}^2$-norm of the origin of the considered system. The result is extended to the quasilinear case in [15].

In this paper, we extend the state feedback in [14] to a particular type of 3×3 linear systems, arising in modelling of multiphase flow. In [7], a model for gas-liquid flow in oil production pipes is proposed, under the form of a 3×3 quasilinear hyperbolic system. The model reproduces an undesirable phenomenon, called *slugging*, featuring large oscillations of the pressure, flow rates and mass hold-ups everywhere inside the pipes. The occurrence of slugging corresponds to the instability of the equilibrium of the model, which can be stabilized by feedback actuation of a valve located at the outlet of the pipes. Remarkably, one of the states of the model, namely the mass fraction of gas, is a Riemann invariant (see e.g. [6]). The linearization of the model proposed in [7] leads to the system considered throughout this article.

The paper is organized as follows. In Section II, we describe the system under consideration. In Section III, we propose a backstepping transformation. The existence of the kernels directly stems from the results presented in [14]. In Section IV, we prove exponential stability in the $\mathcal{L}^2$-norm of the target and original systems. Finally, we discuss the obtained results and perspectives for future improvement in Section V.

II. System definition

Consider the pipe schematically depicted in Figure 1. It is filled with gas and oil coming from a reservoir or a manifold. The geometric distribution of both phases inside the pipes, referred to as flow regime, depends, among other things, on the maturation of the oil field, the geometry of the pipe and the nature of the gas-liquid mixture. One of these flow regimes, called slugging, features periodic oscillations of all the physical quantities (such as pressure, flow rates, hold-ups) inside the pipe. These oscillations are at the birth of production losses and may damage the facilities. A possible solution to suppress them is to use feedback control of the outlet valve, which can be remotely actuated to stabilize the flow. The model proposed in [7], which is of drift-flux type [2], [3] takes the following form

$$\frac{\partial \zeta}{\partial t}(t,x) + A(\zeta)\frac{\partial \zeta}{\partial x}(t,x) = S(x) \qquad (1)$$

on the spatial domain $[0,1]$. The three distributed state variables are the gas mass fraction, pressure and gas velocity. The expressions of the A and S matrices are given in [7].

F. Di Meglio is with the Department of Mechanical and Aerospace Engineering, University of California San Diego, La Jolla, CA 92093-0411, USA.

R. Vazquez is with the Department of Aerospace Engineering, Universidad de Sevilla, Camino de los Descubrimiento s.n., 41092 Sevilla, Spain.

M. Krstic is with the Department of Mechanical and Aerospace Engineering, University of California San Diego, La Jolla, CA 92093-0411, USA.

N. Petit is Professor at MINES ParisTech, 60, Bd St-Michel, 75272 Paris, Cedex 06, France.

[1] i.e., where the two states have transport speeds of opposite signs

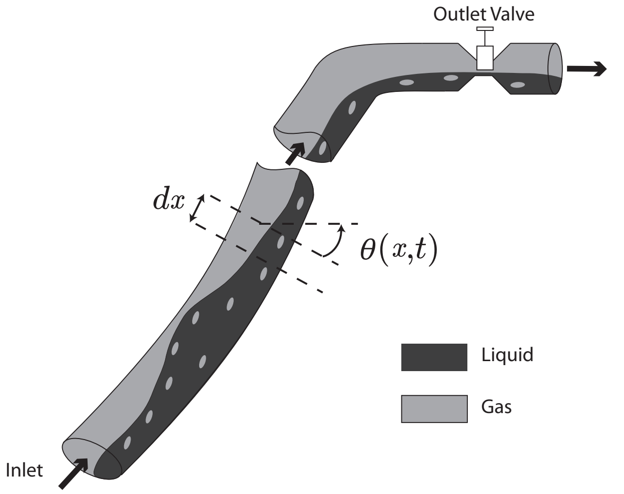

Fig. 1. Schematic view of a pipe conveying oil and gas from a reservoir.

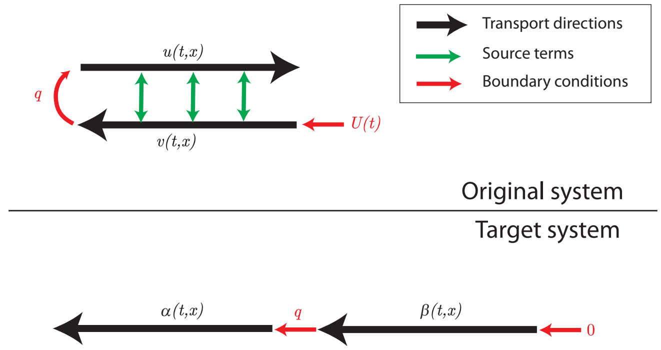

Fig. 2. Schematic view of the control design for 2×2 heterodirectional systems. The backstepping transformation completely removes the internal coupling between both states. The resulting target system is exponentially stable.

The boundary conditions are expressed as follows

$$\begin{pmatrix} h_{l1}(\zeta(t,0)) \\ h_{l2}(\zeta(t,0)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad h_r(\zeta(t,1), Z(t)) = 0 \tag{2}$$

where $Z(t)$ is the opening of the outlet valve, which is the control input. Considering small variations around an equilibrium profile $\bar{\zeta}(x)$ (corresponding to a given valve opening $\bar{Z}$) yields the following linear system with varying coefficients (see Appendix A for the complete linearization)

$$u_{1t}(t,x) + \lambda_1(x) u_{1x}(t,x) = 0 \tag{3}$$

$$u_{2t}(t,x) + \lambda_2(x) u_{2x}(t,x) + \sigma_{21}(x) u_1(t,x) + \sigma_{23}(x) v(t,x) = 0 \tag{4}$$

$$v_t(t,x) - \mu(x) v_x(t,x) + \sigma_{31}(x) u_1(t,x) + \sigma_{32}(x) u_2(t,x) = 0 \tag{5}$$

on the domain $(t,x) \in \mathbb{R}\times[0,1]$. The transport speeds are C^1 functions of space satisfying the following inequalities

$$\forall x \in [0,1] \qquad -\mu(x) < 0 < \lambda_1(x) < \lambda_2(x)$$

and we denote

$$\Lambda(x) = \begin{pmatrix} \lambda_1(x) & 0 & 0 \\ 0 & \lambda_2(x) & 0 \\ 0 & 0 & -\mu(x) \end{pmatrix}$$

$$\Sigma(x) = \begin{pmatrix} 0 & 0 & 0 \\ \sigma_{21}(x) & 0 & \sigma_{23}(x) \\ \sigma_{31}(x) & \sigma_{32}(x) & 0 \end{pmatrix}$$

The linearized boundary conditions read

$$\begin{pmatrix} u_1(t,0) \\ u_2(t,0) \\ v(t,0) \end{pmatrix} = Q_0 \begin{pmatrix} u_1(t,0) \\ u_2(t,0) \\ v(t,0) \end{pmatrix} = \begin{pmatrix} 0 & 0 & q_1 \\ 0 & 0 & q_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1(t,0) \\ u_2(t,0) \\ v(t,0) \end{pmatrix} \tag{6}$$

$$v(t,1) = U(t) \tag{7}$$

where $U(t)$ is the new control input, and q_1 and q_2 are non-zero. System (3)–(5) with boundary conditions (6), (7) forms, along with an appropriate initial condition, a well-posed problem. However, as demonstrated in [7], the equilibrium $u_1 \equiv u_2 \equiv v \equiv 0$ may be unstable, especially for large values of $\bar{Z}$. In the next sections, we propose a stabilizing feedback law following a backstepping approach.

III. Backstepping transformation

A. Analogy with 2×2 heterodirectional systems

System (3)-(4)-(5) is not the most general form of 3×3 hyperbolic systems. Because the first line of matrix $\Sigma(x)$ is filled with zeros (the first state is a Riemann invariant), its structure resembles that of 2×2 heterodirectional systems. In [14], a stabilizing feedback control law is proposed for these systems, along with observers for both the collocated and non collocated cases. We propose to exploit the resemblance with 2×2 heterodirectional systems to design an controller structure for our 3×3 (2×2 + a Riemann invariant) system.

In particular, when designing a backstepping transformation, the kernel equations that allow to suppress the internal coupling between the last two states (u_2 and v) are exactly the same as the kernel equations in the 2×2 case. The coupling between the first state (u_1, the Riemann invariant) and the last state (v, the controlled state) can be suppressed by the backstepping design as well. Importantly, the coupling between the two homodirectional states (u_1 and u_2) *does not need to be suppressed*, since it does not affect the stability of the target system. This idea can be summarized by Figures 2 and 3. This suggests that a generalization of this result to $(n+1)\times(n+1)$ systems, with n states with positive speeds and one controlled state with a negative speed may be possible. We discuss this matter in more detail in Section V.

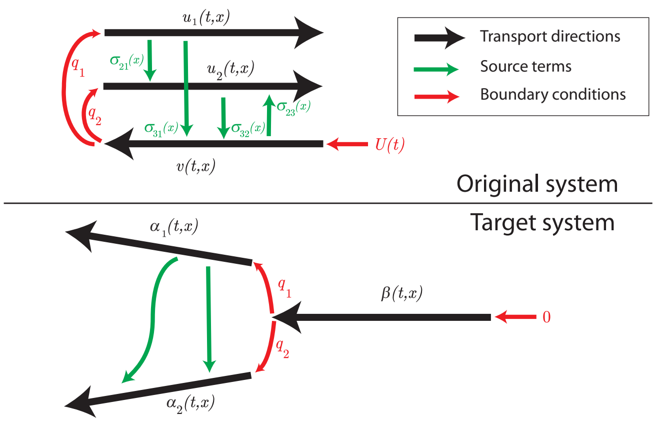

Fig. 3. Schematic view of the control design for a particular 3×3 system (2×2 + a Riemann invariant). The backstepping transformation completely removes the internal coupling between the α_2 and β states, and between α_1 and β. The coupling between α_1 and α_2 (u_1 and u_2 in the original system) is slightly modified, as there is now an integral source term proportional to α_1 in the propagation equation of α_2. The resulting target system is still exponentially stable.

B. Target system

We want to map the original system (3)–(5) to the following target system

$$\alpha_{1t}(t,x) + \lambda_1(x)\alpha_{1x}(t,x) = 0 \tag{8}$$

$$\alpha_{2t}(t,x) + \lambda_2(x)\alpha_{2x}(t,x) + \sigma_{21}(x)\alpha_1(t,x) + \int_0^x c(x,s)\alpha_1(t,s)ds = 0 \tag{9}$$

$$\beta_t(t,x) - \mu(x)\beta_x(t,x) = 0 \tag{10}$$

where c is a C^0 function to be determined defined on the triangular domain

$$\mathcal{T} = \{(x,\xi): \quad 0 \le \xi \le x \le 1\}, \tag{11}$$

along with boundary conditions

$$\alpha(t,0) = Q_0\alpha(t,0) = \begin{pmatrix} 0 & 0 & q_1 \\ 0 & 0 & q_2 \\ 0 & 0 & 1 \end{pmatrix}\alpha(t,0) \quad \beta(t,1) = 0 \tag{12}$$

The $\mathcal{L}^2$-stability of system (8)-(9)-(10) is investigated in Section IV. We now propose a state transformation that maps the original system (3)-(4)-(5) to the target system.

C. Backstepping transformation

To transform system (3)-(4)-(5) into the target system (8)-(9)-(10), we consider a backstepping transformation of the following form.

$$\gamma(t,x) = w(t,x) - \int_0^x K(x,\xi)w(t,\xi)d\xi \tag{13}$$

where $w = (u_1, u_2, v)^T$ and $\gamma = (\alpha_1, \alpha_2, \beta)^T$ and the gains of the kernel matrix

$$K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & k^{22} & k^{23} \\ k^{31} & k^{32} & k^{33} \end{pmatrix}$$

are yet to be determined. Differentiating (13) with respect to space yields

$$\gamma_x(t,x) = w_x(t,x) - K(x,x)w(t,x) - \int_0^x K_x(x,\xi)w(t,\xi)d\xi$$

while differentiating (13) with respect to time, and integrating by parts yields

$$\begin{aligned} \gamma_t(t,x) = &- \Lambda(x)w_x(t,x) - \Sigma(x)w(t,x) \\ &+ K(x,x)\Lambda(x)w(t,x) - K(x,0)\Lambda(0)w(t,0) \\ &- \int_0^x \Big[K_\xi(x,\xi)\Lambda(\xi) + K(x,\xi)\Lambda'(\xi) - K(\xi)\Sigma(\xi)\Big] w(t,\xi)d\xi \end{aligned}$$

Using these expressions into the target system equations, and using the plant equations *and the fact that* $w_1(t,x) = \alpha_1(t,x)$ yields

$$\begin{aligned} 0 = &\Big[\Sigma^0(x) - \Sigma(x) + K(x,x)\Lambda(x) - \Lambda(x)K(x,x)\Big] w(t,x) \\ &- K(x,0)\Lambda(0)w(t,0) - \int_0^x \Big[K_x(x,\xi)\Lambda(x) + K_\xi(x,\xi)\Lambda(\xi) \\ &+ K(x,\xi)\Lambda'(\xi) - K(x,\xi)\Sigma(\xi) + C(x,\xi)\Big] w(t,\xi)d\xi \end{aligned}$$

where $C(x,\xi) = \begin{pmatrix} 0 & 0 & 0 \\ c(x,\xi) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\Sigma^0(x) = \begin{pmatrix} 0 & 0 & 0 \\ \sigma_{21}(x) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Since this has to be verified for all $w(t,x)$, this yields the following equations to be solved

$$0 = K(x,0)\Lambda(0)Q_0 \tag{14}$$

$$0 = \Sigma^0(x) - \Sigma(x) + K(x,x)\Lambda(x) - \Lambda(x)K(x,x) \tag{15}$$

$$0 = \Lambda(x)K_x(x,\xi) + K_\xi(x,\xi)\Lambda(\xi) + K(x,\xi)\Lambda'(\xi) + \Sigma^0(x)K(x,\xi) - K(x,\xi)\Sigma(\xi) + C(x,\xi) \tag{16}$$

on the triangular domain $\mathcal{T}$ defined by (11). Developing the equation (16) yields the following 5 PDE

$$\lambda_2(x)k_x^{22} + \lambda_2(\xi)k_\xi^{22} = -\lambda_2'(\xi)k^{22} + \sigma_{32}(\xi)k^{23} \tag{17}$$

$$\lambda_2(x)k_x^{23} - \mu(\xi)k_\xi^{23} = \sigma_{23}(\xi)k^{22} + \mu'(\xi)k^{23} \tag{18}$$

$$-\mu(x)k_x^{31} + \lambda_1(\xi)k_\xi^{31} = \sigma_{21}(\xi)k^{32} + \sigma_{31}(\xi)k^{33} - \lambda_1'(\xi)k^{31} \tag{19}$$

$$-\mu(x)k_x^{32} + \lambda_2(\xi)k_\xi^{32} = -\lambda_2'(\xi)k^{32} + \sigma_{32}(\xi)k^{33} \tag{20}$$

$$-\mu(x)k_x^{33} - \mu(\xi)k_\xi^{33} = \sigma_{23}(\xi)k^{32} + \mu'(\xi)k^{33} \tag{21}$$

where, for notational convenience, $k^{ij}(x,\xi)$ was abbreviated to k^{ij}. This also yields the following expression of $c(x,\xi)$

$$c(x,\xi) = k^{22}(x,\xi)\sigma_{21}(\xi) + k^{23}(x,\xi)\sigma_{31}(\xi) \tag{22}$$

Equations (14) and (15) give the boundary conditions for equations (17)-(21) as follows

$$q_2\lambda_2(0)k^{22}(x,0) = \mu(0)k^{23}(x,0) \tag{23}$$

$$k^{23}(x,x) = -\frac{\sigma_{23}(x)}{\lambda_2(x)+\mu(x)} \tag{24}$$

$$k^{31}(x,x) = \frac{\sigma_{31}(x)}{\lambda_1(x)+\mu(x)} \tag{25}$$

$$k^{32}(x,x) = \frac{\sigma_{32}(x)}{\lambda_2(x)+\mu(x)} \tag{26}$$

$$\mu(0)k^{33}(x,0) = q_1\lambda_1(0)k^{31}(x,0) + q_2\lambda_2 k^{32}(x,0) \tag{27}$$

A direct application of [14, Theorem 4] with, on the one hand

$$F^1 = k^{22} \qquad\qquad F^2 = k^{23}$$

and, on the other hand

$$F^1 = k^{33} \qquad\qquad F^2 = k^{32} \qquad\qquad F^3 = k^{31}$$

yields the following lemma.

Lemma 3.1: Consider the hyperbolic system (17)-(21) with boundary conditions (23)-(27). Under the assumptions

$$\lambda_1, \lambda_2, \mu \in C^1([0,1]), \quad \forall i,j \quad \sigma_{ij} \in C^0([0,1])$$

there exists a unique continuous solution K.

In particular, $c(x,\xi) = k^{22}(x,\xi)\sigma_{21}(\xi) + k^{23}(x,\xi)\sigma_{31}(\xi)$ is a continuous function on $\mathcal{T}$, and therefore it is bounded on $\mathcal{T}$ which is critical in the proof of stability of Section IV.

D. Inverse transformation

The invertibility of transformation (13) is proved in [12], along with the boundedness of the inverse transformation operator. The inverse transformation reads

$$w(t,x) = \gamma(t,x) + \int_0^x L(x,\xi)\gamma(x,\xi)d\xi \tag{28}$$

where the inverse kernel L is implicitly defined on $\mathcal{T}$ by the following integral equation

$$L(x,\xi) = K(x,\xi) - \int_\xi^x K(x,s)L(s,\xi)ds \tag{29}$$

Given the particular form of K, L has the following form

$$L = \begin{pmatrix} 0 & 0 & 0 \\ l_{2,1} & l_{2,2} & l_{2,3} \\ l_{3,1} & l_{3,2} & l_{3,3} \end{pmatrix}$$

where the l^{ij} are continuous functions on $\mathcal{T}$.

IV. Control law and main result

A. Stability of the target system

Before stating the main result, we prove exponential stability of the target system in the following Lemma.

Lemma 4.1: Consider system (8)-(9)-(10) with boundary conditions (12) and initial conditions u_1^0, u_2^0 and v^0. Under the assumptions

$$\lambda_1, \lambda_2, \mu \in C^1([0,1]), \sigma_{21} \in C^0([0,1]), c \in C^0(\mathcal{T}),$$
$$u_1^0, u_2^0, v^0 \in \mathcal{L}^2([0,1]), \tag{30}$$

the origin is exponentially stable in the $\mathcal{L}^2$-norm.

Proof Consider the following candidate Lyapunov function

$$V(t) = \int_0^1 pe^{-\delta x}\left[\alpha_1(t,x)^2 + \alpha_2(t,x)^2\right] + e^{-\delta x}\beta(t,x)^2 dx \tag{31}$$

where p and δ are strictly positive real numbers to be determined. Differentiating V with respect to time yields

$$\begin{aligned}
\dot V(t) &= \\
&\int_0^1 \Big[-pe^{-\delta x}\left(\lambda_1(x)\alpha_1(t,x)\alpha_{1x}(t,x) + \lambda_2(x)\alpha_2(t,x)\alpha_{2x}(t,x)\right) \\
&\quad - pe^{-\delta x}\sigma_{21}(x)\alpha_2(t,x)\alpha_1(t,x) + e^{\delta x}\mu(x)\beta(t,x)\beta_x(t,x) \\
&\quad - pe^{-\delta x}\alpha_2(t,x)\int_0^x c(x,s)\alpha_1(t,s)ds\Big]dx \\
&= \Big[-pe^{-\delta x}\lambda_1(x)\alpha_1(t,x)^2 - pe^{-\delta x}\lambda_2(x)\alpha_2(t,x)^2 \\
&\quad + e^{\delta x}\mu(x)\beta(t,x)^2\Big]_0^1 + \int_0^1 \Big[p(\lambda_1'(x) - \delta\lambda_1(x))e^{-\delta x}\alpha_1(t,x)^2 \\
&\quad + p(\lambda_2'(x) - \delta\lambda_2(x))e^{-\delta x}\alpha_2(t,x)^2 - (\mu'(x) + \delta\mu(x))e^{\delta x}\beta(t,x)^2 \\
&\quad - 2pe^{-\delta x}\sigma_{21}(x)\alpha_2(t,x)\alpha_1(t,x)\Big]dx \\
&\quad - \int_0^1\int_0^x 2pe^{-\delta x}\alpha_2(t,x)c(x,s)\alpha_1(t,s)dsdx
\end{aligned} \tag{32}$$

Denoting

$$||c||_\infty = \max_{(x,s)\in\mathcal{T}} |c(x,s)|,$$

the last term can be upper-bounded as follows

$$\begin{aligned}
&-\int_0^1\int_0^x 2pe^{-\delta x}\alpha_2(t,x)c(x,s)\alpha_1(t,s)dsdx \\
&\le p||c||_\infty\Bigg(\int_0^1\int_0^x e^{-\delta x}\alpha_1^2(t,s)dsdx \\
&\qquad + \int_0^1\int_0^x e^{-\delta x}\alpha_2(t,x)^2 dsdx\Bigg) \\
&= p||c||_\infty\Bigg(\int_0^1 \alpha_1^2(t,s)\int_s^1 e^{-\delta x}dxds \\
&\qquad + \int_0^1\int_0^x e^{-\delta x}\alpha_2(t,x)^2 dsdx\Bigg) \\
&= p||c||_\infty\Bigg(\int_0^1 \alpha_1(t,x)^2\frac{e^{-\delta x} - e^{-\delta}}{\delta}dx \\
&\qquad + \int_0^1\int_0^x e^{-\delta x}\alpha_2(t,x)^2 dsdx\Bigg) \\
&\le p||c||_\infty\left(\int_0^1 \alpha_1(t,x)^2\frac{e^{-\delta x}}{\delta}dx + \int_0^1\int_0^1 e^{-\delta x}\alpha_2(t,x)^2 dsdx\right) \\
&= p||c||_\infty\left(\int_0^1 \alpha_1(t,x)^2\frac{e^{-\delta x}}{\delta}dx + \int_0^1 e^{-\delta x}\alpha_2(t,x)^2 dsdx\right)
\end{aligned} \tag{33}$$

Plugging (33) into (32) and using the boundary conditions yields

$$\begin{aligned}
\dot V(t) &\le \left[p\lambda_1(0)q_1^2 + p\lambda_2(0)q_2^2 - \mu(0)\right]\beta^2(t,0) \\
&+ \int_0^1 \begin{pmatrix}\alpha_1(t,x)\\ \alpha_2(t,x)\end{pmatrix}^T P(x)\begin{pmatrix}\alpha_1(t,x)\\ \alpha_2(t,x)\end{pmatrix} e^{-\delta x}dx + \int_0^1 \rho(x)e^{\delta x}\beta(t,x)^2 dx
\end{aligned} \tag{34}$$

with

$$P(x) = \begin{pmatrix} p(\lambda_1'(x) - \delta\lambda_1(x)) + \frac{p\|c\|_\infty}{\delta} & -p\sigma_{21}(x) \\ -p\sigma_{21}(x) & p(\lambda_2'(x) - \delta\lambda_2(x)) + p\|c\|_\infty \end{pmatrix} \tag{35}$$

and

$$\rho(x) = (\mu'(x) + \delta\mu(x)) \tag{36}$$

We now seek 2 parameters p and δ such that the following inequalities are satisfied

$$p\left[\lambda_1(0)q_1^2 + \lambda_2(0)q_2^2\right] - \mu(0) < 0 \tag{37}$$

and, for all $x \in [0,1]$

$$\lambda_1'(x) - \boldsymbol{\delta}\lambda_1(x) + \frac{\|c\|_\infty}{\boldsymbol{\delta}} < 0 \tag{38}$$

$$\lambda_2'(x) - \boldsymbol{\delta}\lambda_2(x) + \|c\|_\infty < 0 \tag{39}$$

$$(\mu'(x) + \boldsymbol{\delta}\mu(x)) < 0 \tag{40}$$

$$\left[\lambda_1'(x) - \boldsymbol{\delta}\lambda_1(x) + \frac{\|c\|_\infty}{\boldsymbol{\delta}}\right][\lambda_2'(x) - \boldsymbol{\delta}\lambda_2(x) + \|c\|_\infty] - \sigma_{21}(x)^2 > 0 \tag{41}$$

First, we pick

$$0 < p < \frac{\mu(0)}{\lambda_1(0)q_1^2 + \lambda_2(0)q_2^2}$$

so that (37) is satisfied. Besides, inequalities (38)-(39)-(40)-(41) rewrite

$$\lambda_1(x)\boldsymbol{\delta}^2 - \lambda_1'(x)\boldsymbol{\delta} - \|c\|_\infty > 0 \tag{42}$$

$$\lambda_2(x)\boldsymbol{\delta} - \lambda_2'(x) - \|c\|_\infty > 0 \tag{43}$$

$$\mu(x)\boldsymbol{\delta} + \mu'(x) > 0 \tag{44}$$

$$\begin{aligned}&\lambda_1(x)\lambda_2(x)\boldsymbol{\delta}^3 - \left[\lambda_1(x)\lambda_2'(x) + \|c\|_\infty\lambda_1(x) + \lambda_1'(x)\lambda_2(x)\right]\boldsymbol{\delta}^2 \\ &+ \left[\|c\|_\infty\lambda_1'(x) - \|c\|_\infty\lambda_2(x) + \lambda_1'(x)\lambda_2'(x) - \sigma_{21}(x)^2\right]\boldsymbol{\delta} \\ &+ \|c\|_\infty\left[\lambda_2'(x) + \|c\|_\infty\right] > 0.\end{aligned} \tag{45}$$

Inequalities (42)-(43)-(44)-(45) are satisfied for a sufficiently large δ. Indeed, since assumptions (30) hold, all the transport speeds, their derivatives, and the source terms are upper-bounded in absolute value, and there exists ϵ such that

$$\forall x \in [0,1] \qquad \mu, \lambda_i(x) > \epsilon > 0 \qquad i = 1,2$$

Thus, for all $x \in [0,1]$, $P(x)$ in (35) is positive definite, $\rho(x)$ in (36) is strictly positive and there exists ϵ such that (34) yields

$$\dot{V} \leq -\epsilon V(t) \tag{46}$$

B. Control law and main result

From transformation (13) evaluated at $x = 1$, one gets

$$U(t) = v(t,1) - \int_0^1 k^{31}(1,\xi)u_1(t,\xi) + k^{33}(1,\xi)u_2(t,\xi) + k^{33}(1,\xi)v(t,\xi)d\xi \tag{47}$$

We now state the main result of the paper

Theorem 4.2: Consider system (3)-(4)-(5) with boundary conditions (6)-(7), initial conditions u_1^0, u_2^0, v^0, and the control law defined by (47). Under the following assumptions

$$\lambda_1, \lambda_2, \mu \in C^1([0,1]), \quad \forall i,j \quad \sigma_{ij} \in C^0([0,1]), \quad u_1^0, u_2^0, v^0 \in \mathcal{L}^2([0,1])$$

the origin is exponentially stable in the $\mathcal{L}^2$ sense.

Proof From the continuity of the inverse backstepping transformation, we have the following upper bound (see, e.g., [13])

$$\|w(t,\cdot)\|^2_{\mathcal{L}^2([0,1])} \leq (1 + \|L\|_\infty)\|\gamma(t,\cdot)\|^2_{\mathcal{L}^2([0,1])}$$

Besides, from (46), one gets

$$\|\gamma(t,\cdot)\|^2_{\mathcal{L}^2([0,1])} \leq e^{-\epsilon t}\|\gamma(0,\cdot)\|^2_{\mathcal{L}^2([0,1])}$$

which concludes the proof.

V. Discussion and perspectives

We have presented a backstepping control design for a particular 3×3 linear hyperbolic system with varying coefficients, yielding exponential stability of the origin in the $\mathcal{L}^2$ sense. As explained in Section III-A, the result may be generalized to $(n+1)\times(n+1)$ systems where the controlled state has a negative transport speed and the n other states have a positive speed. Such a result would exploit the fact that, no matter how strong the coupling is,[2] an $n \times n$ homodirectional system with 0 input at the inlet boundary is always stable. This can be seen by considering a Lyapunov function of the form $V(t) = \int_0^1 \sum_{i=1}^n e^{-\delta x} p_i u(t,x)^2 dx$, with sufficiently small p_i and sufficiently large δ. The coupling between the controlled states and the n uncontrolled states would be suppressed by the backstepping transformation. This is a direction for future work.

Importantly, the proposed feedback law requires full-state measurement. This assumption is not realistic for the considered application, where sensors are expensive and difficult to install, and even more difficult to maintain. Boundary measurement is a much more likely scenario. Usually, oil production facilities are relatively well equipped at their outlet, where pressure, flow and density measurements may be available. When bottom pressure sensors are installed, these are used in relatively simple feedback loops (PI controllers) to stabilize the flow, with success [8], [9]. Thus, the case of interest for the design of more advanced control law, is the one where the sensors are located at the outlet, i.e. at the right boundary of the domain.

Unfortunately, we have not been able yet to design a collocated controller for this 3×3 system because the approach we have followed in this paper, relying on the 2×2 case, does not seem to extend to the observer design. However, according to [10], a sufficient condition for exact observability of the quasilinear system is that the sensor is

[2]as long as the transport speeds are C^1 and the coupling terms are C^0 functions.

located where the most quantities "exit" the domain. In our case, it is the right boundary, and the condition is fulfilled. This gives us confidence that it may possible to design an observer for the 3×3 linear case with varying coefficient.

References

[1] G. Bastin and J.-M. Coron. Further results on boundary feedback stabilization of 2x2 hyperbolic systems over a bounded interval. *Proceedings of IFAC Nolcos 2010, Bologna, Italy*, 2010.

[2] S.P.C. Belfroid, R.J. Pvan Der Linden, G.J.N. Alberts, and R. Aasheim R., Schulkes. Severe slugging in oil-well simulations using a drift flux code. pages 357–371, 2010. cited By (since 1996) 0.

[3] C. E. Brennen. *Fundamentals of multiphase flow*. Cambridge Univ Press, 2005.

[4] J.-M. Coron. *Control and Nonlinearity*. American Mathematical Society, 2007.

[5] J.M. Coron, G. Bastin, and B. DAndréa-Novel. Dissipative boundary conditions for one-dimensional nonlinear hyperbolic systems. *SIAM Journal on Control and Optimization*, 47(3):1460–1498, 2008.

[6] R. Courant and D. Hilbert. *Methods of Mathematical Physics, Vol. II*. Wiley-Interscience, New York, 1962.

[7] F. Di Meglio, G.-O. Kaasa, N. Petit, and V. Alstad. Slugging in multiphase flow as a mixed initial-boundary value problem for a quasilinear hyperbolic system. *2011 American Control Conference (to appear)*, 2011.

[8] J.-M. Godhavn, M. P. Fard, and P. H. Fuchs. New slug control strategies, tuning rules and experimental results. *Journal of Process Control*, 15:547–557, 2005.

[9] K. Havre, K.O. Stornes, and H. Stray. Taming slug flow in pipelines. *ABB review*, 4:55–63, 2000.

[10] T. T. Li. *Controllability and Observability for Quasilinear Hyperbolic Systems*, volume 3. Higher Education Press, Beijing, 2009.

[11] X. Litrico and V. Fromion. Boundary control of hyperbolic conservation laws using a frequency domain approach. In *Decision and Control, 2006 45th IEEE Conference on*, pages 5341 –5346, dec. 2006.

[12] W. Liu. Boundary feedback stabilization of an unstable heat equation. *SIAM Journal on Control and Optimization*, Vol. 42, No. 3:1033–1043, 2003.

[13] R. Vazquez and M. Krstic. *Control of Turbulent and Magnetohydrodynamic Channel Flows*. Springer, 2008.

[14] R. Vazquez, M. Krstic, and J.-M. Coron. Backstepping boundary stabilization and state estimation of a 2x2 linear hyperbolic system. *in Proceedings of the 50th Conference on Decision and Control, Orlando (to appear)*, 2011.

[15] R. Vazquez, M. Krstic, J.-M. Coron, and G. Bastin. Local Exponential H^2 Stabilization of a 2x2 Quasilinear Hyperbolic System using Backstepping. *in Proceedings of the 50th Conference on Decision and Control, Orlando (to appear)*, 2011.

[16] C.-Z. Xu and G. Sallet. Exponential stability and transfer functions of processes governed by symmetric hyperbolic systems. *ESAIM: Control, Optimisation and Calculus of Variations*, 7:421–442, 2002.

Appendix

A. Linearization of the drift-flux model for gas-liquid flow

Consider a small variation $\delta\zeta(t, x)$ around an equilibrium profile $\bar{\zeta}(x)$. Neglecting second-order terms in $\delta\zeta$, System (1) becomes

$$\frac{\partial \delta\zeta}{\partial t} + A(\bar{\zeta}(x))\frac{\partial \delta\zeta}{\partial x} + \tilde{S}(x)\delta\zeta = 0 \tag{48}$$

with

$$\tilde{S}(x) = \begin{pmatrix} \frac{\partial A}{\partial \zeta_1}(\bar{\zeta})\bar{\zeta}'(x) & \frac{\partial A}{\partial \zeta_2}(\bar{\zeta})\bar{\zeta}'(x) & \frac{\partial A}{\partial \zeta_3}(\bar{\zeta})\bar{\zeta}'(x) \end{pmatrix}$$

In (48), $A(\bar{\zeta}(x))$ is diagonalizable, i.e.

$$L(x)A(\bar{\zeta}(x)) = \Lambda(\bar{\zeta}(x))L(x)$$

where $L(x)$ is a matrix of left eigenvectors and $\Lambda(x) = \begin{pmatrix} \lambda_1(\bar{\zeta}(x)) & 0 & 0 \\ 0 & \lambda_2(\bar{\zeta}(x)) & 0 \\ 0 & 0 & \mu(\bar{\zeta}(x)) \end{pmatrix}$ the matrix of transport speeds. Thus, considering the change of variables

$$\chi = L(x)\delta\zeta$$

and left-multiplying (48) by $L(x)$ yields

$$\frac{\partial \chi}{\partial t} + \Lambda(x)\frac{\partial \chi}{\partial z} = -\tilde{\Sigma}(x)\chi \tag{49}$$

with

$$\tilde{\Sigma(x)} = L(x)\left(\tilde{S}(x)L^{-1}(x) + A(\bar{u}(x)(L^{-1})'(x)\right)$$

The expression of $\tilde{\Sigma}(x)$ is too complicated to be written in details. Remarkably, the third line of $\tilde{\Sigma}$ is only filled with 0. Indeed, the original state variable ζ_1 is a Riemann invariant. This structure is preserved by the preceding transformation, and $\chi_1 = u_1$ is also a Riemann invariant for (49)[3]. Thus, we denote

$$\tilde{\Sigma} = \begin{pmatrix} 0 & 0 & 0 \\ \tilde{\sigma}_{2,1} & \tilde{\sigma}_{2,2} & \tilde{\sigma}_{2,3} \\ \tilde{\sigma}_{3,1} & \tilde{\sigma}_{3,2} & \tilde{\sigma}_{3,3} \end{pmatrix}$$

Finally, following [1], we define the following expressions

$$\varphi_2(x) = \exp\left(\int_0^z \frac{\tilde{\sigma}_{2,2}(s)}{\lambda_2(s)}ds\right), \ \varphi_3(x) = \exp\left(-\int_0^z \frac{\tilde{\sigma}_{3,3}(s)}{\mu(s)}ds\right),$$
$$\varphi(x) = \frac{\varphi_1(x)}{\varphi_2(x)}$$

and make the following change of variables

$$u_1 = \chi_1, \qquad u_2 = \varphi_2(x)\chi_2, \qquad v = \varphi_3(x)\chi_3$$

This yields system (3)-(4)-(5) with

$$\begin{pmatrix} 0 & 0 & 0 \\ \sigma_{2,1} & 0 & \sigma_{2,3} \\ \sigma_{3,1} & \sigma_{3,2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \varphi_2(x)\tilde{\sigma}_{2,1}(x) & 0 & \varphi(x)\tilde{\sigma}_{2,3}(x) \\ \varphi_3(x)\tilde{\sigma}_{3,1}(x) & \varphi^{-1}(x)\tilde{\sigma}_{3,2}(x) & 0 \end{pmatrix}$$

Denoting $\delta\zeta(t, 0) = \delta\zeta(0)$, the linearized boundary conditions read

$$\frac{\partial h_l}{\partial \zeta_1}(\bar{\zeta}(0))\delta\zeta_1(0) + \frac{\partial h_l}{\partial \zeta_2}(\bar{\zeta}(0))\delta\zeta_2(0) + \frac{\partial h_l}{\partial \zeta_3}(\bar{\zeta}(0))\delta\zeta_3(0) = 0$$
$$\frac{\partial h_l}{\partial \zeta_1}(\bar{\zeta}(L))\delta\zeta_1(L) + \frac{\partial h_l}{\partial \zeta_2}(\bar{\zeta}(L))\delta\zeta_2(L) + \frac{\partial h_l}{\partial \zeta_3}(\bar{\zeta}(L))\delta\zeta_3(L) + \frac{\partial h_l}{\partial \zeta_3}(\bar{\zeta}(L))\delta Z(t) = 0$$

Thus,

$$\frac{\partial h_l}{\partial \zeta}(\bar{\zeta}(0))L^{-1}(\bar{\zeta}(0))\begin{pmatrix} 1 & 0 & 0 \\ 0 & \varphi_2^{-1}(0) & 0 \\ 0 & 0 & \varphi_3^{-1}(0) \end{pmatrix}w(t, 0) = 0$$
$$\frac{\partial h_r}{\partial \zeta}(\bar{\zeta}(L))L^{-1}(\bar{\zeta}(L))\begin{pmatrix} \varphi_1^{-1}(L) & 0 & 0 \\ 0 & \varphi_2^{-1}(L) & 0 \\ 0 & 0 & 1 \end{pmatrix}w(t, L) = -\frac{\partial h_l}{\partial \zeta_3}(\bar{\zeta}(L))\delta Z(t)$$

[3] This can be verified by computing explicitly Σ or, simply, by linearizing the following equation, verified by ζ_1: $\frac{\partial \zeta_1}{\partial t}(t, x) + \lambda_1(t, x)\frac{\partial \zeta_1}{\partial x}(t, x) = 0$.

Eventually, the boundary conditions can be expressed as follows

$$\begin{cases} \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} v(0) \\ \quad u_2(L) = \begin{pmatrix} q & q' \end{pmatrix} \begin{pmatrix} u_1(L) \\ u_2(L) \end{pmatrix} + k\delta Z(t) \end{cases}$$

Setting $U(t) = \begin{pmatrix} q & q' \end{pmatrix} \begin{pmatrix} u_1(L) \\ u_2(L) \end{pmatrix} + k\delta Z(t)$ yields equations (6)-(7).