Witness-Bar Visibility Graphs

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Abstract

Bar visibility graphs were introduced in the seventies as a model for some VLSI layout problems. They have been also studied since then by the graph drawing community, and recently several generalizations and restricted versions have been proposed.

We introduce a generalization, *witness-bar visibility graphs*, and we prove that this class encompasses all the bar-visibility variations considered so far. In addition, we show that many classes of graphs are contained in this family, including in particular all planar graphs, interval graphs, circular arc graphs and permutation graphs.

1 Introduction and preliminary definitions

Given a set S of disjoint horizontal line segments in the plane (called *bars* hereafter) we say that G is a *bar-visibility graph* if there is a bijection between S and the vertices of G, and an edge between two of these if and only if there is a vertical segment (called *line of sight*) between the corresponding bars that does not intersect any other bar. We also say that S is a *bar visibility representation* (or a *bar visibility drawing*) of G.

Bar visibility graphs were introduced by Garey, Johnson and So [15] as a modeling tool for digital circuit design (see also [20]). These representations are also a useful tool for displaying diagrams that convey visual information on relations among data, which is why many variations of these graphs have been considered by the graph drawing community [7, 9, 10, 11, 14, 17, 18].

We need some definitions before we can pose precisely our problem; we use standard terminology as in [6]. We call *v-segment* any vertical segment. We call *ε-segment* any axis aligned rectangle having width $\varepsilon > 0$ (intuitively, a *thick* vertical segment). Let s and t be two horizontal bars. We say that a v-segment connects s and t if its endpoints are in s and t. We say that an ε-segment connects s and t if its horizontal sides are contained in s and t.

Let S be a set of non-overlapping horizontal segments (bars). Two bars $s, t \in S$ are *visible* if, and only if, there is a v-segment connecting s and t intersecting no other segment in S. We say that s and t are *ε-visible* if, and only if there is an ε–segment connecting s and t intersecting no other segment in S.

With the preceding definition, bar visibility graphs as defined in the first paragraph of this section take as nodes a set of disjoint bars, and there is an edge between two nodes if and only if the corresponding bars are visible (this is also called a *strong visibility representation* of the graph [21]). If instead of visibility we require ε-visibility, then we get bar ε-visibility graphs or, equivalently, an *ε-visibility representation* of the graph. The latter have been characterized as those graphs that admit a planar embedding with all cutpoints on the exterior face [21, 22].

A graph G is a *weak bar visibility graph* if its nodes can be put in bijection with a set of disjoint bars and the nodes corresponding to every edge in G are ε-visible (note that not every ε-visibility need be an edge). This family of graphs is exactly the class of all planar graphs [12].

Finally, we say that G is a *bar k-visibility graph* if there is a bijection between a set of bars S and the vertices of G, and an edge between two of these if and only if there is a v-segment joining the corresponding bars that intersects at most k other bars. This generalization has been introduced in recent years [10, 14].

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In this paper we introduce a stronger generalization, *witness-bar visibility graphs*, and we prove that this representation approach encompasses all the bar-visibility variations considered so far. In addition, we show that many classes of graphs are contained in this family, including in particular all planar graphs, interval graphs, circular arc graphs and permutation graphs.

For the definition of witness-bar visibility graphs we consider, in addition to the set S of bars that are in correspondence one-to-one with the vertices of the graph being constructed, a set of green bars that "favor" visibility, and a set of red bars that "obstruct" visibility. Green bars act as *positive witnesses* while red bars correspond to *negative witnesses*. The bars from S neither favor nor obstruct visibilities.

For the ease of description it is useful to consider also *purple bars* that obstruct visibility in a slightly different way than red bars.

Definition 1 *Let S, S_G, S_P and S_R be four sets of horizontal segments (*bars, green-bars, purple-bars, *and* red-bars, *respectively) such that any two elements in $S \cup S_G \cup S_P \cup S_R$ are disjoint. We define:*

1. *The* green-bar visibility graph *of S with respect to S_G has one vertex for every element in S, and two bars $s, t \in S$ are adjacent if and only if there is an ε-segment connecting s and t that* crosses at least one green bar.

2. *The* purple-bar visibility graph *of S with respect to S_P has one vertex for every element in S, and two bars $s, t \in S$ are adjacent if and only if there is an ε-segment connecting s and t that* does not cross any purple bar.

3. *The* witness-bar visibility graph *of S with respect to S_G and S_R has one vertex for every element in S, and two bars $s, t \in S$ are adjacent if and only if there is an ε-segment connecting s and t that* crosses strictly more green bars than red bars.

The class of green, purple and witness-bar visibility graphs are denoted, respectively, by $\mathcal{GBG}$, $\mathcal{PBG}$ and $\mathcal{WBG}$.

An illustration of the three types of graphs is shown in Figure 1 (on a black and white printer, node-bars appear as thin lines, red bars as thick dark lines, purple lines as thick lines colored light grey, and the green lines are seen as thick striped lines).

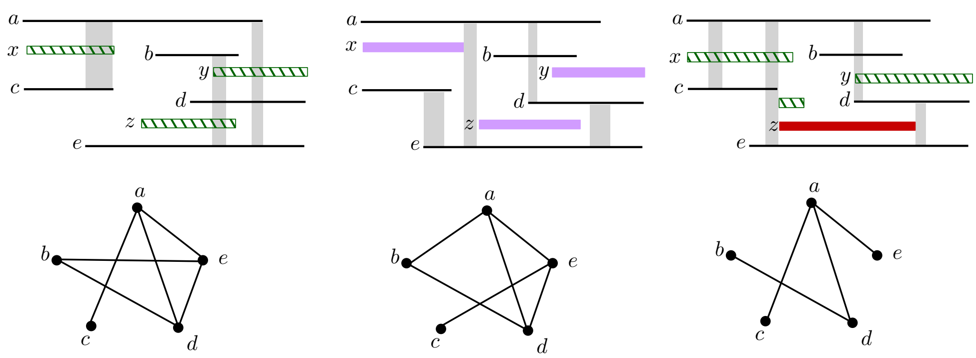


Figure 1: Examples of graphs in the families $\mathcal{GBG}, \mathcal{PBG}$ and $\mathcal{WBG}$, for the set of bars $S = \{a, b, c, d, e\}$.

This work is devoted to the study of the classes of graphs that can be represented via green, purple or bar-visibility graphs and its properties. We start by considering the classes $\mathcal{GBG}$ and $\mathcal{PBG}$, which will be proved to be subclasses of $\mathcal{WBG}$. Then we will enumerate classes of graphs that are contained in $\mathcal{WBG}$, as well as properties of this class related to planarity.

The terminology *witness-bar visibility graphs* is inspired by the concept of *witness proximity graphs*, which focuses on deciding neighborliness relations among points in a finite set according to the presence of some positive and/or negative witness points, a topic that has been studied in recent years [1, 2, 3, 4, 13].

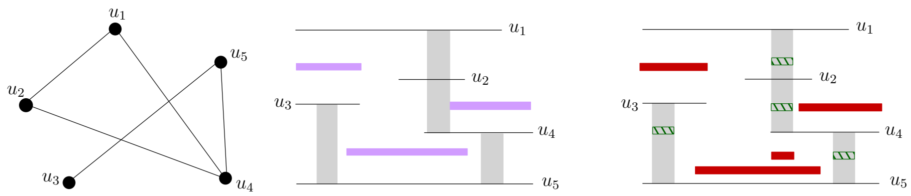
2 $\mathcal{PBG}$ and $\mathcal{GBG}$ are subclasses of $\mathcal{WBG}$

We prove in this section that the classes $\mathcal{PBG}$ and $\mathcal{GBG}$ that we have introduced for the sake of clarity in many proofs, are subclasses of $\mathcal{WBG}$, the class that is our actual subject of study. It is worth noticing first that if we consider a graph in $\mathcal{WBG}$ that has only green bars, then it is exactly a graph in $\mathcal{GBG}$. However, if we have a graph in $\mathcal{WBG}$ that has only red bars, then it is just an empty graph, because for visibility we require the green bars crossed by a line of sight to be *strictly more* than the crossed red bars. This is why we also consider the class $\mathcal{PBG}$ in which purple bars play a stronger obstruction role.

Lemma 1 *The class of graphs $\mathcal{WBG}$ contains the classes $\mathcal{GBG}$ and $\mathcal{PBG}$.*

Proof. *(Sketch)* The fact that $\mathcal{GBG} \subset \mathcal{WBG}$ is trivial; the green-bar visibility representation is also a witness-bar visibility representation.

In Figure 2 we illustrate a method to obtain a $\mathcal{WBG}$ reprsentation of a $\mathcal{PBG}$ graph.



a) *b)* *c)*

Figure 2: a) and b): A graph and one of its purple-bar visibility representations. Visibility corridors (ε-segments) between adjacent vertices are shown. c) The green-bar visibility representation carried out from b). The small red bar between u_4 and u_5 is added to avoid the adjacency between u_1 and u_5 caused by the previously added green bars.

□

In Section 5 we will prove that the two inclusions in Lemma 1 are strict.

3 The graph class $\mathcal{GBG}$

In this section we study the class $\mathcal{GBG}$ having as main objective to obtain properties of the superclass $\mathcal{WBG}$. In our way, we also explore some relationships with other graph classes.

Recall now that an *interval graph* is the intersection graph of a set of (closed) intervals on the real line; this is, it has one vertex for each interval in the set, and an edge between every pair of vertices corresponding to intervals that have nonempty intersection.

Theorem 2 *Let G be a graph. If G is an interval graph, then $G \in \mathcal{GBG}$. The reverse is in general false.*

Proof. Consider the intervals on the real line that define G. First of all, notice that we can extend infinitesimally the intervals in such a way that every pair that intersected overlap in an interval of positive length. Then assign arbitrary to these intervals different heights so that we have a set of bars in the plane. To obtain a green-bar visibility representation of G it suffices to shield each segment with two green bars of the same length, one above and one below. Notice that another representation using less green bars may also exist.

The graph class inclusion is strict, as one can show that C_4 is in $\mathcal{GBG}$ (Figure 3) while the interval graph does not have cycles of length greater than three. □

A difference between the class of interval graphs and $\mathcal{GBG}$ is stated in next proposition, whose proof is omitted here.

Proposition 3 *The girth of any graph in $\mathcal{GBG}$ is at most four, and this value is achievable.*

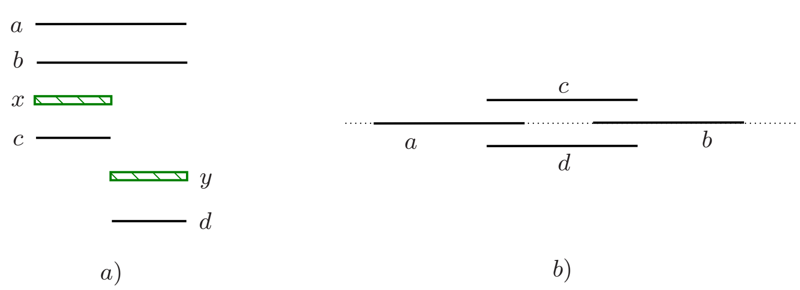


Figure 3: a) C_4 is a $\mathcal{GBG}$ graph but b) it is not an interval graph: a and b must not overlap. Since c is adjacent to both vertices it covers the gap between them, as d should do. Therefore c and d must be adjacent in the interval graph.

As a consequence of the previous result, it follows that the green-bar visibility graph class does not contain any of the bar-visibilities classes described in the introduction of this paper, because C_n can be represented as weak/ε/strong bar visibilty graph for every $n \geq 3$ [21].

Let us recall next some definitions [8].The *dimension* of a partially ordered set P (*poset*) is the smallest possible number of total orders whose intersection is the partial order in P. The *comparability graph* induced by a poset $P = (X, \leq)$ is the graph with vertex set X in which $x, y \in X$ are adjacent if and only if either $x \leq y$ or $y \leq x$ in P; in other words, it is the undirect graph underlying P. Alternatively [16], a comparability graph is a graph such that every generalized cycle of odd length has a triangular chord (a generalized cycle is a closed walk that uses each edge of the graph at most once in each direction). A comparability graph has *dimension* 2 if it is the comparability graph of a poset of dimension 2 (it has been shown that this concept of dimension is well defined [8]).

These definitions can also be considered from a geometric point of view [19]. Let $\mathbb{R}^2$ be the Euclidean 2-dimensional space. A method of ordering the points of $\mathbb{R}^2$ is the following: $(x_i, y_i) \leq (x_j, y_j)$ if, and only if, $x_i \leq x_j$ and $y_i \leq y_j$; we say that (x_i, y_i) is *dominated* by (x_j, y_j). A poset $P = (X, \leq)$ has dimension at most 2 if it can be embedded in $\mathbb{R}^2$ in such a way that the order is preserved. Moreover, changing the coordinates slightly if required, we can assume that the embedding is such that no two elements of X have equal x or y coordinate. As a consequence, any comparability graph of dimension 2 can always be thought as a point set in $\mathbb{R}^2$ with the dominance order. This geometric representation applies in Euclidean d-dimensional space to comparability graphs of dimension d.

Theorem 4 *There are graphs in $\mathcal{GBG}$ that are not comparability graphs.*

Proof. The graph depicted in Figure 4 a) is not a comparability graph since the generalized cycle $v_1 - v_2 - v_4 - v_5 - v_4 - v_3 - v_2 - v_1$ has odd length (seven) but has no triangular chords. Figure 4 b) shows a green-bar visibility representation of this graph. □

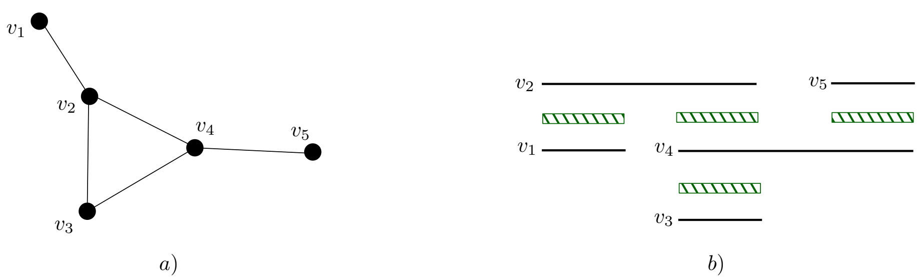


Figure 4: a) A graph that is not a comparability graph and b) a green-bar visibility representation of this graph.

4 The graph class $\mathcal{PBG}$

In this section we turn our attention to the class $\mathcal{PBG}$, having again as main objective to obtain properties of the superclass $\mathcal{WBG}$. In our way we explore as well some relationships with other graph classes.

On may think that the classes $\mathcal{GBG}$ and $\mathcal{PBG}$ are related by complementation, possibly by switching purple and green bar coloring, but it is not the case. For example the union of two disjoint triangular cycles is in $\mathcal{GBG}$, as seen in the preceding section, but its complement is $K_{3,3}$, which is not in $\mathcal{PBG}$, a fact that we prove below.

Reversely, the fact that $G \in \mathcal{PBG}$ does not imply that $G^c \in \mathcal{GBG}$. A simple example is obtained by considering $G = G^c = C_5$, which admits a purple-bar visibility representation (see Figure 5), but we have proved in Proposition 3 that $C_5 \notin \mathcal{GBG}$.

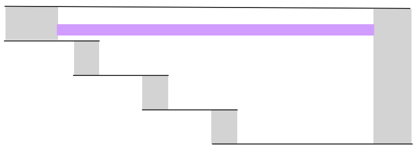


Figure 5: A purple-bar visibility representation of C_5.

It is neither true that the complement of a graph in $\mathcal{PBG}$ is in $\mathcal{PBG}$. Consider $G = K^c_{3,3}$, which consists of the union of two triangles and hence $G \in \mathcal{PBG}$ (triangles are interval graphs) but $G^c = K_{3,3} \notin \mathcal{PBG}$ as will be shown in Theorem 7.

On the other hand, C_5 is also an example that the class $\mathcal{PBG}$ contains some non-perfect graphs. Recall that a *perfect* graph is a graph in which the chromatic number of every induced subgraph equals the size of the largest clique of that subgraph. The largest clique in C_5 is K_2, however $\chi(C_5) = 3$, therefore C_5 is not perfect.

On the positive side, let us see that interval graphs admit a purple-bar visibility representation and prove a lemma that will be useful later.

Theorem 5 *If G is an interval graph, then $G \in \mathcal{PBG}$ (and therefore $G \in \mathcal{WBG}$).*

Proof. Consider the intervals on the real line that define G and lift them to arbitrary different heights so that we have a set of bars in the plane. This is already a purple-bar visibility representation of G. ☐

Lemma 6 *Let G be a triangle-free graph. If $G \in \mathcal{PBG}$ then G is a planar graph.*

Proof. *(Sketch)* Given a graph $G \in \mathcal{PBG}$ we can use the visibility windows in order to obtain a plane representation of G as it is depicted in Figure 6. ☐

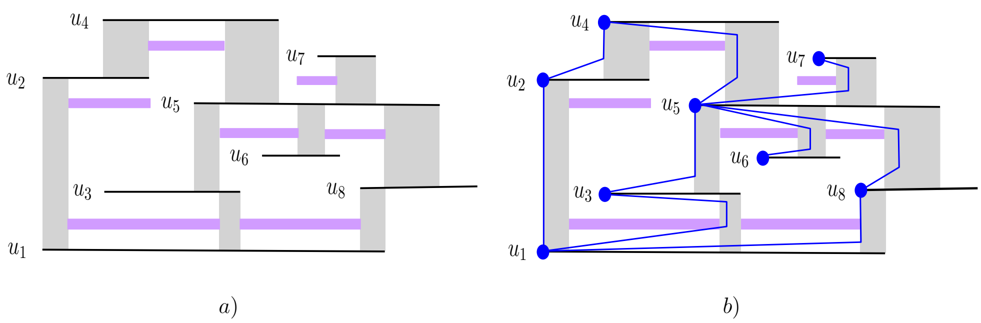


Figure 6: A purple-bar visibility representation of a triangle-free graph and the corresponding construction of its planar embedding -in blue-.

Theorem 7

1) $K_{3,3} \notin \mathcal{PBG}$.

2) $K_n \in \mathcal{PBG}$, $\forall n$.

3) The property of admitting a purple-bar visibility representation is not inherited by subgraphs.

Proposition 8 *There are nonplanar graphs with triangles that do not admit a purple-bar visibility representation.*

An example is shown in Figure 7

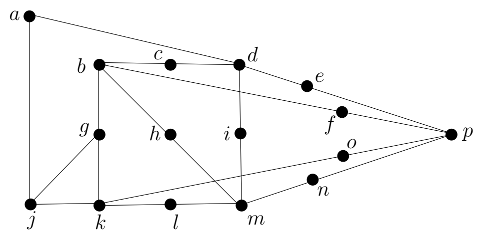


Figure 7: A nonplanar graph G with a triangle ($\triangle gjk$),such that $G \notin \mathcal{PBG}$.

Theorem 9 *Every graph G that can be represented as strong/ε/weak bar visibility graph admits as well a purple-bar visibility representation (and therefore a $\mathcal{WBG}$ representation as well).*

Proof. As every graph that admits the first or second representation is also realizable using weak visibility [21], we only have to prove that the latter class of graphs is contained in $\mathcal{PBG}$.

Let G be a graph realized as a weak bar visibility graph. Each node u of G is a bar, that has some visibility corridors above and below (see Figure 8(a)). Observe that we can shrink all the visibility corridors to still have positive width, yet determining disjoint intervals on u (Figure 8(b)). After the shrinking, we can shield all the bars with purple bars on both sides, yet without crossing the corridors of sight (Figure 8(c) and (d)). This is clearly a purple-bar visibility representation of G. □

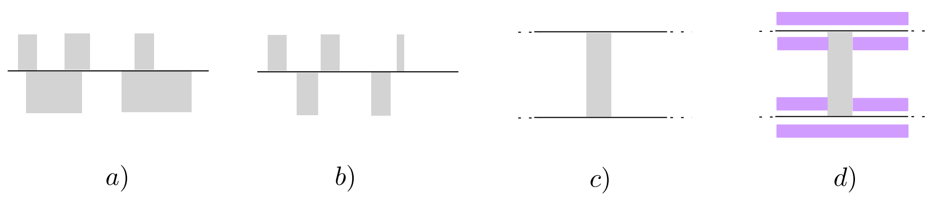


Figure 8: Constructing a purple-bar visibility representation of a bar visibility graph.

5 The class $\mathcal{WBG}$ of witness-bar visibility graphs

We start this section by observing that the contentions in Lemma 1 are strict, that is, the classes $\mathcal{GBG}$ and $\mathcal{PBG}$ are strictly contained in $\mathcal{WBG}$. Figure 9 shows a witness-bar representation of a subdivision of $K_{3,3}$. By Proposition 3, this graph is not in $\mathcal{GBG}$ since it contains chordless cycles of length five. On the other hand, the graph is triangle-free and nonplanar, therefore it is not in $\mathcal{PBG}$ by Lemma 6.

We claimed in the introduction that k-bar visibility is also generalized by witness-bar visibility. This is precisely what we prove in the following theorem:

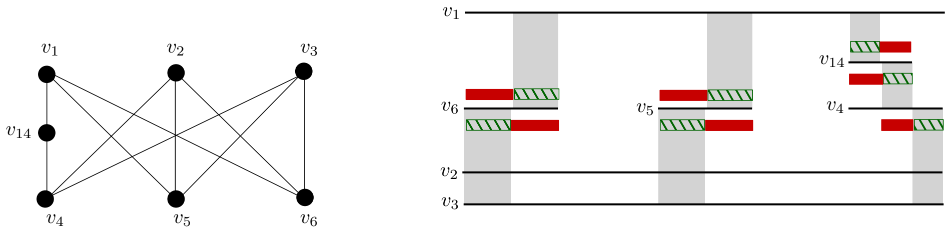


Figure 9: A witness-bar representation of a subdivision of $K_{3,3}$.

Theorem 10 *Every graph G that can be represented as a bar k-visibility graph admits as well a witness-bar visibility representation.*

Proof. Let B be the set of bars in a bar k-visibility representation of G; recall that there is an edge between two of these if and only if there is a v-segment joining the corresponding bars that intersects at most k other bars. For the sake of clarity let us see that, if necessary, we can modify slightly the bars in B to avoid some degeneracies, obtaining again a set of bars whose corresponding bar k-visibility graph is still G.

Leu us consider a vertical line through each endpoint of all bars in B. If for any of these vertical lines ℓ there is a set L of bars in B whose right endpoint is on ℓ, and a set R of bars in B whose left endpoint is also on ℓ, we extend infinitesimally to the right all the bars in L, without jumping over any of the other vertical lines. Clearly the new set of bars still induces G by bar k-visibility, as neither k-visibilities are destroyed, nor new k-visibilities are created. Therefore we can assume, without loss of generality, that no such coverticalities are present in B.

Leu us call consider the set of vertical lines through the endpoints of the bars in B. These lines decompose the planes into strips; inside each strip we have a stack of (portions of) bars of equal length, the width of the strip, placed at different heights. Let us focus in any of these strips (only if it is non empty of bars), which we denote S. Let B_S the set of bars in S and let m be the number of bars in B_S. Notice that at this step we disregard bars or portions of bars outside S.

Case 1: $m \leq k+2$. In this case we use two auxiliary vertical lines to split S into three vertical strips of equal width, which we call *slabs*. In the central slab, between any two consecutive bars, we place a green bar. The graph $\mathcal{WBG}$ induced inside S is the complete graph, which was exactly the situation for bar k-visibility.

Case 2: $m > k+2$. Let us call $b_1, ..., b_m$ the bars in B_S, in order on increasing height. In this case there are $m-k-1$ subsets of $k+2$ consecutive bars in B_S, namely $B_1 = \{b_1, \ldots, b_{k+2}\}, B_2 = \{b_2, \ldots, b_{k+3}\}, \ldots, B_{m-k-1} = \{b_{m-k-1}, \ldots, b_m\}$.

Let us subdivide the strip S into $2(m-k-1)+1 = 2m-2k-1$ vertical slabs, that we denote $S_1, \ldots, S_{2m-2k-1}$. We are dealing inside slab S_{2i} with the $\mathcal{WBG}$-visibility among the bars in B_i, by placing a green bar with the same width than S_{2i} between any two consecutive portions of bar in B_i, and one stack of $k+1$ red bars of the same width just below b_i (but above b_{i-1}) and another stack of $k+1$ red bars of the same width just above b_{i+k+1} (but below b_{i+k+2}). The graph $\mathcal{WBG}$ induced inside S_{2i} for the bars B_i is the complete graph, and no other $\mathcal{WBG}$-visibilities appear in S_{2i}. Repeating the construction, we have mimicked exactly the situation for bar k-visibility using $\mathcal{WBG}$-visibility, as claimed.

□

We prove next that another interesting class of graphs is contained in $\mathcal{WBG}$.

A *circular-arc* graph is the intersection graph of a set of (usually assumed to be open) arcs on the circle. It has one vertex for each arc in the set, and an edge between every pair of vertices corresponding to arcs that intersect.

Theorem 11 *If G is a circular-arc graph, then $G \in \mathcal{WBG}$.*

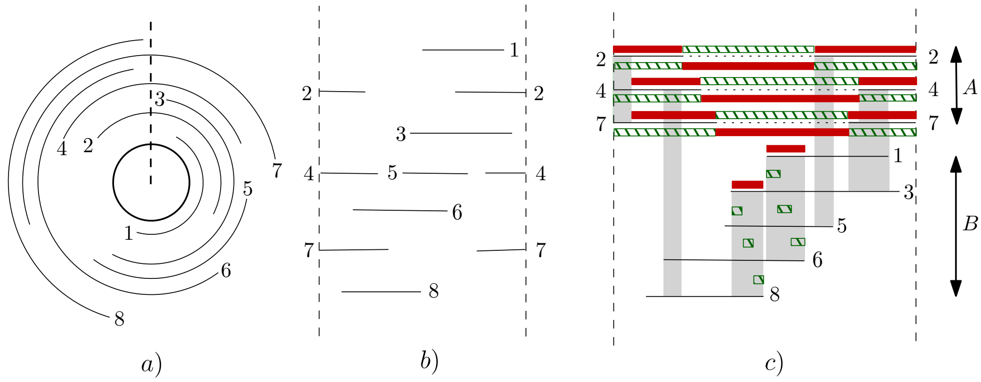


Figure 10: A witness-bar representation of a circular-arc graph.

Proof. *(Sketch)* The realization steps are schematized in the example in Figure 10. □

We prove next another interesting graph class containment in $\mathcal{WBG}$.

Lemma 12 *Let G be a graph. If G is a comparability graph of dimension 2, then $G \in \mathcal{WBG}$.*

Proof. Let $X = \{(x_1, y_1), ..., (x_n, y_n)\}$ be the vertex set of the dominance realization of G in the plane, and assume, without loss of generality, that the vertices have been labeled in such a way that $x_1 < x_2 < ... < x_n$.

For $i = 1, ..., n$, let us now represent vertex (x_i, y_i) as a bar in $\mathbb{R}^2$ given by the segment s_i with endpoints (x_1, y_i) and (x_n, y_i).

We place above s_i $i = 1, ..., n$, a red bar beginning at x-coordinate x_1 and ending at x_i, and a green bar beginning at x-coordinate x_i and ending at x_n. Below s_i, we draw a green bar beginning at x-coordinate x_1 and ending at x_i and a red one beginning at x-coordinate x_i and ending at x_n (Figure 11). It is not difficult to see that this is a witness-bar visibility representation of G.

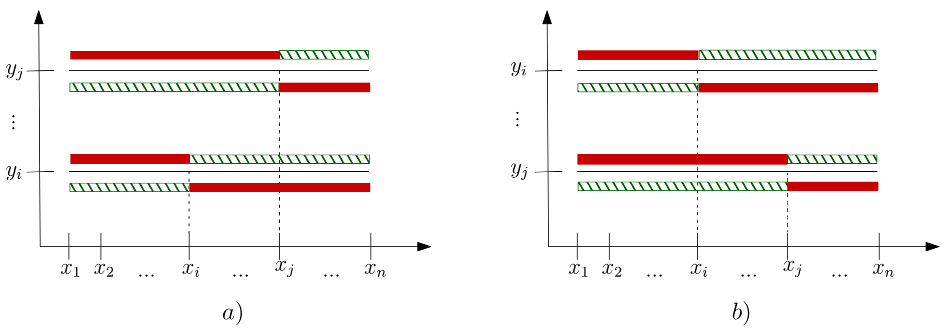


Figure 11: Proof of Lemma 12.

□

A *permutation graph* is the intersection graph of a family of line segments that connect two parallel lines. Equivalently, given a permutation $(\sigma_1, \sigma_2, ..., \sigma_n)$ of the numbers $1, 2, 3, ...n$, a permutation graph has a vertex for each number $1, 2, 3, ...n$ and an edge between any two numbers that are in reversed order in the permutation, i.e., an edge between any two numbers where the segments cross in the permutation diagram. The class of permutation graphs has been widely studied [8], and since they are characterized as comparability graphs underlying partially ordered sets that have dimension at most two, we can easily infer that they are also contained in $\mathcal{WBG}$.

Theorem 13 *If G is a a permutation graph, then $G \in \mathcal{WBG}$.*

Given a graph G, let $\widetilde{G}$ be the graph resulting from subdividing once all edges in G. Then we have:

Lemma 14 *Let G be a graph. If $\widetilde{G} \in \mathcal{WBG}$ then G is a planar graph.*

Lemma 15 $\widetilde{K}_{3,3} \notin \mathcal{WBG}$ *and* $\widetilde{K}^c_{3,3} \in \mathcal{WBG}$.

Theorem 16 *The class of the graphs that admit a witness-bar visibility representation is not closed under complementation.*

We conclude this section with another result on the class $\mathcal{WBG}$, that discards the possibility of characterizing the class by forbidden minors:

Theorem 17 *The property of admitting a witness-bar visibility representation is not inherited by subgraphs.*

Proof. We know that $K_6 \in \mathcal{WBG}$ from Theorem 7 and Lemma 1. On the other hand $\widetilde{K}_{3,3}$ is a subdivision of a subgraph of K_6, but we know from Lemma 15 that $\widetilde{K}_{3,3}$ is not in $\mathcal{WBG}$. This settles the claim. □

6 Concluding remarks

Let us summarize the properties we have proved for the class $\mathcal{WBG}$ of witness-bar visibility graphs:

- Every graph G that can be represented as strong/ε/weak bar visibility graph admits as well a witness-bar visibility representation
- Every graph G that can be represented as a bar k-visibility graph admits as well a witness-bar visibility representation
- The class of interval graphs is contained in the class $\mathcal{WBG}$.
- If G is a circular arc graph, then $G \in \mathcal{WBG}$
- If G is a permutation graph, then $G \in \mathcal{WBG}$
- The class of the graphs that admit a witness-bar visibility representation is not closed under complementation.
- The property of admitting a witness-bar visibility representation is not inherited by subgraphs, which discards the possibility of characterizing the graph class $\mathcal{WBG}$ by forbidden minors.

We conclude that the graph class $\mathcal{WBG}$ is very rich and encompasses many other classes. However, to obtain a characterization or a recognition algorithm appear to be quite challenging problems.

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