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On the connectivity of infinite graphs and 2-complexes

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Abstract

This paper contains a study of the connectivity of infinite graphs and 2-complexes. Various connectivity types are defined and relationships among them are given. In addition new Menger–Whitney type theorems are stated for both graphs and 2-complexes. © 1999 Elsevier Science B.V. All rights reserved

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0. Introduction

Many results concerning the notion of connectivity can be found in graph theory. The classical result on connectivity is the well-known Menger–Whitney Theorem (MWT for short) which shows that for any two vertices of a graph, the maximum number of pairwise disjoint paths joining them is equal to the minimum number of vertices needed to separate them [11,13,9]. A two-dimensional analogue of the MWT for finite 2-complexes is given by Woon in [14], where a 2-path is defined as an ordered sequence of pairwise adjacent 2-simplices. In addition, Woon posed the question of extending his results to infinite 2-complexes.

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The aim of this paper is to answer Woon's question. We introduce various types of connectivity concerning the ideal points at infinity of an infinite 2-complex K and then we prove several MWT type theorems for such connectivity types. These results are contained in Sections 2 and 3. Incidentally, we provide a more general and shorter proof of Woon's main theorem in Section 2.

In pursuing our aim we have found and used several theorems related to already known extensions of the MWT concerning the ideal points at infinity of an infinite graph [12,5,8]. These results seem to be new in the literature and we have included them in Section 1.

Finally, in Appendix A, we give several relationships among the different connectivity types introduced in this paper for both graphs and 2-complexes.

We next give the basic notation we shall use along this paper. We recall that a *simplicial complex*, K, is a set of simplices such that:

(a) If $\sigma \in K$ and τ is a face of σ ($\tau < \sigma$, for short) then $\tau \in K$.

(b) If $\sigma, \sigma' \in K$ then $\sigma \cap \sigma'$ is empty or a common face of σ and σ'.

The complex K is *locally finite* if any $\sigma \in K$ is the face of only finitely many simplices of K. For $\sigma \in K$ the *star* of σ in K is the subcomplex $\mathrm{st}(\sigma;K) = \{\mu;\ \exists\tau \in K \textit{ with } \mu < \tau \textit{ and } \sigma < \tau\}$. The *link* of σ in K is the subcomplex $\mathrm{lk}(\sigma;K) = \{\mu \in \mathrm{st}(\sigma;K);\ \sigma \cap \mu = \emptyset\}$.

A *subcomplex* L of K is a complex whose simplices are simplices of K. Given a subcomplex $L \subseteq K$, the notation $K \backslash L$ will stand for the subcomplex of K generated by $K - L$; that is, $K \backslash L = \{\tau \in K;\ \tau < \rho \text{ and } \rho \notin L\}$. The *i-skeleton* of K is the subcomplex $\mathrm{sk}^i K \subseteq K$ consisting of all simplices $\sigma \in K$ with $\dim \sigma \leqslant i$. We say that K is *purely n-dimensional* when any simplex $\sigma \in K$ is the face of some n-simplex of K. For the sake of simplicity, we shall say that K is an *n-complex* when K is a purely n-dimensional locally finite connected complex. Let σ be an $(n-1)$-simplex of an n-complex K. The *valence* of σ, $\mathrm{val}(\sigma)$, is the number of n-simplices in $\mathrm{st}(\sigma;K)$. The valence of K is the number

$$\mathrm{val}(K) = \min\{\mathrm{val}(\sigma);\ \dim \sigma = n-1\}.$$

An $(n-1)$-simplex $\sigma \in K$ is said to be a *boundary simplex* when $\mathrm{val}(\sigma) = 1$. Otherwise we say that σ is an *interior simplex*. The *boundary* of K, ∂K, is the smallest subcomplex of K containing the boundary simplices. The boundary ∂K is said to be *full* when any simplex in K meets ∂K in a (possibly empty) face.

Given an increasing sequence of finite n-subcomplexes $K_i \subseteq \mathrm{int}\, K_{i+1}$ $(i \geqslant 1)$ with $K = \bigcup_{i=1}^{\infty} K_i$, a *Freudenthal end* of K is a decreasing sequence $(C_i)_{i\geqslant 1}$ of infinite connected components $C_i \subseteq K - K_i$ $(i \geqslant 1)$. We recall that there are only finitely many infinite connected components in $K - K_i$ for each $i \geqslant 1$. Let $\mathscr{F}(K)$ be the set of Freudenthal ends of K. It easy to check that $\mathscr{F}(K) = \mathscr{F}(\mathrm{sk}^1 K)$. Moreover, it can be proved that $\mathscr{F}(K)$ can be topologized in such a way that $\mathscr{F}(K)$ is homeomorphic to a closed subset of the Cantor set. See [4] for more details on the space $\mathscr{F}(K)$.

1. Some Menger–Whitney type theorems for infinite graphs

For a *graph* we mean a connected 1-complex G. In particular G will always be locally finite. Let $V(G)$ denote the set of vertices of G. A *path* $\alpha : a_0 - a_n$ between two vertices $a_0, a_n \in G$ is a finite sequence of vertices $\{a_0, \ldots, a_n\}$ such that $a_i \neq a_j$ $(i \neq j)$ and the segment $\langle a_i,\ a_{i+1}\rangle$ is an edge of G $(0 \leqslant i \leqslant n-1)$. A path α between the sets $A, B \subseteq V(G)$ is a path $\alpha : a_0 - a_n$ with $\alpha \cap A = \{a_0\}$ and $\alpha \cap B = \{a_n\}$. A (one-way) *ray* $R : a_0 - \infty$ starting at $a_0 \in G$ is a sequence of vertices $\{a_0, \ldots\}$ such that $a_i \neq a_j$ $(i \neq j)$ and $\langle a_i,\ a_{i+1}\rangle$ is an edge of G. A ray between $\Pi \subseteq V(G)$ and infinity (∞, for short) is a ray R starting at some $e \in \Pi$, with $\Pi \cap R = \{e\}$. It is clear that a ray defines a unique Freudenthal end. Moreover it is not hard to show that the Freudenthal ends of G can be described as equivalence classes of rays, where two rays R and R' are related if there exists a ray R'' whose intersections with R and R' are infinite (see [12] for details).

A ray $R : a_0 - \varepsilon$ between $a_0 \in V(G)$ and $\varepsilon \in \mathscr{F}(G)$ is a ray starting at a_0 which defines the end ε. Similarly we can define a ray between the sets $\Pi \subseteq V(G)$ and $F \subseteq \mathscr{F}(G)$.

Finally, a *biray* (or two-way ray) with source ε and target ε' $R : \varepsilon - \varepsilon'$ is a sequence of vertices indexed by the set of integer numbers $\mathbb{Z}\,\{\ldots a_{-2}, a_{-1}, a_0, a_1, a_2 \ldots\}$ such that $a_i \neq a_j$ $(i \neq j)$, the segment $\langle a_i, a_{i+1}\rangle$ is an edge of K, and R defines the ends $\varepsilon, \varepsilon' \in \mathscr{F}(G)$. Similarly we can define a biray between $F, F' \subseteq \mathscr{F}(G)$.

Two paths (rays, respectively) $\alpha, \beta : a - b$ ($\alpha, \beta : a - \infty$, resp.) are said to be *independent* when $\alpha \cap \beta = \{a, b\}$ ($\alpha \cap \beta = \{a\}$, resp.). Two birays are independent when they are disjoint.

The dual notion to independent paths is the notion of cut-set. When we introduce the ends and the infinity point ∞ of a graph, different notions of *cut-set* can be considered. Namely, given a (finite) set of vertices $J \subseteq V(G)$ we say that J is a cut-set for $a, b \in V(G)$ if a and b lie in different connected components of $G - J$. Notice that J exists since G is locally finite. Moreover, the set J is a cut-set for $a \in V(G)$ and ∞ if a lies in a finite connected component of $G - J$. Given $a \in V(G)$ and $\varepsilon \in \mathscr{F}(G)$ we say that J is a cut-set for a and ε when a does not lie in the connected component $V_\varepsilon \subseteq G - J$ which defines ε. Finally, J is a cut-set of $\varepsilon, \varepsilon' \in \mathscr{F}(G)$ when $V_\varepsilon \neq V_{\varepsilon'}$.

For the sake of simplicity by a *trajectory* we shall mean a path, a ray, or a biray accordingly to the context.

Various Menger–Whitney type theorems relating the different cut-sets and the corresponding sets of independent trajectories can be found in the literature. In order to deal with them in a simple way we consider the set of symbols $\{V(G), \infty, \mathscr{F}(G)\}$ and we choose the pairs $(V(G), V(G))$, $(V(G), \infty)$, $(V(G), \mathscr{F}(G))$, and $(\mathscr{F}(G), \mathscr{F}(G))$. Any of these pairs is called a *connectivity pair*.

Given a connectivity pair (A, B) and $a \in A$ and $b \in B$ with $a \neq b$, the connectivity order of (a, b) is the maximum number $\mathrm{Conn}(a, b)$ of independent trajectories from a to b. The connectivity order of the pair $A_0 \subseteq A$, $B_0 \subseteq B$ is the number $\mathrm{Conn}(A_0, B_0) = \min\{\mathrm{Conn}(a, b);\ a \in A_0, b \in B_0,\ a \neq b\}$. The *connectivity order of type* (A, B) of G is the number $\mathrm{Conn}(A,B)$. We say that G is *n-connected of type* (A,B) if $\mathrm{Conn}(A,B)\geqslant n$. Notice that for one-ended graphs only the connectivity orders of type $(V(G),V(G)$ and $(V(G),\infty))=(V(G),\mathscr{F}(G))$ are defined.

For the connectivity pair (A,B), let $\mathscr{S}(A,B)$ denote the family of all cut-sets of type (A,B), i.e. $\mathscr{S}(A,B)=\cup\{\mathscr{S}(a,b);\ a\in A, b\in B, a\neq b\}$ where $\mathscr{S}(a,b)$ is the family of all the cut-sets for a,b in G. Moreover, the cut-order of (a,b) is the number $\mathrm{Sep}(a,b)=\min\{|J|;\ J\in\mathscr{S}(a,b)\}$. Here $|J|$ denotes the cardinal number of J. Notice that these numbers are finite since G is locally finite. Then the following general form of the Menger–Whitney theorem holds. Indeed for the pairs $(V(G),V(G))$, $(V(G),\infty)$, $(V(G),\mathscr{F}(G))$, and $(\mathscr{F}(G),\mathscr{F}(G))$ the corresponding proofs can be found in [9,7,8,12] respectively.

Theorem 1.1. *For any connectivity pair* (A,B) *and* $a\in A$, $b\in B$ *with* $a\neq b$, $\mathrm{Sep}(a,b)=\mathrm{Conn}(a,b)$. *In particular, the connectivity order* $\mathrm{Conn}(A_0,B_0)$ *coincides with the cut-order* $\mathrm{Sep}(A_0,B_0)=\min\{|J|;\ J\in\mathscr{S}(a,b);\ a\in A_0, b\in B_0;\ a\neq b\}$ *for any pair* $A_0\subseteq A$ *and* $B_0\subseteq B$.

In order to keep this section within a sensible length we shall give the relationships among the different connectivity types in Appendix A. We now proceed to give some consequences and variations of the Menger–Whitney theorem stated above for the various connectivity types defined here. They are the analogues of already known results involving the connectivity type $(V(G),V(G))$ and the classical Menger–Whitney theorem.

For the connectivity type $(V(G),\infty)$ we can prove the following theorems

Theorem 1.2 (Dirac [3, Theorem B]). *Let G be an infinite graph, then the following statements are equivalent*:

(a) *G is n-connected of type* $(V(G),\infty)$.

(b) *Let* $A=\{v_1,\dots,v_p\}$ *be a finite set of vertices of G. Given any family* $\{a_1,\dots,a_p\}$ *of positive integers with* $\sum_{i=1}^{p}a_i=n$ *there exist n independent rays from A to* ∞ *such that* a_i *of them start from* v_i, *for all i.*

(c) *Given any set of vertices* $A\subseteq V(G)$ *with* $|A|=n$ *there exists a family of n disjoint rays from A to* ∞.

Proof. (a) ⇒ (b): It is a particular case of Proposition 1.7 below; see Remark 1.8.

(b) ⇒ (c): It is obvious.

(c) ⇒ (a): Clearly, condition (c) implies that $\mathrm{val}(G)\geqslant n$. Otherwise, if $v\in V(G)$ is a vertex with $\mathrm{val}(v)\leqslant n-1$, we can form a set $A\subseteq V(G)$ containing $\{v\}\cup \mathrm{lk}(v;G)$ with $|A|=n$ and (c) does not hold for A.

As $\mathrm{val}(G)\geqslant n$, then $|\mathrm{lk}(v;G)|\geqslant n$ for all $v\in V(G)$, and by condition (c) we can find n disjoint rays starting at n vertices in $\mathrm{lk}(v;G)$, and these rays yield n independent rays starting at v. This finishes the proof. □

Theorem 1.3 (Linck [10]). *Let G be an infinite graph. Then the following statements are equivalent*:

(a) *G is n-connected of type $(V(G), \infty)$.*

(b) *Given a set $A \subseteq V(G)$ with $|A| = n-1$, for any vertex $v \in A$ there exists a biray $R \subseteq G$ with $R \cap A = \{v\}$.*

(c) *Given a set $B \subseteq V(G)$ with $|B| = n$, for any vertex $v \in B$ there exists a ray $R \subseteq G$ with $R \cap B = \{v\}$.*

Proof. (a)$\Rightarrow$(b): Given $A \subseteq V(G)$ with $|A| = n-1$, we take $v \in A$. Since G is n-connected of type $(V(G), \infty)$ we can find n independent rays starting at v. Hence at least two rays do not contain vertices in A other than v. These two rays define a biray R with $R \cap A = \{v\}$.

(b)$\Rightarrow$(c): There exists a biray R which contains v and at most a vertex $v_i \in B$. Then it is clear that we can find a ray $R' \subseteq R$ containing v with $R' \cap B = \emptyset$.

(c)$\Rightarrow$(a): Let $J \subseteq V(G)$ be any set of vertices with $|J| \leqslant n-1$. Given any vertex $v \in G - J$, by using (c) we can find a ray $R \subseteq G$ such that $v \in R$ and $R \cap J = \emptyset$. Therefore, the connected component of v in $G - J$ is infinite, and $J \notin \mathscr{S}(V(G), \infty)$. $\square$

Theorem 1.4 (Dirac [2] and Halin [6]). *Let G be an infinite s-connected graph. Then the following statements are equivalent*:

(a) *G is n-connected of type $(V(G), \infty)$.*

(b) *For $A = \{v_1, \dots, v_{n-1}\} \subseteq V(G)$ and $1 \leqslant m \leqslant \min\{s+1, n-1\}$ there exists a biray $R \subseteq G$ with $R \cap A = \{v_1, \dots, v_m\}$.*

(c) *For $B = \{v_1, \dots, v_n\} \subseteq V(G)$ and $1 \leqslant m \leqslant \min\{s+1, n\}$ there exists a ray $R \subseteq G$ with $R \cap A = \{v_1, \dots, v_m\}$.*

Proof. (a)$\Rightarrow$(b) The case $m = 1$ is Theorem 1.3. Assume we have already proved (b) for $m \leqslant k-1 \leqslant s$. Let R be a biray with $R \cap A = \{v_1, \dots, v_{k-1}\}$. Given $v_k \in A$, by using Theorem 1.2b we can find $k+1$ independent rays $L_1, \dots, L_{k+1}$ from v_k *to* ∞ which do not meet $\{v_{k+1}, \dots, v_{n-1}\}$. When $R \cap L_i \neq \emptyset$, let $a_i \in V(L_i)$ denote the first vertex in R $(1 \leqslant i \leqslant k+1)$.

The vertices $v_1, \dots, v_{k-1}$ define a decomposition of R into $k-2$ paths $R_1, \dots, R_{k-2}$ and two rays $R_{-\infty}$, R_∞. Assume that two vertices a_s, a_t lie in the same path (ray) R_j $(1 \leqslant j \leqslant k-2,\ j = \pm\infty)$. Then a biray can be found in $R \cup L'_s \cup L'_t$ containing $\{v_1, \dots, v_k\}$. Here $L'_i \subseteq L_i$ denotes the path from v_k to a_i. Therefore, we can now assume that at least one ray L_i does not meet R.

In case that only L_1 misses R, we can assume that each a_i $(2 \leqslant i \leqslant k+1)$ defines a unique path or ray $R_{j(i)}$ $(1 \leqslant j(i) \leqslant k-2,\ j(i) = \pm\infty)$. Hence, a suitable biray can be found in $R \cup L_1 \cup L'_{i_0}$ where $j(i_0) = \pm\infty$. At this point it will suffice to assume that at least two rays L_i miss R.

As G is s-connected we can also find a set of s independent paths $\gamma_j : v_k - v_j$ $(j \neq k)$. Let $b_i \in \gamma_i$ $(1 \leqslant i \leqslant k-1)$ denote the first vertex in $\gamma_i \cap R$. If no b_i lies in $R_{-\infty} \cup R_\infty$, there must exist a path R_j containing two vertices b_p, b_q and hence a biray

$R' \subseteq R \cup \gamma'_p \cup \gamma'_q$ can be easily found with $\{v_1, \dots, v_k\} \subseteq R'$. Here $\gamma'_p \subseteq \gamma_p$ denotes the path from v_k to b_p.

Assume now $b_1 \in R_\infty$. As there are two rays L_p, L_q which are disjoint with R, we can assume without loss of generality that $L_p \cap \gamma_1 = \{v_k\}$, and it is clear that a biray can be found in $R \cup L_p \cup \gamma_1$ containing $\{v_1, \dots, v_k\}$.

The proof of (a) $\Rightarrow$ (c) is similar and we omit it. Moreover, (b) $\Rightarrow$ (a) as well as (c) $\Rightarrow$ (a) follow from Theorem 1.3. □

Example 1.5. Any infinite tree G with all vertices of valence $n \geqslant 4$ shows that Theorem 1.4 does not hold without the hypothesis on the connectivity of G. Indeed, G is 1-connected of type $(V(G), V(G))$ but n-connected of type $(V(G), \infty)$. Moreover, G does not satisfy either (b) or (c) in Theorem 1.4 for $m = 2$.

We next give similar theorems for the connectivity type $(V(G), \mathscr{F}(G))$. We start with the following characterization

Theorem 1.6 (Dirac [3, Theorem B]). *Let G be an infinite graph, then the following statements are equivalent*:

(a) *G is n-connected of type $(V(G), \mathscr{F}(G))$.*

(b) *Given two sets $A = \{v_1, \dots, v_p\} \subseteq V(G)$ and $B = \{\varepsilon_1, \dots, \varepsilon_q\} \subseteq \mathscr{F}(G)$ and two sets of positive integers $\{a_1, \dots, a_p\}$ and $\{b_1, \dots, b_q\}$ with $\sum_{k=1}^p a_k = \sum_{h=1}^q b_h = n$, there exist n independent rays from A to B such that a_k of them start at v_k and b_h of them define ε_h for all k, h.*

Theorem 1.6 is an immediate consequence of the following more general proposition which will be used also for 2-complexes in Section 3 below.

Proposition 1.7. *Let G be an infinite graph $A = \{v_1, \dots, v_p\} \subseteq V(G)$, and $\mathscr{B} = \{B_1, \dots, B_q\}$ a family of pairwise disjoint closed sets of Freudenthal ends of G. Assume that for each pair (k, h) there exist n rays running from v_k to B_h. Then given two sets of positive integers $\{a_1, \dots, a_p\}$ and $\{b_1, \dots, b_q\}$ with $\sum_{k=1}^p a_k = \sum_{h=1}^q b_h = n$, there exist n independent rays from A to $\bigcup_{h=1}^q B_h$ such that a_k of them start at v_k and b_h of them end at B_h for all k, h.*

Proof. First we consider a family $\mathscr{R}(k, h)$ of n pairwise disjoint rays from v_k to B_h and let $\mathscr{F}(k, h) \subseteq B_h$ denote the set of ends defined by the rays in $\mathscr{R}(k, h)$. Then we choose a connected finite subgraph $K \subseteq G$ such that $\mathrm{st}(v_k; G) \subseteq K$ for all $v_k \in A$ and moreover the connected components $V_\varepsilon \subseteq G - K$ determined by the ends $\varepsilon \in \bigcup \{\mathscr{F}(k, h);\ 1 \leqslant k \leqslant p, 1 \leqslant h \leqslant q\}$ are pairwise disjoint in such a way that the rays in $\mathscr{R} = \bigcup \{\mathscr{R}(k, h);\ 1 \leqslant k \leqslant p, 1 \leqslant h \leqslant q\}$ which meet V_ε are exactly those determining ε. Furthermore, for each $R \in \mathscr{R}(k, h)$ with end $\varepsilon \in \mathscr{F}(k, h)$ let $T_R \subseteq R \cap V_\varepsilon$ be a subray of R.

Given $\varepsilon \in \mathscr{F}(k, h)$ we form the family $\mathscr{T}_\varepsilon$ consisting of all rays T_R where $R \in \mathscr{R}$, and $\mathscr{F}(R) = \varepsilon$ and we choose a subfamily $\mathscr{M}_\varepsilon \subseteq \mathscr{T}_\varepsilon$ such that

$$|\mathscr{M}_\varepsilon| = \max\{|\mathscr{H}|;\ \mathscr{H} \subseteq \mathscr{T}_\varepsilon \text{ and the rays in } \mathscr{H} \text{ are pairwise disjoint}\}.$$

Next for each $R \in \mathscr{R}$ with $\mathscr{F}(R) = \varepsilon$ and $T_R \notin \mathscr{M}_\varepsilon$ we choose n pairwise disjoint paths in V_ε joining R to all rays in $\mathscr{M}_\varepsilon$. We call them the *net* of R and we denote it by $\mathscr{N}_R$. Notice that some paths in $\mathscr{N}_R$ may be degenerate one-point paths.

We take a new finite subgraph $G' \subseteq G$ containing K, all paths in the net $\mathscr{N}_R$ of each R, and moreover a subpath in each R (in $\mathscr{M}_\varepsilon$ or not) passing through all points in R obtained as intersection of R with the paths in the nets. Furthermore, we require that each $T \in \mathscr{M}_\varepsilon$ meets the frontier $Fr(G') = G' \cap (G \backslash G')$ in just one vertex. Here $G \backslash G'$ is the subgraph generated by $G - G'$, see Introduction.

Then for each k and h we consider the sets of vertices $\Gamma_k = \mathrm{lk}(v_k; G) - A$ and $\Theta_h = Fr(G') \cap (\bigcup \{T;\ T \in \mathscr{M}_\varepsilon \text{ and } \varepsilon \in B_h\})$. Notice that $\Gamma_k \neq \emptyset$ for all k since $p \leqslant n$.

We now construct a new graph G_0 as follows. We take pairwise disjoint sets $\{D_k\}_{1 \leqslant k \leqslant p}$ and $\{E_h\}_{1 \leqslant h \leqslant q}$ with $|D_k| = a_k$ and $|E_h| = b_h$. Then we form the complete bipartite graphs $L_k = K(D_k, \Gamma_k)$ and $L'_h = K(E_h, \Theta_h)$. Finally, we consider two further vertices c and c' and the complete bipartite graphs $C = K(c, \bigcup_{k=1}^p D_k)$ and $C' = K(c', \bigcup_{h=1}^q E_h)$ and we set

$$G_0 = \left(G' \Big\backslash \bigcup_{k=1}^{p} \mathrm{st}(v_k; G)\right) \cup \left(\bigcup_{k=1}^{p} L_k\right) \cup \left(\bigcup_{h=1}^{q} L'_h\right) \cup C \cup C'.$$

We claim that $\mathrm{Conn}(c, c') \geqslant n$ in G_0. Indeed, let $J \subseteq G_0$ be a set of vertices with $|J| \leqslant n-1$. Then one finds $x_0 \in D_{k_0} - J$ and $y_0 \in E_{h_0} - J$ for some k_0 and h_0. Moreover, there exists at least one ray $R \in \mathscr{R}(k_0, h_0)$ which does not meet J. However, R may contain some other vertices $v_j \in A - \{v_{k_0}\}$. We proceed to show that it is always possible to choose R in such a way that for any $v_j \in A \cap R$ we have $D_j - J \neq \emptyset$. Otherwise, all rays in $\mathscr{R}(k_0, h_0)$ which avoid $J \cap G = J \cap G'$ contain some $v_j \in A$ with $D_j \subseteq J$. Let $I = \{k; D_k \subseteq J\}$. Then one gets necessarily $n = |\mathscr{R}(k_0, h_0)| \leqslant |J \cap G \cup \{v_k;\ k \in I\}|$. Moreover, since $D_k \subseteq J$ for all $k \in I$ one gets also $|J \cap G| \leqslant n - 1 - \sum_{k \in I} a_k$. This leads to the contradiction $n \leqslant |J \cap G| + |I| \leqslant n - 1$.

Therefore, we have proved the existence of a non-empty subset $\mathscr{L}(k_0, h_0) \subseteq \mathscr{R}(k_0, h_0)$ consisting of rays R which do not meet J and for all $v_j \in R \cap A$ we have $D_j - J \neq \emptyset$. We consider the family of ends $\mathscr{C}(k_0, h_0) = \{\varepsilon \in \mathscr{F}(k_0, h_0); \varepsilon = \mathscr{F}(R) \text{ with } R \in \mathscr{L}(k_0, h_0)\}$. We call an end in $\mathscr{C}(k_0, h_0)$ a *clean end*.

We next show that there exists a clean end ε_0 such that some $T_0 \in \mathscr{M}_{\varepsilon_0}$ does not meet J. Otherwise, if $m_\varepsilon = |\mathscr{M}_\varepsilon|$ and $I = \{k; D_k \subseteq J\}$ we have we have

$$\sum_{\varepsilon \in \mathscr{C}(k_0, h_0)} m_\varepsilon \leqslant (n-1) - \sum_{k \in I} a_k - r,$$

where $r = |J - \bigcup_{\varepsilon \in \mathscr{C}(k_0, h_0)} V_\varepsilon|$. In addition, if w_ε is the number of rays in $\mathscr{R}(k_0, h_0)$ which defines the end ε, the maximality of $\mathscr{M}_\varepsilon$ and the above inequality yield

$$\sum_{\varepsilon \in \mathscr{C}(k_0, h_0)} w_\varepsilon \leqslant \sum_{\varepsilon \in \mathscr{C}} (k_0, h_0) m_\varepsilon \leqslant (n-1) - \sum_{\varepsilon' \in \mathscr{F}(k_0, h_0) - \mathscr{C}(k_0, h_0)} w_{\varepsilon'}$$

since $\sum_{\varepsilon' \in \mathscr{F}(k_0, h_0) - \mathscr{C}(k_0, h_0)} w_{\varepsilon'} \leqslant r + |I|$. Hence $n = \sum_{\varepsilon \in \mathscr{F}(k_0, h_0)} w_\varepsilon \leqslant n - 1$ which is a contradiction.

Therefore, we have proved that there exists $\varepsilon_0 \in \mathscr{C}(k_0, h_0)$ and $T_0 \in \mathscr{M}_{\varepsilon_0}$ with $T_0 \cap J = \emptyset$. Hence, there exists a ray $R_0 \in \mathscr{L}(k_0, h_0)$ with $\mathscr{F}(R_0) = \varepsilon_0$. In addition, the construction of the graph G' allows us to choose a path $\gamma_0 \subseteq G'$ from R_0 to T_0 such that the union $U = \gamma_0 \cup R_0 \cup T_0$ misses the set J. Moreover, $A \cap (T_0 \cap \gamma_0) = \emptyset$ and if $(A - \{v_{k_0}\}) \cap R_0 \neq \emptyset$ we can replace v_{k_0} by the last vertex in $R_0 \cap A$ since $D_j - J \neq \emptyset$ for all $v_j \in R_0 \cap A$. For this we use that $R_0 \in \mathscr{L}(k_0, h_0)$. So, we can assume, without loss of generality, that $U \cap A = \{v_{k_0}\}$ and then we easily connect c to c' in G_0 by a path passing through x_0, U, and y_0.

We have checked that $\mathrm{Conn}(c, c') \geqslant n$ in G_0 and the Menger–Whitney theorem for the connectivity pair $V(G), V(G)$ in (1.1) provides n independent paths $\gamma_1, \gamma_2, \ldots, \gamma_n$ in G_0 from c to c'. In particular, $\gamma_j \cap G'$ $(1 \leqslant j \leqslant n)$ are pairwise disjoint paths with $\gamma_j \cap G'$ running from some $\Gamma_{k(j)}$ to some $\Theta_{h(j)}$. Moreover, for each k and h only a_k and b_h, respectively, of the paths $\gamma_j \cap G'$ verify $k(j) = k$ and $h(j) = h$, respectively. Now it is clear that $\{\gamma_j \cap G'\}_{1 \leqslant j \leqslant n}$ can be extended to a family of n independent rays with the required properties. This finishes the proof. □

Remark 1.8. Since a ray starting at v is just a ray running from v to $\mathscr{F}(G)$, one gets $(a) \Rightarrow (b)$ in Theorem 1.2 as a particular case of Proposition 1.7 by setting $q = 1$ and $B_1 = \mathscr{F}(G)$.

Other results concerning the connectivity pair $(V(G), \mathscr{F}(G))$ are the following.

Theorem 1.9 (Linck [10]). *Let G be an infinite graph. Then the following statements are equivalent*:

(a) *G is n-connected of type $(V(G), \mathscr{F}(G))$.*
(b) *Given $A \subseteq V(G)$ with $|A| = n - 1$, for any vertex $v \in A$ and any end $\varepsilon \in \mathscr{F}(G)$ there exists a biray $R \subseteq G$ with both ends ε and $R \cap A = \{v\}$.*
(c) *Given a set $B \subseteq V(G)$ with $|B| = n$, for any vertex $v \in B$ and any end $\varepsilon \in \mathscr{F}(G)$ there exists a ray $R \subseteq G$ whose end is ε and such that $R \cap B = \{v\}$.*

The proof of Theorem 1.9 follows the same pattern as the proof of Theorem 1.3 and we omit it.

Moreover, since $\mathrm{Conn}(V(G), \mathscr{F}(G)) = \mathrm{Conn}(V(G), V(G))$ (see Proposition A.2) we get the following analogue of the (1.4) above

Theorem 1.10 (Dirac [2] and Halin [6]). *Let G be an infinite graph. Then the following statements are equivalent*:

(a) *G is n-connected of type $(V(G), \mathscr{F}(G))$.*
(b) *For $A = \{v_1, \ldots, v_{n-1}\} \subseteq V(G)$, $\varepsilon \in \mathscr{F}(G)$, and $1 \leqslant m \leqslant n - 1$ there exists a biray R whose only end is ε and such that $R \cap A = \{v_1 \ldots v_m\}$.*
(c) *For $B = \{v_1, \ldots, v_n\} \subseteq V(G)$, $\varepsilon \in \mathscr{F}(G)$, and $1 \leqslant m \leqslant n$ there exists a ray R whose only end is ε and such that $R \cap A = \{v_1, \ldots, v_m\}$.*

The proof is similar to the proof of Theorem 1.4. We leave it to the reader.

Theorem 1.11. *Let G be an infinite graph. Then the following statements are equivalent* ($n \geqslant 2$ *and* $|\mathscr{F}(G)| \geqslant 2$):
(a) *G is n-connected of type* $(V(G), \mathscr{F}(G))$.
(b) *For* $A = \{v_1, \ldots, v_{n-1}\} \subseteq V(G)$, $\varepsilon, \varepsilon' \in \mathscr{F}(G)$, *and* $1 \leqslant m \leqslant n-1$ *there exists a biray R whose ends are* ε *and* ε' *and such that* $R \cap A = \{v_1, \ldots, v_m\}$.

Proof. (a)$\Rightarrow$(b): By using (a) we can find two rays R_1, R_2 from v_1 to ε such that $R_i \cap A = \{v_1\}$ $(i = 1,2)$. Similarly there are two rays R'_j $(j = 1,2)$ from v_1 to ε' with the same property. As $\varepsilon \neq \varepsilon'$ each intersection $R_i \cap R_j$ is finite. It is now easy to construct a biray $R \subseteq R_1 \cup R_2 \cup R'_1 \cup R'_2$ with $\mathscr{F}(R) = \{\varepsilon, \varepsilon'\}$ and $R \cap A = \{v_1\}$. Hence, we have shown (b) for $m = 1$. At this point we can follow the pattern of the proof of Theorem 1.4 to prove (b). The converse (b)$\Rightarrow$(a) follows from Theorem 1.9. $\square$

For the connectivity pair $(\mathscr{F}(G), \mathscr{F}(G))$ we can prove an analogue of Theorem 1.6. Actually we shall use Proposition 1.7 to prove a more general result. Namely

Proposition 1.12. *Let G be an infinite graph and* $\mathscr{A} = \{A_1, \ldots, A_p\}$ *and* $\mathscr{B} = \{B_1, \ldots, B_q\}$ *two families of closed sets of Freudenthal ends of G such that the elements of* $\mathscr{A} \cup \mathscr{B}$ *are pairwise disjoint. Assume that for each pair* (k,h) *there exist n birays running from* A_k *to* B_h. *Then given two sets of positive integers* $\{a_1, \ldots, a_p\}$ *and* $\{b_1, \ldots, b_q\}$ *with* $\sum_{k=1}^{p} a_k = \sum_{h=1}^{q} b_h = n$, *there exist n independent birays from* $\bigcup_{k=1}^{p} A_k$ *to* $\bigcup_{h=1}^{q} B_h$ *such that* a_k *of them start at* A_k *and* b_h *of them end at* B_h *for all* k, h.

Proof. First for each pair (k,h) we choose a family $\mathscr{R}(k,h)$ of n pairwise disjoint birays from A_k to B_h. Let $\mathscr{F}(k,h)^- \subseteq A_k$ and $\mathscr{F}(k,h)^+ \subseteq B_h$ denote the sets of *left* and *right ends*, respectively, defined by the birays in $\mathscr{R}(k,h)$. Then we consider a connected finite subgraph $K \subseteq G$ such that the connected components $V_\alpha \subseteq G - K$ defined by the ends $\alpha \in \bigcup_{k,h}(\mathscr{F}(k,h)^- \cup \mathscr{F}(k,h)^+)$ are pairwise disjoint. Moreover, we also assume that the rays which meet V_α are exactly those birays determining α. Then if $R \in \mathscr{R}(k,h)$ defines the left end η, let $T_R \subseteq R \cap V_\eta$ be a subray of R contained in the component $V\eta$. Given the left end $\eta \in \mathscr{F}(k,h)^-$ we form the family $\mathscr{T}_\eta$ consisting of all rays T_R with $R \in \bigcup\{\mathscr{R}(k,h);\ 1 \leqslant k \leqslant p, 1 \leqslant h \leqslant q\}$ and such that η is the left end of R. Then we choose a maximal subfamily $\mathscr{M}_\eta \subseteq \mathscr{T}_\eta$ as in the proof of (1.7) as well as nets of paths $\mathscr{N}_R$ from all $R \notin \mathscr{M}_\eta$ to the rays in $\mathscr{M}_\eta$.

We now extend the infinite subgraph $K \cup \{V_\varepsilon;\ \varepsilon \in \cup_{k,h}\mathscr{F}(k,h)^+\}$ to a new graph G' by adding finite subgraphs in each V_η with $\eta \in \cup_{k,h}\mathscr{F}(k,h)^-$ in such a way that G' contains all paths in the net $\mathscr{N}_R$ of each R as well as a subray in R passing through all points obtained as intersection of R with the paths in $\mathscr{N}_R$. Furthermore, we require that for every left end η each ray $T \in \mathscr{M}_\eta$ meets the frontier $\mathrm{Fr}(G') = G' \cap (G \backslash G')$ in just one vertex. Then for each k we consider the set of vertices $\Gamma_k = \mathrm{Fr}(G') \cap (\bigcup\{T;\ T \in \mathscr{M}_\eta \text{ and } \eta \in A_k\})$. We now construct a new graph G_0 as follows. We take pairwise disjoint sets $\{D_k\}_{1 \leqslant k \leqslant p}$ with $|D_k| = a_k$. Then we form the complete bipartite graphs $L_k = K(D_k, \Gamma_k)$ and $C = K(c, \bigcup_{k=1}^{p} D_k)$ for some new vertex c. Then we set $G_0 = G' \cup (\bigcup_{k=1}^{p} L_k) \cup C$. Notice that for each h the ends in B_h are also ends of G_0.

We claim that for each $h \leqslant q$ there exist n rays in G_0 running from c to B_h. Indeed, let J be a set of vertices of G_0 with $|J| \leqslant n-1$. Then one finds $x \in D_{k_0} - J$ for some k_0. Moreover, there exists at least one biray $R \in \mathscr{R}(k_0, h)$ which does not meet J. Let $\mathscr{L}(k_0, h) \subseteq \mathscr{R}(k_0, h)$ be the subset consisting of such birays. A left end $\eta \in \mathscr{F}(k_0, h)$ will be called a *clean end* if there exists $\{R \in \mathscr{L}(k_0, h)\}$ with $\eta \in \mathscr{F}(R)$. Let $\mathscr{C}(k_0, h)$ denote the set of clean ends for the pair (k_0, h). A similar argument as in the proof of Proposition 1.7 above shows that there exist a clean end η_0 and a ray $T_0 \in \mathscr{M}_{\eta_0}$. Therefore one finds a biray R_0 whose left end is η_0 as well as a path γ joining T_0 to R_0 in G_0 such that the union $U = \gamma \cup T_0 \cup R_0$ does not meet J. Now it is easy to construct a ray from c to B_h by using U.

We have checked that for each h there exist at least n rays in G_0 running from c to B_h, and Proposition 1.7 applied to G_0 with $A = \{c\}$ provides n independent rays from c to $\bigcup_{h=1}^{q} B_h$ such that b_h of them end at B_h. Clearly a_k of them necessarily pass through vertices in Γ_k and so we can extend these rays to birays in G with left ends in $\bigcup_{k=1}^{p} A_k$ satisfying the required properties. This finishes the proof. □

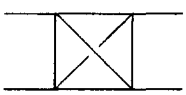
We now have as an immediate consequence of Proposition 1.12 the analogue of Theorem 1.6 for the connectivity pair $(\mathscr{F}(G), \mathscr{F}(G))$. Namely,

Theorem 1.13 (Dirac [3, Theorem B]). *Let G be an infinite graph, then the following statements are equivalent*:

(a) *G is n-connected of type $(\mathscr{F}(G), \mathscr{F}(G))$.*

(b) *Given two disjoint sets of ends $F = \{\eta_1, \ldots, \eta_p\}$, and $F' = \{\varepsilon_1, \ldots, \varepsilon_q\}$, and two sets of positive integers $\{a_1, \ldots, a_p\}$ and $\{b_1, \ldots, b_q\}$ with $\sum_{k=1}^{p} a_k = \sum_{h=1}^{q} b_h = n$, there exist n independent birays from F to F' such that a_k of them define η_k and b_h of them define ε_h for all k, h.*

Remark 1.14. For the usual connectivity type $(V(G), V(G))$, the single case $p = q = n$ $a_i = b_j = 1$ in (b) above is equivalent to (a) (see [14, Theorem 7A]). But this is not the case for the connectivity type $(\mathscr{F}(G), \mathscr{F}(G))$ as the following graph G shows:

$G \equiv$ 

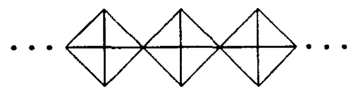
The graph G satisfies (b) for $p = q = n = 2$, $a_i = b_j = 1$ but it is only 1-connected of type $(\mathscr{F}(G), \mathscr{F}(G))$. Similarly for the pair $(V(G), \mathscr{F}(G))$.

2. Connectivity of infinite 2-complexes

Give a 2-complex P, a *2-path* in P, $\alpha : e_0 - e_n$, is a finite sequence of edges and triangles $\{e_0, t_1, e_1, t_2, \ldots, t_n, e_n\}$ such that t_i are triangles, e_i are edges and $e_i, e_{i-1} < t_i$ $(1 \leqslant i \leqslant n)$. The corresponding notion of *2-ray* joining e_0 *to* ∞ (or $\varepsilon \in \mathscr{F}(P)$) is now clear. Similarly the notion of 2-biray joining two Freudenthal ends $\varepsilon, \varepsilon' \in \mathscr{F}(P)$ is straightforward.

The 2-paths in P induce a stronger notion of connectedness in any 2-complex P. Namely P is said to be *2-path connected* when any two edges e, e' can be joined by a 2-path in P. The definition of 2-path connected component is now clear. Moreover, 2-rays in P induce a new class of ends in P. Namely a *2-end* Δ is the equivalence class of 2-rays, where two 2-rays R, R', are equivalent if there exists a ray R'' which meets both R and R' in infinitely many edges. This is equivalent to say that for any finite subcomplex $K \subseteq P$ we have $C_K(R) = C_K(R')$ where $C_K(R) \subseteq P - K$ denotes the 2-path connected component containig a subray of R.

Let $\mathscr{F}_2(P)$ denote the set of 2-ends of P. There exist clearly infinite connected 2-complexes P with $\mathscr{F}_2(P) = \emptyset$, for instance the following sequence P of squares:

$P \equiv$ 

The following lemma shows that 2-path connected 2-complex P verifies $\mathscr{F}_2(P) \neq \emptyset$. For the sake of simplicity, we shall say that P is an *admissible 2-complex* when P is a 2-path connected 2-complex such that any triangle in P contains at most one boundary edge. Notice that this is the case when ∂P is full in P.

Lemma 2.1. *Let P be a 2-path connected infinite 2-complex. Then $\mathscr{F}_2(P) \neq \emptyset$. More explicitly, any connected component of the complement of a finite subcomplex defines at least one 2-end.*

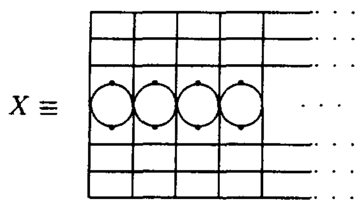
Proof. Let $K \subseteq P$ be a finite subcomplex. Given a connected component $L \subseteq P - K$ we can find a subcomplex $L' \subseteq L$ which is purely two-dimensional and $L - L'$ is finite. We claim that L' defines at least one 2-end. Otherwise, all 2-path connected components $A_i \subseteq L'$ are finite. For each A_i we take an edge $e_i \in A_i$ Since P is 2-path connected we can find 2-paths in P $\alpha_i : e_i - e_{i+1}$. Moreover all these 2-paths meet $P - L'$ in edges. Let σ_i be the first edge in $\alpha_i \cap (P - L')$. As $P - L'$ is finite we find an edge $\sigma = \sigma_{i_1} = \sigma_{i_2} = \cdots$ which appears in infinite many 2-rays. This yields a contradition since P is locally finite. □

Definition 2.2. Given an admissible 2-complex P, we denote by $\mathscr{E}(P)$ the set of interior edges of P (see Introduction). The *bipartite graph of P*, $G(P)$, is defined as follows. Let $V(G(P)) = E \cup T$ where E is the set consisting of the barycenters $\bar{e}$ with $e \in \mathscr{E}(P)$, and T is the set of barycenters of triangles of P. Now $\bar{e} \in E$ is joined in $G(P)$ to $\bar{t} \in T$ when $e < t$. Clearly $G(P)$ is a subcomplex of the 1-skeleton, $\mathrm{sk}^1 P^{(1)}$, of the first barycentric subdivision $P^{(1)}$ of P.

Any 2-path (2-ray, 2-biray) in P yields a path (ray, biray) in $G(P)$ and vice versa. In particular, P is 2-path connected if and only if $G(P)$ is connected. Moreover, the Freudenthal ends of $G(P)$ are in 1-1 correspondence with the 2-ends of P. Indeed, it is easy to define the following bijection $g : \mathscr{F}(G(P)) \simeq \mathscr{F}_2(P)$. Given $\varepsilon \in \mathscr{F}(G(P))$ defined by the ray $\{\overline{e_0}, \overline{t_1}, \overline{e_1}, \overline{t_2}, \ldots\}$ in $G(P)$, we define $g(\varepsilon)$ as the 2-end defined by the 2-ray $\{e_0, t_1, e_1, t_2, \ldots\}$ in P.

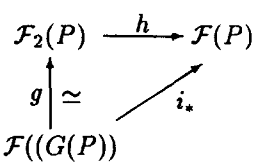
In general $\mathscr{F}(G(P))$ does not coincide with $\mathscr{F}(P)$ as the following example shows.

Example 2.3. Let $X \subseteq \mathbb{R}^2$ be the subspace $X = [2, \infty) \times [-4, 4] - \bigcup\{C_n; n \geqslant 1\}$ where C_n is the open disk of center $2n + 1$ and radius 1 $(n \geqslant 1)$. Then P is the admissible 2-complex obtained as a subdivision of the following cellular decomposition of X without new vertices:



The 2-complex P verifies $\mathscr{F}_2(P) = 2 > 1 = \mathscr{F}(P)$. In any case, we have

Proposition 2.4. *Let P be an admissible 2-complex. Then there exists a surjective map $h : \mathscr{F}_2(P) \to \mathscr{F}(P)$ such that the following diagram is commutative*:



where g was defined above and i_* is induced by the (topological) inclusion $i : G(P) \subseteq P$.

Proof. The map h is defined as $i_* g^{-1}$. In order to show that h is onto, it will suffice to check that i_* is onto. Let $\varepsilon \in \mathscr{F}(P) = \mathscr{F}(\mathrm{sk}^1 P^{(1)})$. We construct the following sequence of vertices $S \subseteq V(G(P))$. Given an edge $e \subseteq P$ we take $\bar{e} \in S$ if $e \in \mathscr{E}(P)$. Otherwise, $\bar{t} \in S$ where t is the unique triangle $t \in P$ with $e < t$. The sequence S defines at least one Freudenthal end $\eta \in \mathscr{F}((G(P))$. It is not hard to check that $i_*(\eta) = \varepsilon$. $\square$

Proposition 2.5. *The map h in* (2.4) *is injective* (*and hence bijective*) *if and only if for any finite subcomplex $K \subseteq P$ there exists another finite subcomplex $L \subseteq P$ such that $K \subseteq L$ and for any connected component $C \subseteq P - L$ two edges e, e' in C can be joined by a 2-path in $P - K$.*

Proof. Assume that the condition does not hold. Then we can find a finite subcomplex K and an increasing sequence of finite subcomplexes $\{L_i;\, i \geqslant 0\}$ with $L_0 = K$, $P = \bigcup_{i \geqslant 0} L_i$, and $L_i \subseteq \operatorname{int} L_{i+1}$ such that for each $i \geqslant 1$ there exists a connected component $C_i \subseteq P - L_i$ and two edges e_i, $e'_i \in C_i$ such that any 2-path $\alpha : e_i -\ e'_i$meets K.

After choosing a subsequence, if necessary, we can assume without loss of generality that $C_{i+1} \subseteq C_i$ for all $i \geqslant 1$.

Clearly, each C_i has two or more 2-path connected components. In fact, at least two of them are infinite. Otherwise, if C'_i is the only infinite 2-path connected component of C_i, we could find L_j with j large enough to guarantee $C_j \subseteq C'_i$, and the two edges e_j, e'_j could be joined outside K by a 2-path in $P - L_i$.

Therefore, we can choose two decreasing sequences of infinite 2-path connected component $C_i^1, C_i^2 \subseteq C_i$ which define two different 2-ends $\Delta_1, \Delta_2 \in \mathscr{F}_2(P)$, respectively, such that $h(\Delta_1) = h(\Delta_2) \in \mathscr{F}(P)$ is the Freudenthal end defined by the sequence $\{C_i\}_{i \geqslant 1}$. Hence, h is not injective.

Conversely, assume now that the condition in Proposition 2.5 holds. Given two 2-ends $\Delta_1, \Delta_2 \in \mathscr{F}_2(P)$ with $h(\Delta_1) = h(\Delta_2) = \varepsilon$, let $\{L_i\}_{i \geqslant 1}$ be an increasing sequence of finite subcomplexes as above and let $\{C_i\}$ be the decreasing sequence of connected components $C_i \subseteq P - L_i$ which defines the Freudenthal end ε.

Let $R_1 = \{e_0, t_1, e_1, \ldots\}$ and $R'_1 = \{e'_0, t'_1, e'_1, \ldots\}$ be two 2-rays defining Δ_1 and Δ_2, respectively. By the definition of h, and by choosing a subsequence, if necessary, we can assume, without loss of generality, that the baricenters $\overline{e_i}$, $\overline{e'_i}$ belong to C_i. By connectedness $e_i, e'_i \in C_i$, and then for each L_i we can find L_j such that e_i and e'_i can be joined by a 2-path outside L_i. Therefore, the 2-path connected components of C_i which contain subrays of R_1 and R_2 coincide, and so $\Delta_1 = \Delta_2$. That is, h is injective. □

This rather technical proposition has the following interesting consequence:

Corollary 2.6. *Let P be an admissible 2-complex such that $\operatorname{lk}(v; P)$ is connected for all $v \in V(G)$ except possibly a finite set. Then the map h in* (2.4) *is bijective.*

As an immediate consequence we have

Corollary 2.7. *Let M^n be a triangulated n-manifold with $n \geqslant 2$. Then $\mathscr{F}_2(\mathrm{sk}^2 M) = \mathscr{F}(\mathrm{sk}^2 M)$.*

Example 2.8. Corollary 2.6 does not give a sufficient condition as the following example shows. Let P be a triangulation of $\mathbb{R}^2 - \bigcup\{C_n;\ n \geqslant 0\}$ where C_n is the unit open disk of center $(2n-1,0)$. Then $\mathscr{F}_2(P)=\mathscr{F}(P)=\{*\}$ but the links of the points $(2n,0)$ are not connected.

Proof of Corollary 2.6. Let A denote the set of vertices whose links are not connected. Given an increasing sequence of finite subcomplexes $\{L_i\}_{i\geqslant 1}$ as in the proof of Proposition 2.5 we can assume that $A\subseteq L_1$. Let N_i be the simplicial neighbourhood of $P-L_i$, $N_i=\{\sigma\in P;\ \sigma\cap(P-L_i)\neq\emptyset\}$. Given L_i, we can choose L_j with $N_j\cap L_i=\emptyset$. Then any connected component $C\subseteq P-L_j$ is contained in a 2-path connected component of L_i. Otherwise, let $\{C'_r\}$ be the set of 2-path connected components which meet C. By connectedness of C, given C'_s there exists at least a vertex v in an intersection $C'_s\cap C'_{s'}\cap(P-L_j)$. Therefore, $\mathrm{lk}(v;P)=\mathrm{lk}(v;N(L_j))=\bigcup\{\mathrm{lk}(v;C'_s);\ v\in C'_s)\}$. As $\mathrm{lk}(v;C'_s)\cap\mathrm{lk}(v;C'_{s'})=\emptyset$ when $s\neq s'$, $v\in A$ which is a contradiction.

Hence, for any pair of edges $e,e'\in C$ we have a 2-path $\alpha: e-e'$ outside L_i and we can now apply Proposition 2.5. □

We are now ready to define various connectivity types for an infinite admissible 2-complex P. First, notice that three new connectivity pairs from the set of symbols $\{\mathscr{E}(P),\infty,\mathscr{F}(P),\mathscr{F}_2(P)\}$ can be now added to the corresponding analogues of those in Section 1. Namely $(\mathscr{E}(P),\mathscr{F}_2(P))$, $(\mathscr{F}(P),\mathscr{F}_2(P))$, and $(\mathscr{F}_2(P),\mathscr{F}_2(P))$. Given a connectivity pair (A,B), and $a\in A$, $b\in B$ with $a\neq b$, the connectivity order of (a,b) is the maximum number $\mathrm{Conn}(a,b)$ of independent 2-trajectories from a to b. Here, by a *2-trajectory* we mean a 2-path, a 2-ray or a 2-biray according to the nature of a and b. Notice that the type $(\mathscr{F}(P),\mathscr{F}_2(P))$ is defined for any pair $\varepsilon\in\mathscr{F}(P)$, $\Delta\in\mathscr{F}_2(P)$ with $h(\Delta)\neq\varepsilon$. The *connectivity order of type* (A,B) of P, $\mathrm{Conn}(A,B)$, is now define in a similar way as in Section 2, that is $\mathrm{Conn}(A,B)=\min\{\mathrm{Conn}(a,b);\ a\in A,\ b\in B;\ a\neq b\}$. The 2-complex is said to be *n-connected of type* (A,B) if $\mathrm{Conn}(A,B)\geqslant n$.

A set J of triangles and/or edges of P is a *cut-set* for a, b if any 2-trajectory from a to b in P contains some element of J. If $\mathscr{S}(a,b)$ is the family of cut-sets for a,b, the cut-order $\mathrm{Sep}(a,b)$ is define as in Section 2. The following proposition is immediate from the definition of the bipartite graph $G(P)$.

Proposition 2.9. *Let P be an infinite admissible 2-complex. Then the connectivity order of type (A,B) with $A,B\neq\mathscr{F}(P)$ coincides with* $\mathrm{Conn}(G(A),G(B))$. *Here $G(\mathscr{E}(P))$ is the set $E\subseteq V(G(P))$ associated to interior edges, $G(\infty)=\infty$ and $G(\mathscr{F}_2(P))=\mathscr{F}(G(P))$.*

The following result is an immediate consequence of Proposition 2.9 and Theorem 1.1.

Proposition 2.10. *Let P be an infinite admissible 2-complex and let A,B as in Proposition 2.9. Then the following statements are equivalent*:

(a) *There is no cut-set J of type (A,B) with $|J|<n$.*

(b) *There exist n independent trajectories between any two elements $a\in A$, $b\in B$. That is, G is n-connected of type (A,B).*

The corresponding result when one considers Freudenthal ends of P is also true. But it is not an immediate consequences of Proposition 2.9 since we cannot identify $\mathscr{F}(G(P))$ with $\mathscr{F}(P)$. In fact, we have the following Menger–Whitney-type theorems for ends of P.

Proposition 2.11. *Let P be an infinite admissible 2-complex. Given a finite set of edges $\Pi \subseteq \mathscr{E}(P)$ and any set $F \subseteq \mathscr{F}(P)$ of ends the following two statements are equivalent:*
(a) *The sets Π and F cannot be separated by a cut-set with fewer than n edges and/or triangles.*
(b) *There exist n independent 2-rays from Π to F. That is, P is n-connected of type $(\mathscr{E}(P), \mathscr{F}(P))$.*

Proof. Assume (a). Then there is no cut-set $J \subseteq V(G(P))$ with $|J| \leqslant n-1$ for $\overline{\Pi} = \{\bar{e};\ e \in \Pi\} \subseteq V(G(P))$ and $i_*^{-1}(F) \subseteq \mathscr{F}(G(P))$, where i_* is the map in Proposition 2.4. By [8] we can find n independent rays from $\overline{\Pi}$ to $i_*^{-1}(F)$ in $G(P)$, and hence these rays define n independent 2-rays from Π to $i_*(i_*^{-1}(F)) = F$ since i_* is onto by Proposition 2.4.

The converse (b) $\Rightarrow$ (a) is obvious. □

Proposition 2.12. *Let P be an infinite admissible 2-complex, and let $F, H \subseteq \mathscr{F}(P)$ be two set of Freudenthal ends such that $F \cap \mathrm{cl}(H) = H \cap \mathrm{cl}(F) = \emptyset$ where $\mathrm{cl}(H)$, and $\mathrm{cl}(F)$ denote the corresponding topological closures. Then the following statements are equivalent:*
(a) *F and H cannot be separated by a cut-set with fewer than n edges and/or triangles.*
(b) *There exist n independent 2-birays from F to H. That is, P is n-connected of type $(\mathscr{F}(P), \mathscr{F}(P))$.*

Proof. It is similar to the proof of Proposition 2.11. The continuity of the map i_* in Proposition 2.4 yields the inclusion $\mathrm{cl}(i_*^{-1}(A)) \subseteq i_*^{-1}(\mathrm{cl}(A))$ which implies $\mathrm{cl}(i_*^{-1}(A)) \cap i_*^{-1}(B) = \emptyset$ when $\mathrm{cl}(A) \cap B = \emptyset$ for arbitrary subsets $A, B \subseteq \mathscr{F}(P)$. By the main theorem of [12] there exist n-independent birays from $i_*^{-1}(F)$ to $i_*^{-1}(H)$. These birays define n independent 2-birays from F to H. □

Proposition 2.13. *Let P be an infinite admissible 2-complex, and let $F \subseteq \mathscr{F}(P)$, and $H \subseteq \mathscr{F}_2(P)$ be two subsets with $F \cap \mathrm{cl}(h(H)) = h(H) \cap \mathrm{cl}(F) = \emptyset$. Then the following statements are equivalent:*
(a) *F and H cannot be separated by a cut-set with fewer than n edges and/or triangles.*
(b) *There exist n independent 2-birays from F to H. That is, P is n-connected of type $(\mathscr{F}(P), \mathscr{F}_2(P))$.*

Proof. By using the same arguments as in the proof of Proposition 2.12 we find n independent birays from $i_*^{-1}(h(H))$ to $i_*^{-1}(F)$. These birays define n independent 2-birays in P joining H to F. $\square$

The previous propositions from Propositions 2.11 to 2.13 can be summarized in the following general Menger–Whitney Theorem for admissible 2-complexes:

Theorem 2.14. *Let P be an admissible 2-complex P. For any connectivity pair (A,B), if $a\in A$, $b\in B$ and $a\neq b$ then* $\mathrm{Sep}(a,b)=\mathrm{Conn}(a,b)$. *In particular, the connectivity order of type (A,B) coincides with the cut-order* $\mathrm{Sep}(A,B)=\min\{\mathrm{Sep}(a,b);\ a\in A,\ b\in B;\ a\neq b\}$. *Notice that these numbers are finite since P is locally finite.*

We finish this section with a theorem which allows us to consider cut-sets containing only edges for any type of connectivity. This theorem was originally proved by Woon [14, Theorem 3] for finite 2-complexes and connectivity pair $(\mathscr{E}(P),\mathscr{E}(P))$. We give here a more general and simpler proof.

Theorem 2.15. *Let P be an infinite admissible 2-complex such that* $\mathrm{val}(e)\geqslant n$ *for any interior edge $e\in\mathscr{E}(P)$. Then P is n-connected of type (A,B) if and only if there exists no cut-set $J\in\mathscr{S}(A,B)\cap\mathscr{E}(P)$ with $|J|<n$.*

Theorem 2.15 is an immediate consequence of the following

Lemma 2.16. *If J is a minimal cut-set of type (A,B) for P with $|J|=k<n$, then there exists a cut-set $J'\in\mathscr{S}(A,B)\cap\mathscr{P}(\mathscr{E}(P))$ with $|J'|=k$.*

Proof. We shall prove the lemma inductively on the number $m\geqslant 0$ of triangles in J. The case $m=0$ is trivial. Assume that the result holds for m, and let $J=\{t,t_1,t_2,\dots,t_m\}\cup\{a_{m+1},\dots,a_{k-1}\}$ be a cut-set of type (A,B) with triangles $t,t_1,t_2,\dots,t_m$.

Given $c\in C$, for any $C\in\{\mathscr{E}(P),\infty,\mathscr{F}(P),\mathscr{F}_2(P)\}$, let A_c denote the set consisting of all edges in P which can be joined to c by trajectories which do not meet J. If J is a cut-set for $a\in A$ and $b\in B$ we have $A_a\cap A_b=\emptyset$. Moreover, since J is minimal there exists a set $\{\gamma_\alpha\}_{\alpha\in J}$ of independent trajectories with $\gamma_\alpha\cap J=\{\alpha\}$ for each $\alpha\in J$. Given γ_t, let e_a (e_b respectively) denote the edge of t which appears in $A_a\cap\gamma_t$ ($A_b\cap\gamma_t$ respectively). Finally, let e be the third edge of t.

Assume $A,B=\mathscr{E}(P)$.

Case 1: $A_e\cap A_a=\emptyset$. If $e_a=a$, as $\mathrm{val}(a)\geqslant n$ there exist p triangles $s_1,\dots,s_p$ in $\mathrm{st}(a;P)-J$. Moreover, since $p>k-m-1$ there exists an edge $a'<s_j$ with $a'\notin J$. Thus $a'\in A_a$, and any 2-path from a' to b must meet J. Furthermore, the assumption $A_e\cap A_a=\emptyset$ yields that any 2-path $\xi: a'-b$ with $\xi\cap J=\{t\}$ must contain a. Therefore $J_1=\{a,t_1,\dots,t_m\}\cup\{a_{m+1},\dots,a_{k-1}\}$ is a cut-set for a' and b with only m triangles.

If $a \neq e_a$, the assumption $A_e \cap A_a = \emptyset$ implies that $J_1 = \{e_a, t_1, \ldots, t_m\} \cup \{a_{m+1}, \ldots, a_{k-1}\}$ is a cut-set for a and b.

Case 2: $A_e \cap A_a \neq \emptyset$. As $A_a \cap A_b = \emptyset$, it follows that $A_e \cap A_b = \emptyset$, and we proceed in the same way by replacing a by b.

Assume $A = \mathscr{E}(P)$, and $B \neq \mathscr{E}(P)$.

In Case 1, the proof is the same as above.

In Case 2, the set $J_2 = \{e_b, t_1, \ldots, t_m\} \cup \{a_{m+1}, \ldots, a_{k-1}\}$ is a cut-set for a, b with m triangles.

Finally, assume $A, B \in \{\mathscr{F}_2(P), \mathscr{F}(P)\}$. In Case 1 the set $J_3 = \{e_a, t_1, \ldots, t_m\} \cup \{a_{m+1}, \ldots, a_{k-1}\}$ is a cut-set for a, b. In Case 2 the set J_2 above is a cut-set.

We now apply the induction hypothesis to finish the proof. □

3. Some Menger–Whitney type theorems for 2-complexes

This section contains the two-dimensional analogues of the results stated at the end of Section 1. We recall that the analogues for $(\mathscr{E}(P), \mathscr{E}(P))$-connectivity are given in [14, Section 4] for finite 2-complexes. Actually, the same proofs work for infinite 2-complexes. We shall start with the following theorems concerning the connectivity type $(\mathscr{E}(P), \infty)$.

Theorem 3.1. *Let P be an admissible infinite 2-complex with* $\mathrm{val}(P) \geqslant n$, *then the following statements are equivalent*:

(a) *P is n-connected of type $(\mathscr{E}(P), \infty)$.*

(b) *Given $A = \{e_1, \ldots, e_p\} \subseteq \mathscr{E}(P)$ and any family $\{a_1, \ldots, a_p\}$ of positive integers with $\sum_{i=1}^{p} a_i = n$ there exist n independent 2-rays from A to ∞ such that a_i of them start from e_i, for all i.*

(c) *Given any set of edges $A \subseteq \mathscr{E}(P)$ with $|A| = n$ there exists a family of n independent 2-rays from A to ∞.*

Proof. $(a) \Rightarrow (b)$: According to Proposition 2.9 $\mathrm{Conn}(E, \infty) = n$ for the set E of vertices of $G(P)$ associated to interior edges. Moreover, the set A yields a set $\bar{A} = \{\bar{e}_1, \ldots, \bar{e}_n\} \subseteq E$ and Proposition 1.7 applied to $G(P)$ (see Remark 1.8) shows that there exist n rays in $G(P)$ a_i of them starting at $\bar{e}_i$. Clearly these rays defined the required 2-rays in P.

(b) $\Rightarrow$ (c): It is obvious.

(c) $\Rightarrow$ (a): Let J be a cut-set for P. As $\mathrm{val}(P) \geqslant n$ we can assume that $J \subseteq \mathscr{E}(P)$ by Theorem 2.15. If $|J| \leqslant n-1$ and $e \in \mathscr{E}(P) - J$ we can apply (c) to $J \cup \{e\}$ to get a 2-ray R from e to ∞ with $R \cap J = \emptyset$ which is a contradiction. So $|J| \geqslant n$, and P is n-connected of type $(\mathscr{E}(P), \infty)$. □

Theorem 3.2. *Let P be an admissible infinite 2-complex with* $\mathrm{val}(P) \geqslant n$. *Then the following statements are equivalent*:

(a) *P is n-connected of type $(\mathscr{E}(P), \infty)$.*
(b) *Given a set $A \subseteq \mathscr{E}(P)$ with $|A| = n-1$, any edge $e \in A$ is contained in a 2-biray which avoids the other $n-2$ edges of A.*
(c) *Given a set $B \subseteq \mathscr{E}(P)$ with $|B| = n$, any edge in B is contained in a 2-ray which avoids the other $n-1$ edges of B.*

Proof. (a) $\Rightarrow$ (b): By using Proposition 2.9 we get $\mathrm{Conn}(E, \infty) = n$ in the the bipartite graph $G(P)$. Here E is the set of vertices of $G(P)$ corresponding to interior edges. Moreover, the set A defines a set $\bar{A} \subseteq E$. The same proof as in (a) $\Rightarrow$ (b) of Theorem 1.3 yields a biray R in $G(P)$ which contains a vertex $\bar{e} \in \bar{A}$ and avoids the rest of vertices of $\bar{A}$. The biray R clearly defines a 2-biray in P with the required properties.

(b) $\Rightarrow$ (c): It is obvious.

(c) $\Rightarrow$ (a): It is similar to the proof of (c) $\Rightarrow$ (a) in Theorem 1.3. □

Theorem 3.3. *Let P be an admissible infinite 2-complex. Assume that P is s-connected of type $(\mathscr{E}(P), \mathscr{E}(P))$ and $\mathrm{val}(P) \geqslant n$. Then the following statements are equivalent*:
(a) *P is n-connected of type $(\mathscr{E}(P), \infty)$.*
(b) *For $A = \{e_1, \ldots, e_{n-1}\} \subseteq \mathscr{E}(P)$ and $1 \leqslant m \leqslant \min\{s+1, n-1\}$ there exists a 2-biray $R \subseteq P$ with $R \cap A = \{e_1, \ldots, e_m\}$.*
(c) *For $A = \{e_1, \ldots, e_n\} \subseteq \mathscr{E}(P)$ and $1 \leqslant m \leqslant \min\{s+1, n\}$ there exists a 2-ray $R \subseteq P$ with $R \cap A = \{e_1, \ldots, e_m\}$.*

Proof. (a) $\Rightarrow$ (b): We know by Proposition 2.9 that $\mathrm{Conn}(E, \infty) = n$ and $\mathrm{Conn}(E, E) = s$ for the set E of vertices of $G(P)$ corresponding to interior edges of P. Moreover, the set A defines a set $\bar{A} = \{\bar{e}_1, \ldots, \bar{e}_{s+1}\} \subseteq E$ and the inductive proof of (a) $\Rightarrow$ (b) in Theorem 1.4 can be carried out here to obtain a biray R in $G(P)$ with $R \cap \bar{A} = \{\bar{e}_1, \ldots, \bar{e}_m\}$. The biray R yields the required 2-biray in P.

The proof of (a) $\Rightarrow$ (c) is similar and we omit it. Moreover (b) $\Rightarrow$ (a) as well as (c) $\Rightarrow$ (a) follow from Theorem 3.2. □

For the connectivity type $\mathrm{Conn}(\mathscr{E}(P), \mathscr{F}_2(P))$ we have the following result which follows the pattern of Theorem 3.2. We leave the proof to the reader. Notice that $\mathscr{F}_2(P))$ is identified with $\mathscr{F}(G(P))$ by Proposition 2.4. Compare with Theorem 1.9.

Theorem 3.4. *Let P be an admissible infinite 2-complex with $\mathrm{val}(P) \geqslant n$. Then the following statements are equivalent*:
(a) *P is n-connected of type $(\mathscr{E}(P), \mathscr{F}_2(P))$.*
(b) *Given a set $A \subseteq \mathscr{E}(P)$ with $|A| = n-1$, for any edge $e \in A$ and any end $\delta \in \mathscr{F}_2(P)$ there is a 2-biray R with both 2-ends Δ and such that $R \cap A = \{e\}$.*
(c) *Given a set $B \subseteq \mathscr{E}(P)$ with $|B| = n$, any $e \in B$ and any 2-end Δ there is a 2-ray R whose 2-end is Δ and such that $B \cap R = \{e\}$.*

Since Conn($\mathscr{E}(P)$, $\mathscr{F}_2(P)$) = Conn($\mathscr{E}(P), \mathscr{E}(P)$), we also get the following theorem.

Theorem 3.5. *Let P be an admissible infinite 2-complex with* val$(P)\geqslant n$. *Then the following statements are equivalent*:
(a) *P is n-connected of type* Conn($\mathscr{E}(P)$, $\mathscr{F}_2(P)$).
(b) *For $A=\{e_1,\dots,e_{n-1}\}\subseteq\mathscr{E}(P)$, $\Delta\in\mathscr{F}_2(P)$, and $1\leqslant m\leqslant n-1$ there exists a 2-biray R whose only 2-end is Δ and such that $R\cap A=\{e_1,\dots, e_m\}$.*
(c) *For $B=\{e_1,\dots,e_n\}\subseteq\mathscr{E}(P)$, $\Delta\in\mathscr{F}_2(P)$, and $1\leqslant m\leqslant n$ there exists a 2-ray R whose 2-end is Δ and such that $R\cap B=\{e_1,\dots,e_m\}$.*

Proof. We know by Theorem 2.9 that Conn$(E,\mathscr{F}(G(P))$ = Conn$(E,E)=n$. Now we can argue as in Theorem 3.3 to show that (a) implies both (b) and (c). The converses follow from Theorem 3.4. □

Another result for the same connectivity type is the following theorem (compare Theorem 1.11) whose proof is also omitted.

Theorem 3.6. *Let P be an admissible infinite 2-complex with* val$(P)\geqslant n$ *and* $|\mathscr{F}_2(P)|\geqslant 2$. *Then the following statements are equivalent*:
(a) *P is n-connected of type* Conn($\mathscr{E}(P), \mathscr{F}_2(P)$).
(b) *For $A=\{e_1,\dots,e_{n-1}\}\subseteq\mathscr{E}(P)$, $\Delta,\Delta'\in\mathscr{F}_2(P)$, and $1\leqslant m\leqslant n-1$ there exists a 2-biray R with 2-ends Δ,Δ' and such that $R\cap A=\{e_1,\dots,e_m\}$.*

For the connectivity type $(\mathscr{F}_2(P),\mathscr{F}_2(P))$ we can prove the two-dimensional analogue of (1.6) by applying (1.6) to the bipartite graph $G(P)$ of P. We leave the details to the reader.

We finish this section by considering connectivity types of a 2-complex P involving Freudenthal ends. In general there is no bijection between the Freudenthal end of P and the Freudenthal ends of $G(P)$. However, the analogues of Theorems 3.2 and 3.3 for the connectivity pair Conn($\mathscr{E}(P),\mathscr{F}(P)$) hold. We leave to the reader the task of state and prove them. Next example shows that the hypothesis on the connectivity type $(\mathscr{E}(P),\mathscr{E}(P))$ is necessary for the analogue of Theorem 3.3.

Example 3.7. Let M be any admissible 2-complex which is 3-connected of type ($\mathscr{E}(P)$, $\mathscr{E}(P)$) and with only one 2-end (e.g. the 2-skeleton of any one-ended open 3-manifold. See Corollary 2.7, and [1]). Let $A=\{x_0,\dots,x_n,\dots\}$ be any sequence of non-adjacent vertices of M. We consider three disjoint copies $A_i=\{x_n^i\}\subseteq M_i$ $(1\leqslant i\leqslant 3)$ of A and M respectively and we construct the 2-complex P_0 by identifying to a point y_n the three points x_n^i for each n. Given $y_0\in P_0$ we take three edges adjacent to y_0, $\gamma_i=\langle y_0,v_i\rangle\subseteq M_i\subseteq P_0$, one in each copy M_i of M. Let $c\notin P_0$ and we consider the complex K_0 with three triangles $\langle c,y_0,v_i\rangle$ $(1\leqslant i\leqslant 3)$. We take $P=K_0\cup P_0$. Then P has only one Freudenthal end but three 2-ends. Moreover, P is 3-connected of type

$(\mathscr{E}(P), \mathscr{F}(P))$ and only 1-connected of type $(\mathscr{E}(P), \mathscr{E}(P))$. Clearly, P satisfies condition (a) but not condition (b) in the analogue of Theorem 3.3. A similar example can be constructed for condition (c).

For Freudenthal ends the two-dimensional analogue of Theorem 1.13 also holds. More explictly, we have

Theorem 3.8. *Let P be an admissible infinite 2-complex; then the following statements are equivalent:*
(a) *P is n-connected of type $(\mathscr{F}(P), \mathscr{F}(P))$.*
(b) *Given any two disjoint sets of Freudenthal ends $F = \{\eta_1, \dots, \eta_q\}$ and $F' = \{\varepsilon_1, \dots, \varepsilon_p\}$ and two sets of positive integers $\{a_1, \dots, a_p\}$ and $\{a'_1, \dots, a'_q\}$ with $\sum_{i=1}^{p} a_i = \sum_{i=1}^{q} a'_j = n$, there exist n independent 2-birays from F to F' such that a_i of them define η_i and a'_j of them define ε_j, for all i, j.*

Proof. Clearly only (a) $\Rightarrow$ (b) needs to be checked. For this we observe that according to Proposition 2.4 each Freudenthal end $\alpha \in \mathscr{F}(P)$ defines a closed set $A_\alpha = i_*^{-1}(\alpha) \subseteq \mathscr{F}(G(P))$ where $G(P)$ is the bipartite graph of P. Therefore the sets F and F' determine two families $\mathscr{A}_F$ and $\mathscr{A}_{F'}$ of pairwise disjoint closed sets of Freudenthal ends of $G(P)$. As P is n-connected of type $(\mathscr{F}(P), \mathscr{F}(P))$ it follows that Proposition 1.12 can be applied to $\mathscr{A}_F$ and $\mathscr{A}_{F'}$ in $G(P)$ to show the existence of n disjoint birays in $G(P)$ such that a_i of them start at A_{η_i} and b_j of them end at A_{ε_j}. Moreover birays in $G(P)$ can be regarded as 2-birays in P via the bijection g in Proposition 2.4 and now the diagram in Proposition 2.4 yields the result. □

Remark 3.9. We leave to the reader the statements and the proofs of the corresponding theorems for the connectivity pairs $(\mathscr{E}(P), \mathscr{F}(P))$ and $(\mathscr{F}(P), \mathscr{F}_2(P))$ by using Propositions 1.7 and 1.12, respectively.

Appendix. Some relationships among the various connectivity types

Here we give some relationships among the various connectivity orders already defined for graphs and 2-complexes in Sections 1 and 2, respectively. In this appendix we shall use the identity $\mathrm{Sep}(A, B) = \mathrm{Conn}(A, B)$ provided by Theorems 1.1 and 2.14 without any further comment. We shall start with the results concerning graphs.

Lemma A.1. *For a graph G the following equalities hold:*
(a) $\mathscr{S}(V(G), V(G)) = \mathscr{S}(V(G), \mathscr{F}(G))$.
(b) *If* $|\mathscr{F}(G)| \geqslant 2$ *then* $\mathscr{S}(V(G), \infty) \cup \mathscr{S}(\mathscr{F}(G), \mathscr{F}(G)) = \mathscr{S}(V(G), \mathscr{F}(G))$.

Proof. (a) Let $J \in \mathscr{S}(V(G), V(G))$ be a cut-set for the vertices $v, w \in G$. Let C_v and C_w be the connected components of v, and w in $G - J$. If both C_v and C_w are finite there must exist a third infinite connected component C_∞ since J is finite. Then J separates v and w of any end ε defined by C_∞, and so $J \in \mathscr{S}(V(G), \mathscr{F}(G))$. If C_v (C_w) is infinite we proceed in the same way with $C_w = C_\infty$ ($C_v = C_\infty$, respectively). We have shown $\mathscr{S}(V(G), V(G)) \subseteq \mathscr{S}(V(G), \mathscr{F}(G))$. Conversely, if J separates $v \in V(G)$ of $\varepsilon \in \mathscr{F}(G)$, it is obvious that J separates v of any vertex $w \in V_\varepsilon$.

(b) If $J \in \mathscr{S}(\mathscr{F}(G), \mathscr{F}(G))$ then J separates two ends $\varepsilon, \varepsilon' \in \mathscr{F}(G)$, and therefore it separates ε from any vertex in the connected component $V'_\varepsilon \subseteq G - J$ which defines ε'. Hence, $\mathscr{S}(\mathscr{F}(G), \mathscr{F}(G)) \subseteq \mathscr{S}(V(G), \mathscr{F}(G))$. Moreover, if $L \in \mathscr{S}(V(G), \infty)$ then J leaves some vertex v in a finite connected component $C_v \subseteq G - L$. Therefore, L separates v from the whole set $\mathscr{F}(G)$, and so $L \in \mathscr{S}(V(G), \mathscr{F}(G))$.

Now if $K \in \mathscr{S}(V(G), \mathscr{F}(G)) - \mathscr{S}(\mathscr{F}(G), \mathscr{F}(G))$, K separates an end $\varepsilon \in \mathscr{F}(G)$ of a vertex $v \in V(G)$ but the connected component $C_v \subseteq G - K$ which contains v must be finite. Therefore $K \in \mathscr{S}(V(G), \infty)$. Hence, equality (b) holds. □

Proposition A.2. (a) $\text{Conn}(V(G), V(G)) = \text{Conn}(V(G), \mathscr{F}(G)) \leqslant \text{Conn}(\mathscr{F}(G), \mathscr{F}(G))$.
(b) $\text{Conn}(V(G), \mathscr{F}(G)) \leqslant \text{Conn}(VG), \infty)$.
(c) *In fact, when* $|\mathscr{F}(G)| \geqslant 2$ *we have*

$$\text{Conn}(V(G), \mathscr{F}(G)) = \min\{\text{Conn}(V(G), \infty), \text{Conn}(\mathscr{F}(G), \mathscr{F}(G))\}.$$

Corollary A.3. $\text{Conn}(V(G), V(G)) = \text{Conn}(V(G), \mathscr{F}(G))$ *is the smallest connectivity order of G. Moreover, for a one-ended graph the connectivity order* $\text{Conn}(\mathscr{F}(G), \mathscr{F}(G))$ *is not defined and the other connectivity orders are the same.*

Proof of Proposition A.2. The parts (a) and (b) are direct consequences of Lemma A.1.

(c) Assume $\text{Sep}(V(G), \mathscr{F}(G)) = n < \text{Sep}(\mathscr{F}(G), \mathscr{F}(G))$. Then any $J \in \mathscr{S}(V(G), \mathscr{F}(G))$ with $|J| = n$ does not belong to $\mathscr{S}(\mathscr{F}(G), \mathscr{F}(G))$ and so $J \in \mathscr{S}(V(G), \infty)$ by Lemma A.1(b). Hence $\text{Sep}(V(G), \infty) \leqslant n$, and by (b) we get $\text{Sep}(V(G), \infty) = n$.

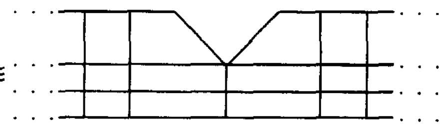
If $\text{Sep}(V(G), \mathscr{F}(G)) = n < \text{Sep}(V(G), \infty)$, clearly $J \notin \text{Sep}(V(G), \infty)$ when $|J| = n$. Therefore $J \in \mathscr{S}(\mathscr{F}(G), \mathscr{F}(G))$ by Lemma A.1(b) and so $\text{Conn}(\mathscr{F}(G), \mathscr{F}(G)) \leqslant n$, and (a) yields $\text{Conn}(\mathscr{F}(G), \mathscr{F}(G)) = n$. □

The *width* of the end ε, $w(\varepsilon)$, is the maximum number of pairwise disjoint rays which define ε. The number $w(\varepsilon)$ is attained [7], and it is also called multiplicity in [12]. The width of $\mathscr{F}(G)$ is the number $w(G) = \min\{w(\varepsilon);\ \varepsilon \in \mathscr{F}(G)\}$. We can add to Proposition A.2 the following proposition whose proof is immediate.

Proposition A.4. *If* $w(G)$ *and* $\text{val}(G)$ *are the width and valence of G, respectively, then* $\text{Conn}(\mathscr{F}(G), \mathscr{F}(G)) \leqslant w(G)$ *and* $\text{Conn}(V(G), \infty) \leqslant \text{val}(G)$.

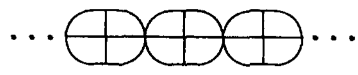
Remarks. A.5. (1) The simple examples below show that the inequalites in Propositions A.2 and A.4 can be strict.

(a)

$G \equiv$ 

$$\mathrm{Conn}(V(G),\mathscr{F}(G))=2 < \mathrm{Conn}(\mathscr{F}(G),\mathscr{F}(G)) = 3 < w(G) = 4.$$

(b)

$G \equiv$ 

$$\mathrm{Conn}(V(G),\mathscr{F}(G))=1 < \mathrm{Conn}(V(G),\infty) = 2 < \mathrm{val}(G) = 3.$$

(2) Notice that for fixed values $\mathrm{val}(G)$ and $|\mathscr{F}(G)|$ it can be found arbitrary large values for $\mathrm{Conn}(\mathscr{F}(G),\mathscr{F}(G))$. We give an example with $\mathrm{val}(G)=2=|\mathscr{F}(G)|$. Let C_n be a cycle with n edges. Then the graph $G=C_n\times\mathbb{Z}\cup\{v\times\mathbb{R};\ v\in V(C_n)\}$ verifies $\mathrm{Conn}(\mathscr{F}(G),\mathscr{F}(G))=n$.

Now we turn our interest to 2-complexes. For them we have the following.

Proposition A.6. *Any infinite admissible 2-complex P with $|\mathscr{F}(P)|\geqslant 2$ verifies*:

(a) $\mathrm{Conn}(\mathscr{E}(P),\mathscr{F}_2(P))=\mathrm{Conn}(\mathscr{E}(P),\mathscr{E}(P))=\min\{\mathrm{Conn}(\mathscr{E}(P),\mathscr{F}(P)),\mathrm{Conn}(\mathscr{F}_2(P),\mathscr{F}_2(P))\}$.

(b) $\max\{\mathrm{Conn}(\mathscr{E}(P),\mathscr{F}(P)),\mathrm{Conn}(\mathscr{F}_2(P),\mathscr{F}_2(P))\}\leqslant\mathrm{Conn}(\mathscr{F}(P),\mathscr{F}_2(P))\leqslant\min\{\mathrm{Conn}(\mathscr{F}(P),\mathscr{F}(P)),w_2(P)\}$.

Here $w_2(P)=\min\{w(\Delta),\Delta\in\mathscr{F}_2(P)\}$ denotes the width of P and $w(\Delta)$ is the width of the 2-end Δ; i.e. the maximal number of independent 2-rays defining the 2-end Δ.

On the other hand, it is clear that $\mathrm{Conn}(\mathscr{E}(P),\infty)\leqslant\mathrm{val}(P)$. Furthermore we can prove

Proposition A.7. *For an infinite admissible 2-complex P with $|\mathscr{F}_2(P)|\geqslant 2$ the following equalities hold*:

(a) $\min\{\mathrm{Conn}(\mathscr{E}(P),\infty),\mathrm{Conn}(\mathscr{F}_2(P),\mathscr{F}_2(P))\}=\mathrm{Conn}(\mathscr{E}(P),\mathscr{F}_2(P))=\mathrm{Conn}(\mathscr{E}(P),\mathscr{E}(P))$.

(b) $\min\{\mathrm{Conn}(\mathscr{E}(P),\infty),\mathrm{Conn}(\mathscr{F}(P),\mathscr{F}_2(P))\}=\mathrm{Conn}(\mathscr{E}(P),\mathscr{F}(P))$.

Corollary A.8. *The number* $\mathrm{Conn}(\mathscr{E}(P),\mathscr{E}(P))=\mathrm{Conn}(\mathscr{E}(P),\mathscr{F}_2(P))$ *is the smallest connectivity order of P. In addition, if P is an infinite admissible 2-complex with only one 2-end then all the connectivity orders defined for P are the same.*

The proof of Propositions A.6 and A.7 need the following lemmas involving the family of cut-sets $\mathscr{S}(A,B)$.

Lemma A.9. *For P as above we have*:
(a) $\mathscr{S}(\mathscr{E}(P),\mathscr{F}(P))\subseteq\mathscr{S}(\mathscr{E}(P),\mathscr{F}_2(P))=\mathscr{S}(\mathscr{E}(P),\mathscr{E}(P))$.
(b) $\mathscr{S}(\mathscr{F}(P),\mathscr{F}(P))\subseteq\mathscr{S}(\mathscr{F}(P),\mathscr{F}_2(P))=\mathscr{S}(\mathscr{F}_2(P),\mathscr{F}_2(P))$.
(c) $\mathscr{S}(\mathscr{E}(P),\mathscr{E}(P))=\mathscr{S}(\mathscr{E}(P),\mathscr{F}(P))\cup\mathscr{S}(\mathscr{F}_2(P),\mathscr{F}_2(P))$.

Proof. (a) Clearly, if J separates the edge e from the end ε then J separates e from any 2-end $\Delta\in h^{-1}(\varepsilon)$ (see Proposition 2.4). In order to show the equality in (a) we just mimic the proof of Lemma A.1 by using the identification $\mathscr{F}_2(P)=\mathscr{F}(G(P))$ in Proposition 2.4.

(b) If J separates ε and ε' in $\mathscr{F}(P)$ then J also separates ε (ε' respectively) of any 2-end $\Delta'\in h^{-1}(\varepsilon')$ ($\Delta\in h^{-1}(\varepsilon)$ respectively).

(c) The inclusion $\mathscr{S}(\mathscr{F}_2(P),\mathscr{F}_2(P))\subseteq\mathscr{S}(\mathscr{E}(P),\mathscr{E}(P))$ follows from Proposition 2.4 and the same proof as in (A.1(a)), and so $\mathscr{S}(\mathscr{F}_2(P),\mathscr{F}_2(P))\cup\mathscr{S}(\mathscr{E}(P)),\mathscr{F}(P))\subseteq\mathscr{S}(\mathscr{E}(P),\mathscr{E}(P))$ by (a).

Assume now $J\in\mathscr{S}(\mathscr{E}(P),\mathscr{E}(P))$. If $J\notin\mathscr{S}(\mathscr{E}(P),\mathscr{F}(P))$ the complement $P-J$ has at least two 2-path connected components and all of them are infinite, otherwise J would be a cut-set in $\mathscr{S}(\mathscr{E}(P)),\mathscr{F}(P))$. Hence $J\in\mathscr{S}(\mathscr{F}_2(P),\mathscr{F}_2(P))$.

If $J\notin\mathscr{S}(\mathscr{F}_2(P),\mathscr{F}_2(P))$ the complement $P-J$ has at least two 2-path components but only one of them can be infinite. Therefore $J\in\mathscr{S}(\mathscr{E}(P),\mathscr{F}(P))$. □

Lemma A.10. *For P as above we have*
(a) $\mathscr{S}(\mathscr{E}(P),\infty)\cup\mathscr{S}(\mathscr{F}_2(P),\mathscr{F}_2(P))=\mathscr{S}(\mathscr{E}(P),\mathscr{F}_2(P))$.
(b) $\mathscr{S}(\mathscr{E}(P),\infty)\cup\mathscr{S}(\mathscr{F}(P),\mathscr{F}_2(P))=\mathscr{S}(\mathscr{E}(P),\mathscr{F}(P))=\mathscr{S}(\mathscr{E}(P),\infty)\cup\mathscr{S}(\mathscr{F}(P),\mathscr{F}(P))$.

Proof. (a) This follows from the identification $\mathscr{F}_2(P)=\mathscr{F}(G(P))$ in Proposition 2.4 and by the same proof as in Lemma A.1(b).

(b) Given a cut-set J for $e\in\mathscr{E}(P)$ and ∞, it is clear that J separates e from any Freudenthal end $\varepsilon\in\mathscr{F}(P)$. Similarly, given a cut-set J for $\varepsilon\in\mathscr{F}(P)$ and $\Delta\in\mathscr{F}_2(P)$ we have that J separates ε from any interior vertex in the connected component $C_\Delta\subseteq P-J$ which defines Δ.

Moreover, if $J\in\mathscr{S}(\mathscr{E}(P),\mathscr{F}(P))-\mathscr{S}(\mathscr{F}(P),\mathscr{F}_2(P))$, J separates an edge $e\in\mathscr{E}(P)$ of a Freudenthal end $\varepsilon\in\mathscr{F}(P)$, and by Lemma 2.1 e must lie in a finite connected component of $P-J$. Therefore $J\in\mathscr{S}(\mathscr{E}(P),\infty)$, and the first equality is proved. The second equality is checked in a similar way. □

Proof of Proposition A.6. (a) By using Lemma A.9(a) and (c) we can easily check

$$\begin{aligned}\mathrm{Conn}(\mathscr{E}(P),\ \mathscr{F}_2(P)) &= \mathrm{Conn}(\mathscr{E}(P),\mathscr{E}(P))\\ &\leqslant \min\{\mathrm{Conn}(\mathscr{E}(P),\mathscr{F}(P)),\mathrm{Conn}(\mathscr{F}_2(P),\mathscr{F}_2(P))\}.\end{aligned}$$

Furthermore, if $\mathrm{Conn}(\mathscr{E}(P),\mathscr{E}(P))=n<\mathrm{Conn}(\mathscr{E}(P),\mathscr{F}(P))$, by Theorem 2.14 there exists a cut-set $J\in\mathscr{S}(\mathscr{E}(P),\mathscr{E}(P))$ with $|J|=n$. In particular, $J\notin\mathscr{S}(\mathscr{E}(P),\mathscr{F}(P))$, and hence $J\in\mathscr{S}(\mathscr{F}_2(P),\mathscr{F}_2(P))$ by Lemma A.9(c). So $\mathrm{Conn}(\mathscr{F}_2(P),\mathscr{F}_2(P))=n$.

In case $n<\mathrm{Conn}(\mathscr{F}_2(P),\mathscr{F}_2(P))$, we can use Lemma A.9(c) again to show $\mathrm{Conn}(\mathscr{E}(P),\ \mathscr{F}(P))=n$.

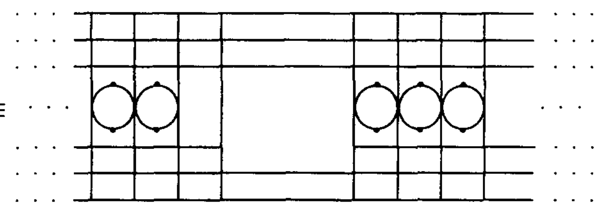
(b) By using Lemmas A.9(b) and A.10(b) one gets $\mathrm{Conn}(\mathscr{F}_2(P),\mathscr{F}_2(P))\leqslant\mathrm{Conn}(\mathscr{F}(P),\mathscr{F}_2(P))\leqslant\mathrm{Conn}(\mathscr{F}(P),\mathscr{F}(P))$ and $\mathrm{Conn}(\mathscr{E}(P),\mathscr{F}(P))\leqslant\mathrm{Conn}(\mathscr{F}(P),\mathscr{F}_2(P))$.

Finally, one easily checks $\mathrm{Conn}(\mathscr{F}(P),\mathscr{F}_2(P))\leqslant w_2(P)$. □

Proof of Proposition A.7. (a) Here we follow the same arguments as in the proof of A.2(c) by using Lemmas A.9(a), (c) and A.10(a).

(b) It follows the same pattern as the proof of A.2(c) by using now A.10(b). □

Example A.11. The following admissible 2-complex P shows that the inequalities in Proposition A.6 can be strict. Let X be as in Example 2.3 and let X' be the symmetric copy of X with respect to the axis OY. We consider the space $Y=[-2,-1]\times[-4,4]\cup X\cup X'\cup[-1,2]\times([2,4]\cup[-4,-3])$. Then P is the admissible 2-complex obtained as a subdivision of the following cellular decomposition of X without new vertices:

$Y\equiv$ 

We have $\mathrm{Conn}(\mathscr{E}(P),\mathscr{F}(P))=\mathrm{Conn}(\mathscr{F}_2(P),\mathscr{F}_2(P))=1$, $\mathrm{Conn}(\mathscr{F}(P),\mathscr{F}_2(P))=2$ and $\mathrm{Conn}(\mathscr{F}(P),\mathscr{F}(P))=w_2(P)=3$.

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