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MOORE SPACES IN PROPER HOMOTOPY

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Abstract. Moore spaces are defined in proper homotopy theory. Some results on the existence and uniqueness of those spaces are proven. An example of two non properly equivalent Moore spaces is given.

Introduction.

The purpose of this paper is to provide the correct statements and details of the results announced in [4]. Namely, we prove the existence of proper Moore spaces of types of type $(\mathcal{S}; n)$ for certain objects $\mathcal{S}$ in the abelian category of towers of groups (tow-$\mathcal{Ab}$, $\mathcal{Ab}$) and $n \geqq 2$ (Theorem 2.9). Nevertheless objects can be of projective dimension 2 in (tow-$\mathcal{Ab}$, $\mathcal{Ab}$), and this fact determines an obstruction to the uniqueness of proper Moore spaces (Theorem 3.2). In fact, an example of two non properly equivalent Moore spaces is given in Appendix A. As a consequence of Theorem 3.2 two sufficient conditions for the uniqueness of proper Moore spaces are stated (Corollary 3.7 and Proposition 3.9).

The existence of Moore spaces in proper homotopy was already announced in [4], but in that paper the obstruction from Theorem 3.2 was not considered and the uniqueness of such spaces was wrongly asserted.

For towers of projective dimension 1, proper Moore spaces behave in a very similar way to ordinary Moore spaces. In particular, for projective dimension 1, proper Moore spaces define proper homotopy groups, and a coefficient exact sequence which generalize the various proper homotopy groups known in the literature and their corresponding Milnor exact sequences (Examples 2.15).

1. Preliminaries and Notations

CATEGORIES OF TOWERS. Given a category C, the category of towers of C, tow-C, is the category of inverse sequences $\mathcal{S}=\{A_1 \leftarrow A_2 \leftarrow \cdots\}$ in C where a (tow-C)-morphism $f: \mathcal{S} \rightarrow \mathcal{S}'$ is represented by a sequence of C-morphisms

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$f_k: A_{n_k} \to A'_k$, $n_1 < n_2 < \cdots$, such that given $r > s$ there exists $j > n_r$, n_s making commutative the diagram

$$\begin{array}{ccc}
A_{n_r} & \xrightarrow{\quad f_r \quad} & A'_r \\
\uparrow & & \Big\downarrow \\
A_j & & \\
\downarrow & & \\
A_{n_s} & \xrightarrow{\quad f_s \quad} & A'_s
\end{array}$$

where the maps without name are bonding maps.

We are interested in the full subcategory of $\mathcal{M}or$(tow-C) whose objects are arrows $f: \mathcal{X} \to A$ where $\mathcal{X}$ is a (tow-C)-object and A is a C-object regarded as a constant tower whose bonding maps are the identity. This category is denoted (tow-C, C). A (tow-C, C)-morphism from $f: \mathcal{X} \to A$ to $g: \mathcal{Y} \to B$ can be regarded as a C-morphism between A and B and a (tow-C)-morphism from $\mathcal{X}$ to $\mathcal{Y}$ such that both morphisms are compatible via the bonding maps.

It is convenient to represent (tow-C, C) as follows. Objects are towers $X = \{X_0 \leftarrow X_1 \leftarrow \cdots\}$; a morphism consists of a map $f: X \to Y$ in tow-C, together with a compatible map $f_0: X_0 \to Y_0$ in C.

We shall specially use the above constructions for $C = \mathcal{T}op$, $\mathcal{G}r$, $\mathcal{A}b$; the categories of topological spaces, groups and abelian groups respectively.

Since tow-$\mathcal{A}b$ and (tow-$\mathcal{A}b$, $\mathcal{A}b$) are abelian categories (see [1]) we can define kernels and images and state exact sequences in a natural way. In particular, we can use projective objects and define the functor Ext. See [10] for details.

PROPER CATEGORY. A *proper map* (*p-map*) is a continuous map $f: X \to Y$ such that $f^{-1}(K)$ is compact for each compact subset $K \subseteq Y$. Proper homotopy (p-homotopy), proper homotopy equivalence, etc$\cdots$ can be defined in the natural way.

We shall deal with the category $\mathcal{P}$ of T_2-locally compact σ-compact spaces and p-maps. One can check that $\mathcal{P}$ is a cofibration category in the sense of H. Baues (see [5] and [3]) whose cofibrations are the p-maps with the Proper Homotopy Extension Property. We call these p-maps *proper cofibrations* (*p-cofibrations*). It can be shown that any p-cofibration is a closed embedding. We denote p-cofibrations by arrows "$\rightarrowtail$".

Given a space X in $\mathcal{P}$, *a system of ∞-neighbourhoods of* X is the object of tow-$\mathcal{T}op$, $\varepsilon(X)=\{U_1 \leftarrow U_2 \leftarrow \cdots\}$ where $\overline{X-U_j}=K_j$ is compact, $K_j \subseteq K_{j+1}$ and $X=\bigcup\{\operatorname{int} K_j\,;\, j\geqq 1\}$.

We recall that a CW-complex X is said to be *strongly locally finite* if X can be covered in a locally finite way by finite subcomplexes. In that case, it is known that X admits a countable locally finite cover by finite subcomplexes and so the ∞-neighbourhoods of X can be chosen to be subcomplexes. Finite dimensional locally finite CW-complexes and locally finite simplicial complexes are strongly locally finite (see [11]).

If $f: X \to Y$ is a p-map and $\{U_j\}$ and $\{W_j\}$ are systems of ∞-neighbourhoods of X and Y respectively, for each j there exists $k(j)$ such that $f(U_{k(j)}) \subset W_j$ and therefore we get a morphism $\varepsilon(f): \varepsilon(X) \to \varepsilon(Y)$ (see [10] for more details).

Given a space X in $\mathcal{P}$ a *Freudenthal end of* X is an element of the set $\mathcal{F}(X)=\varprojlim \pi_0(U_j)$ where $\pi_0(-)$ denotes the set of connected components.

Now let $\mathcal{P}^J$ be the category $\mathcal{P}$ under $J=[0, \infty)$ such that for every $\mathcal{P}^J$-object $J \xrightarrow{i} X$, i is a p-cofibration. The category $\mathcal{P}^J$ is a category of cofibrations where the notion of proper wedge (V_p) is defined in a natural way. The set of p-homotopy classes in $\mathcal{P}^J$ will be denoted by $[-, -]_p^J$.

(1.0.1) The category over J, $\mathcal{P}_J$, is again a cofibration category, and it allows the definition of proper quotients. More explicitly, if $r: X \to J$ is a $\mathcal{P}_J$-object, and $A \rightarrowtail X$ the proper quotient $X/_pA(r)$ is defined by the push-out in $\mathcal{P}$

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, "r"'] & X \arrow[d, dashed, "q"] \\ J \arrow[r, tail, dashed, "\bar{i}"] & X/_pA(r) \end{tikzcd}$$

Notice that $\bar{i}$ is a p-cofibration whose image is $q(A)$. In particular, $q(A)$ is homeomorphic to J. Furthermore, this shows that the proper quotient has the same ordinary homotopy type as the usual topological quotient.

(1.0.2) It can be shown that any space in $\mathcal{P}$ admits an onto p-may $h: X \to J$ ([10; 6.5.3]). Actually, h can be chosen to be a retraction of a given p-cofibration $i: J \subseteq X$. Indeed, since all p-maps on J are p-homotopic one can find a p-homotopy $H: J\times I \to J$ connecting $h|J$ to id_J. Then by using the proper H.E.P. one gets a p-homotopy $H': X\times I \to J$ extending H, and H'_1 is a p-retraction of i.

(1.0.3) The proper homotopy type of $X/_pA(r)$ does not depend on the map

$r: X \to J$. Moreover, given another p-map $r': X \to J$, there exists a p-homotopy equivalence $\xi: X/_pA(r) \to X/_pA(r')$ such that $q \circ \xi = q'$. This is due to the homotopy invariance of push-outs in cofibration categories ([5: II. 1.2b)]), since all proper maps on J are properly homotopic.

When it is clear which map r is involved in the quotient, we shall drop the map r from the notation.

(1.0.4) Finally, let $\mathcal{P}^*$ denote the category under and over J. It can be shown that $\mathcal{P}^*$ is a cofibration category where we can define proper wedges and proper quotients as well as proper cones (c_p), and proper suspensions (Σ_p). For a space $J \rightarrowtail X \to J$ in $\mathcal{P}^*$, these constructions do not depend on the p-retraction r (up to p-homotopy equivalence in $\mathcal{P}^J$). Moreover, the set $[\Sigma_p X, Y]_p^J$ is endowed of a natural group structure for any space Y in $\mathcal{P}^J$. See [3] for details.

A strongly locally finite CW-complex X will be considered as a space in $\mathcal{P}^*$ by choosing a cellular embedding $i: J \subseteq X^1$, and a p-retraction of i. See (1.0.2) above.

If (X, α) is a space in $\mathcal{P}^J$ the (tow-$\mathcal{G}_t$, $\mathcal{G}_t$)-object

$$\Pi_n(X, \alpha) = \{\pi_n(X, *_0) \longleftarrow \pi_n(U_1, *_1) \longleftarrow \pi_n(U_2, *_2) \cdots\} \qquad (n \geqq 1)$$

is called the n-th homotopy tower of the pair (X, α), where $\alpha(t_j) = *_j$ with $\alpha([t_j, \infty)) \subseteq U_j$ and the bonding maps are induced by the inclusions and the base-point change isomorphisms.

(1.0.5) *A space* X *is said to be* ***properly*** *k-connected* if $\mathcal{F}(X) = \{*\}$ and $\Pi_r(X, \alpha) \cong 0$ in (tow-$\mathcal{G}_t$, $\mathcal{G}_t$) $(0 \leqq r \leqq k)$. Similarly for proper pairs (X, A) with $\mathcal{F}(X) = \mathcal{F}(A) = \{*\}$. It is worth pointing out that although α and α' represent the same Freudenthal end, the towers $\Pi_n(X, \alpha)$ and $\Pi_n(X, \alpha')$ need not to be isomorphic (see [23; p. 13]). Nevertheless, if X is properly 1-connected there is no dependence on the ray α. In fact, any p-homotopy $H: \alpha \simeq \alpha'$ induces a pro-isomorphism $H_\#: \Pi_n(X, \alpha') \to \Pi_n(X, \alpha)$. In addition, if $H: f \simeq g$ is a p-homotopy there is a commutative diagram

$$\begin{tikzcd}
\Pi_x(X, \alpha) \arrow[r, "f_*"] \arrow[dr, "g_*"'] & \Pi_n(Y, f \circ \alpha) \arrow[d, "G_*"] \\
& \Pi_n(Y, g \circ \alpha)
\end{tikzcd}$$

where $G = H \circ (\alpha \times id)$.

The towers $\Pi_n(X, A, \alpha)$ $(n \geqq 2)$ are also defined for pairs (X, A) in $\mathcal{P}^J$. In addition, if $A=J$ we have the identification $\Pi_n(X, J; \alpha)=\Pi_n(X, \alpha)$ $(n\geqq 1)$.

We recall that E. Brown in [7] gives a functor $\boldsymbol{P}$: tow-$\mathcal{G}_s\to\mathcal{G}_t$ which carries the tower $\Pi_n(X, \alpha)$ to the Brown-Grossman group $\pi_n^\infty(X, \alpha)$. In a similar way we can define a functor $\boldsymbol{P}_0$: (tow-$\mathcal{G}_t$, $\mathcal{G}_t$)$\to\mathcal{G}_t$ which maps $\Pi_n(X, \alpha)$ to the global Brown-Grossman group $\pi_n(X, \alpha)$ (see [17]).

The n-th homology tower of X can be defined as the tower $(n\geqq 0)$

$$\boldsymbol{H}_n(X)=\{H_n(X)\longleftarrow H_n(U_1)\longleftarrow H_n(U_2)\longleftarrow\cdots\}$$

where the bonding maps are induced by the inclusions.

The chain complex of towers of X, $C_*(X)$ is $\{\partial\colon \boldsymbol{C}_n(X)\to\boldsymbol{C}_{n-1}(X)\}$ where

$$\boldsymbol{C}_n(X)=\{C_n(X)\longleftarrow C_n(U_1)\longleftarrow C_n(U_2)\longleftarrow\cdots\}.$$

Now a proper cohomology theory, $\boldsymbol{H}^n$, with coefficients in a (tow-$\mathcal{Ab}$, $\mathcal{Ab}$)-object $\mathcal{S}$ is defined as the homology of the complex

$$\cdots\longleftarrow C^n(X)\longleftarrow C^{n-1}(X)\longleftarrow\cdots$$

where $C^n(X)=$(tow-$\mathcal{Ab}$, $\mathcal{Ab}$)$(\boldsymbol{C}_n(X), \mathcal{S})$ (see [14] for details).

Also relative versions of these functors for pairs (X, A) in $\mathcal{P}$ are defined.

Notice that the above functors (including Π_n) are well defined up to (tow-$\mathcal{G}_t$, $\mathcal{G}_t$)-isomorphisms.

Fundamental results on homotopy groups like the Blakers-Massey Theorem, the Freudenthal Theorem or the Hurewicz Theorem can be translated to proper homotopy by using the following proposition

1.1. PROPOSITION.—([2; 1.1]) *Let (X, A) be a connected strongly locally finite CW-pair with only one Freudenthal end and assume that (X, A) is properly k-connected. Then there exists a strongly locally finite CW-pair (X', A') such that*

i) *X (respectively A) is a strong deformation p-retract subcomplex of X' (respectively A').*

ii) $(X')^k\subseteqq A'$.

In particular if $A=J$, X has the same homotopy type as a CW-complex X' with $(X')^k=J$.

We recall that for any pair in $\mathcal{P}$ and any proper map $\rho\colon X\to J$, there is a natural homeomorphism $J\cong q(A)$, where $q\colon X\to X/_pA(\rho)$ is the quotient (see (1.0.1)). Therefore, for any ray $\alpha\colon J\to X$ the map q induces morphisms of towers

$$q_* : \Pi_r(X, A, \alpha) \longrightarrow \Pi_r(X/_pA(\rho), q \circ \alpha)$$

1.2. THEOREM.—*Let* (X, A) *be a strongly locally finite CW-pair such that* (X, A) *is properly n-connected and* A *is properly m-connected* $(n, m \geqq 1)$. *Then the morphism* q_* *defined above is an isomorphism if* $2 \leqq r \leqq m+n$, *and an epimorphism if* $r=m+n+1$.

PROOF. Firstly, we shall prove the theorem when $X^n \subseteqq A$ and $A^m = J$.

Let $i : J \to A$ be the inclusion. According to (1.0.2), we can find a proper retraction $r : X \to J$ of i. Let $\{U'_j\}$ be a system of ∞-neighbourhoods of X consisting of subcomplexes. Without loss of generality we can assume that $r(U'_j) \subseteqq [t_j, \infty)$. Let $U_j = U'_j \cup [t_j, \infty)$. It is clear that $\{U_j\}$ is a new system of ∞-neighbourhoods with $U^1_j = [t_j, \infty)$, $(U_j, U_j \cap A)$ is n-connected and $A_j = U_j \cap A$ is m-connected for any $j \geqq 0$. Then, if $X/_pA$ is the proper quotient constructed with the retraction r, it is easily checked that $\{U_j/_pA_j\}$ is a system of ∞-neighbourhoods of $X/_pA$, where $U_j/_pA_j$ is constructed by using the restriction $r|U_j : U_j \to [t_j, \infty)$. Since proper quotients has the same ordinary homotopy type as ordinary quotients, we can levelwise apply the ordinary Blakers-Massey Theorem [24; 6.22] to get isomorphisms

$$q_{j*} : \pi_r(U_j, A_j) \longrightarrow \pi_r(U_j/_pA_j)$$

if $r \leqq n+m$ and epimorphisms if $r=m+n+1$. Now the result follows when $\alpha = i$ and $\rho = r$. Moreover, we can use the naturality of the base ray change isomorphisms and the homotopical invariance of proper quotients (see (1.0.3), and (1.0.5)) to prove the result for (X, A) as above and arbitrary α and ρ.

The general case can be reduced to the previous case by using Theorem 1.1.

Using Theorem 1.2 and a proof similar to the ordinary case ([24; 6.23]) we obtain

1.3. THEOREM.—*Let* X *be a properly n-connected strongly locally finite CW-complex. Then there is a natural suspension* (*tow*-$\mathcal{G}_t$, $\mathcal{G}_t$)-*morphism*

$$\Sigma_* : \Pi_k(X) \longrightarrow \Pi_{k+1}(\Sigma_p X)$$

which is an isomorphism if $k \leqq 2n$ *and an epimorphism if* $k=2n+1$.

1.4. REMARK.—The results stating that Σ and q induce isomorphisms between the corresponding homology towers can be proved in a straightforward way without using Proposition 1.1.

Proposition 1.1 and the above remark give an easy proof of the following theorem.

1.5. THEOREM.—([21; II.4.2.7]) *Let X be a properly n-connected strongly locally finite CW-complex. Then the natural Hurewicz morphism $h: \Pi_k(X) \to \boldsymbol{H}_k(X)$ is an isomorphism if $k=n+1$ and an epimorphism if $k=n+2$.*

Furthermore Theorem 1.2 and Remark 1.4 provide a relative version of Theorem 1.5 for a proper pair (X, A) of strongly locally finite CW-complexes with A properly 1-connected and X properly n-connected.

Finally Theorem 1.5, the Brown-Grossman functor $\boldsymbol{P}_0$ in (1.0.5), and [7; p, 43] lead to

1.6. THEOREM.—*Given a p-map $f: X \to Y$ where X and Y are properly 1-connected finite dimensional, locally finite CW-complexes such that $f_*: \boldsymbol{H}_r(X) \to \boldsymbol{H}_r(Y)$ is an isomorphism for each r, then the map f is a p-homotopy equivalence.*

2. Proper Moore Spaces.

We start with some notions in (tow-$\mathcal{Ab}$, $\mathcal{Ab}$).

2.1. DEFINITION.—A *free tower* in (tow-$\mathcal{Ab}$, $\mathcal{Ab}$) is a tower

$$F(\mathcal{L}) = \{F(L_0) \longleftarrow F(L_1) \longleftarrow \cdots\}$$

where the following four conditions hold; i) $\mathcal{L}$ is a filtration $\mathcal{L} \equiv L_0 \supseteqq L_1 \supseteqq \cdots$ with L_0 a countable set. ii) $\bigcap_{j=1}^{\infty} L_j = \emptyset$. iii) The differences $L_k \backslash L_{k+1}$ are finite. iv) $F(L_i)$ is the free group generated by L_i and the bonding morphism are induced by the inclusions.

Given the towers $F(\mathcal{L})$ and $F(\mathcal{L}')$, it can easily be checked that any bijection $L_0 \cong L_0'$ induces an isomorphism $F(\mathcal{L}) \cong F(\mathcal{L}')$ in (tow-$\mathcal{Ab}$, $\mathcal{Ab}$). So the isomorphism class of $F(\mathcal{L})$ is determined by the cardinality of L_0.

2.2. REMARKS.—a) Free towers are projective objects in (tow-$\mathcal{Ab}$, $\mathcal{Ab}$) (see [14]).

b) Given a strongly locally finite CW-complex X, and a system of ∞-neighbourhoods $\{U_j\}$ consisting of subcomplexes, the tower of cellular n-chains of X,

$$\boldsymbol{C}_n(X) = \{C_n(X) \longleftarrow C_n(U_1) \longleftarrow C_n(U_2) \longleftarrow \cdots\}$$

is obviously a free tower. Also the tower of cellular n-cycles

$$\boldsymbol{Z}_n(X)=\{Z_n(X)\longleftarrow Z_n(U_1)\longleftarrow Z_n(U_2)\longleftarrow\cdots\}$$

is a free tower. More generally, the kernel of any morphism between two free towers is always a free tower (see [14; 5.1]).

2.3. DEFINITION.—A tower $\mathcal{S}$ is said to be *geometrically admissible* if there exists an exact sequence in (tow-$\mathcal{Ab}$, $\mathcal{Ab}$)

$$0\longrightarrow F(\mathcal{L}_3)\xrightarrow{j_3}F(\mathcal{L}_2)\xrightarrow{j_2}F(\mathcal{L}_1)\xrightarrow{j_1}\mathcal{S}\longrightarrow 0$$

When $F(\mathcal{L}_3)$ is trivial we say that $\mathcal{S}$ has *geometrical projective dimension* (g.p.d.) 1. Otherwise, we write g. p. d. $\mathcal{S}=2$.

2.4. REMARK.—For any strongly locally finite CW-complex X the n-th homology tower of X is geometrically admissible since the exact sequence of towers

$$0\longrightarrow \boldsymbol{Z}_{n+1}(X)\longrightarrow \boldsymbol{C}_{n+1}(X)\longrightarrow \boldsymbol{Z}_n(X)\longrightarrow \boldsymbol{H}_n(X)\longrightarrow 0$$

is a free resolution. However, the short exact sequence

$$0\longrightarrow \operatorname{Im}\partial_{n+1}\longrightarrow \boldsymbol{Z}_n(X)\longrightarrow \boldsymbol{H}_n(X)\longrightarrow 0$$

is not always a free resolution. Indeed, let X be the CW-complex obtained from the cylinder $S^n\times[0,\infty)$ by attaching an $(n+1)$-cell at $S^n\times\{j\}$ by a map $f\colon S^n\to S^n$ of degree 2^j $(j\geqq 1)$. Then one can check that $\operatorname{Im}\partial_{n+1}$ is not free since ∂_{n+1} has no right inverse.

2.5. REMARK.—In [9] Dymov introduced the notion of copresentation of a tower $\mathcal{S}$. More explicitly, following [9] we say that $\mathcal{S}=\{G_0\leftarrow G_1\leftarrow\cdots\}$ admits a copresentation if there exist a levelwise epimorphism $\phi\colon F(\mathcal{L})\to\mathcal{S}$ with $F(\mathcal{L})$ a free tower and subsets $R_j\subseteqq F(L_j)$ with R_j-R_{j+1} finite and $\bigcap_{j=1}^{\infty}R_j=\emptyset$, such that $\operatorname{Ker}\phi_j$ is the subgroup $\langle R_j\rangle$ generated by R_j. In general, the tower $\langle\mathcal{R}\rangle\equiv\langle R_0\rangle\leftarrow\langle R_1\rangle\leftarrow\cdots$ needs not to be projective.

It is easy to check that a tower $\mathcal{S}$ is geometrically admissible if and only if admits a copresentation (up to isomorphism). In fact, if $\langle\mathcal{R}\rangle\xrightarrow{f}F(\mathcal{L})\to\mathcal{S}\to 0$ is a Dymov copresentation, and $F(\mathcal{R})$ denotes the free tower consisting of the free groups $F(R_i)$, there is a levelwise epimorphism $F(\mathcal{R})\xrightarrow{k}\langle\mathcal{R}\rangle$ and an exact sequence $F(\mathcal{R})\xrightarrow{fk}\langle\mathcal{R}\rangle\to F(\mathcal{L})$. By Remark 2.2(b), $\operatorname{Ker}(f\circ k)$ is a free tower, and so $\mathcal{S}$ is geometrically admissible. Conversely, for any free resolution

$$0\longrightarrow F(\mathcal{L}_3)\xrightarrow{j_3}F(\mathcal{L}_2)\xrightarrow{j_2}F(\mathcal{L}_1)\xrightarrow{j_1}\mathcal{S}\longrightarrow 0$$

let $\{\varphi_i : F(L^2_{n(i)}) \to F(L^1_i)\}$ be a levelwise representative of j_2. Then $\mathcal{S}' = \{\text{Coker}\,\varphi_i\}$ is isomorphic to $\mathcal{S}$ by exactness and admits the following copresentation. Let $\phi: F(\mathcal{L}_1) \to \mathcal{S}'$ be the natural levelwise quotient morphism. We take $R_i = \varphi_i(L^2_{n(i)}) \subseteq F(L^1_i)$. Now, it is clear that $R_i - R_{i+1} \subseteq \varphi_i(L^2_{n(i)} - L^2_{n(i+1)})$ is finite.

If $\tilde{S}^n$ ($\tilde{B}^n$ respectively), is the space obtained by attaching finitely many copies (possibly no copy) of S^n (B^n respectively) ($n \geqq 2$) at each $m \in \boldsymbol{N} \subseteq [0, \infty)$, one checks that $\Pi_n(\tilde{S}^n)$ ($\Pi_n(\tilde{B}^n, \tilde{S}^{n-1})$ respectively) can be identified in a natural way with some $F(\mathcal{L})$, with $L_0 \subseteq \boldsymbol{N}$. Moreover the following result holds:

2.6. Lemma.—*Given a space X in $\mathcal{P}^J$, there is a natural bijection $\rho: [\tilde{S}^n, X]^J_p \cong (\text{tow-}\mathcal{A}b, \mathcal{A}b)(F(\mathcal{L}) \to \Pi_n(X))$ $(n \geqq 2)$, given by $\rho([f]) = f_*$. Moreover, if Z is the mapping cone $X \cup_f \tilde{B}^{n+1}$ in the cofibration category $\mathcal{P}^J$, then f_* can be also regarded as the boundary operator $d_{n+1}: \Pi_{n+1}(Z, X) \to \Pi_n(X)$.*

Proof.—Take $\varphi: F(\mathcal{L}) \to \Pi_n(X)$. After identifying $\Pi_n(\tilde{S}^n)$ with $F(\mathcal{L})$, let

$$\begin{tikzcd}
\pi_n(\tilde{S}^n) \arrow[d] & \pi_n(\tilde{S}^n_{k(1)}) \arrow[l] \arrow[d] & \pi_n(\tilde{S}^n_{k(2)}) \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\pi_n(X) & \pi_n(U_1) \arrow[l] & \pi_n(U_2) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

be a levelwise representative of φ, where $\tilde{S}^n_{k(t)}$ is obtained by deleting from $\tilde{S}^n$ the copies, S^n_j, of S^n placed at the points $1 \leqq j < k(t)$, $k(1) < k(2) < \cdots$.

We define $f \mid S^n_j$ as a representative of $\varphi_{k(j)}[l_j]$ where $l_j: S^n_j \subseteq \tilde{S}^n_{k(t)}$, $k(t) \leqq j < k(t+1)$. One easily checks that $f_* = \varphi$. This proves that ρ is onto. The injectivity follows in a similar way.

Finally, we have the diagram

$$\begin{tikzcd}
\Pi_{n+1}(Z, X) \arrow[d] & \Pi_{n+1}(\tilde{B}^{n+1}, \tilde{S}^n) \arrow[l, "p_*"', "\cong"] \arrow[d, "\cong"] \\
\Pi_n(X) \arrow[r] & \Pi_n(\tilde{S}^n) = F(\mathcal{L})
\end{tikzcd}$$

where $p: \tilde{B}^{n+1} \to Z$ is the canonical p-map and p_* is an isomorphism by Theorem 1.2 since $\tilde{B}^{n+1}/_p\tilde{S}^n \cong \tilde{S}^{n+1} \cong Z/_pX$.

For the sake of simplicity, we shall use the single notation $\tilde{S}^n$ ($\tilde{B}^n$) for all the "strings" of spheres (balls) described above. The particular objects $\tilde{S}^n$ ($\tilde{B}^n$) we are using in the future will be clear from the context. Similarly for $F(\mathcal{L})$.

2.7. REMARK.—Let X be a strongly locally finite one-ended CW-complex. Given a cellular embedding $J \subseteqq X^1$, the $(n+1)$-skeleton X^{n+1} turns out to be properly equivalent to the mapping cone $X^n \cup_f \tilde{B}^{n+1}$ in $\mathcal{P}^J$ of a p-map $f: \tilde{S}^n \to X^n$. Indeed, up to a p-homotopy equivalence (see [18; 6.7] or [5; II. 1.2]), we can assume that the attaching map $f_\alpha: S^n_\alpha \to X^n$ of the $(n+1)$-cell e^{n+1}_α verifies $f_\alpha(*) = m_\alpha \in \boldsymbol{N} \subseteqq J$, where $* \in S^n$ is the base point.

Then we define $\tilde{S}^n$ by attaching at $k \in \boldsymbol{N}$ all the spheres S^n_α with $m_\alpha = k$. In this way we can gather together all the attaching maps f_α to get a proper map $f: \tilde{S}^n \to X^n$ extending the embedding $J \subseteqq X^n$. It is clear that $X^{n+1} \cong_p X^n \cup_f \tilde{B}^{n+1}$ in $\mathcal{P}^J$.

2.8. DEFINITION.—Given a tower $\mathcal{S}$ and $n \geqq 2$, a proper Moore space $R(\mathcal{S}, n)$ is a properly 1-connected finite dimensional, locally finite CW-complex such that its q-th (reduced) homology tower is isomorphic to $\mathcal{S}$ when $q=n$ and trivial otherwise.

In $R(\mathcal{S}, n)$ is a proper Moore space, the tower $\mathcal{S}$ is geometrically admissible by Remark 2.4. Conversely, we have

2.9. THEOREM.—*Given a tower* $\mathcal{S}$ *with free resolution*

$$0 \longrightarrow F(\mathcal{L}_3) \xrightarrow{j_3} F(\mathcal{L}_2) \xrightarrow{j_3} F(\mathcal{L}_1) \xrightarrow{j_1} \mathcal{S} \longrightarrow 0$$

there exists a proper Moore space $R(\mathcal{S}, n)$, *for any* $n \geqq 2$.

PROOF. By Lemma 2.6 we may regard j_2 as the induced morphism $f_{2*}: \Pi_n(\tilde{S}^n) \to \Pi_n(\tilde{S}^n)$ of a p-map $f_2: \tilde{S}^n \to \tilde{S}^n$. Let $X^n = \tilde{S}^n$ and $X^{n+1} = C_{f_2}$. Now, by Lemma 2.6 we may identify $\boldsymbol{C}_{n+1}(C_{f_2}) = \Pi_{n+1}(X^{n+1}, X^n)$ with $F(\mathcal{L}_2)$, and $\partial: \boldsymbol{C}_{n+1}(C_{f_2}) \to \boldsymbol{C}_n(C_{f_2}) = \Pi_n(X^n)$ with the boundary operator.

Since Ker j_2=Im j_3, from the diagram of unbroken arrows

$$\begin{array}{ccc}
\Pi_{n+1}(X^{n+1}) & \xleftarrow{\ \varphi_3\ } & F(\mathcal{L}_3) \\
\big\downarrow i_* & & \big\downarrow j_3 \\
\Pi_{n+1}(X^{n+1}, X^n) & = & F(\mathcal{L}_2) \\
\big\downarrow d_{n+1} & & \big\downarrow j_2 \\
\Pi_n(X^n) & = & F(\mathcal{L}_1)
\end{array}$$

we get Im $i_* \subseteqq$ Im j_3, and the projectiveness of $F(\mathcal{L}_3)$ yields a (tow-$\mathcal{Ab}$, $\mathcal{Ab}$)-morphism $\varphi_3: F(\mathcal{L}_3) \to \Pi_{n+1}(X^{n+1})$. Again by Lemma 2.6, $F(\mathcal{L}_3)$ may be regarded as $\Pi_{n+1}(\tilde{S}^{n+1})$, and we pick up a representative $f_3: \tilde{S}^{n+1} \to X^{n+1}$ of $\varphi_3([\mathrm{id}])$. If we define $X^{n+2}=C_{f_3}$, the boundary morphism

$$\Pi_{n+2}(X^{n+2}, X^{n+1}) \xrightarrow{d_{n+2}} \Pi_{n+1}(X^{n+1}, X^n)$$

is now identified with j_3. So $\boldsymbol{H}_m(X^{n+2})$ is $\mathcal{S}$ if $m=n$ and trivial otherwise.

Applying Theorem 1.6 we get the following results as corollaries of Theorem 2.9.

2.10. COROLLARY.—*Given two towers $\mathcal{S}$ and $\mathcal{S}'$ as in Theorem 2.9 and proper Moore spaces $R(\mathcal{S}, n)$ and $R(\mathcal{S}', n)$, the wedge $R(\mathcal{S}, n)\vee R(\mathcal{S}', n)$ is a proper Moore space of type $(\mathcal{S}\oplus\mathcal{S}', n)$, where $\mathcal{S}\oplus\mathcal{S}'$ represents the coproduct of $\mathcal{S}$ and $\mathcal{S}'$ in $(tow\text{-}\mathcal{A}b, \mathcal{A}b)$.*

2.11. COROLLARY.—*Given a tower $\mathcal{S}$ as in Theorem 2.9, if K is a proper Moore space of type $(\mathcal{S}, n)$ then $\Sigma_p K$ is a proper Moore space of type $(\mathcal{S}, n+1)$.*

For $n\geqq 3$, Corollary 2.11 has the following converse

2.12. PROPOSITION.—*Any proper Moore space of type $(\mathcal{S}, n)$ with $n\geqq 3$ is the proper suspension of a a proper Moore space of type $(\mathcal{S}, n-1)$. If g.p.d. $\mathcal{S}=1$ then the result also holds for $n\geqq 2$.*

PROOF.—Let X be of type $(\mathcal{S}, n)$ as in the proof of Theorem 2.9. Then X^{n+1} is the proper mapping cone of some $f_2 : \tilde{S}^n \to \tilde{S}^n = X^n$, and there exists g such that $X^{n+1} \cong_p \Sigma_p Y^n$, with $Y=C_g$, and $\Sigma_p g \cong_p f_2$. Since $n\geqq 3$, we can apply Theorem 1.3 and Lemma 2.6 to

$$\Sigma_* : \Pi_n(Y^n) \longrightarrow \Pi_{n+1}(\Sigma_p Y^n)$$

and we can find a proper map $h : \tilde{S}^{n+1} \to X^{n+1} \cong_p \Sigma_p Y^n$, with $\Sigma_p h$ properly homotopic to the attaching map f_3 of the $(n+2)$-cells of X. Therefore $X=C_{f_3} \cong_p \Sigma_p C_h$, and obviously C_h is a proper Moore space of type $(\mathcal{S}, n-1)$.

2.13. DEFINITION.—Given a geometrically admissible tower $\mathcal{S}$ and $n\geqq 3$ ($n\geqq 2$ if g. p. d. $\mathcal{S}=1$), we know from Proposition 2.12 and (1.0.4) that for any proper Moore space $R(\mathcal{S}, n)$, the set $[X(\mathcal{S}, n), X]_p^J$ can be endowed of a group structure for any X in $\mathcal{P}^J$. When the space $R(\mathcal{S}, n)$ is unique (up to p-homotopy), the above group is called the n-th proper homotopy group of X with coefficients in $\mathcal{S}$ and it will be denoted by $\pi_n(X; \mathcal{S})$.

When g. p. d. $\mathcal{S}=1$ we get the uniqueness of proper Moore spaces of type $(\mathcal{S}, n)$ $(n\geqq 2)$ just by imitating the proof in ordinary homotopy and using Theorem 1.6. See also Corollary 3.5 below. Moreover, as in ordinary homotopy one gets

2.14. PROPOSITION.—*Given a space X in $\mathcal{P}^J$ and a tower $\mathcal{S}$ with g.p.d. $\mathcal{S}=1$, there exists an exact sequence*

$$0 \to \mathrm{Ext}(\mathcal{S}, \Pi_{n+1}(X, A)) \to \pi_n(X, A; \mathcal{S}) \to (\text{tow-}\mathcal{A}b, \mathcal{A}b)(\mathcal{S}, \Pi_n(X, A)) \to 0$$

We omit the proof which can be done by mimicing the ordinary case in [18; Ch. 5]. However we shall give the following applications of this result

2.15. EXAMPLES.—Throughout these statements X is a $\mathcal{P}^J$-space with only one Freudenthal end.

a) If $\mathcal{S}\equiv\{0\leftarrow \boldsymbol{Z} \xleftarrow{\text{id}} \boldsymbol{Z} \xleftarrow{\text{id}} \cdots\}$ then $R(\mathcal{S}, n)=\boldsymbol{R}^{n+1}$ and $\pi_n(X; \mathcal{S})$ agrees with the n-th proper homotopy group defined in [16] and [6]. Using basic properties of the functor Ext, Proposition 2.14 yields the diagram of exact sequences

$$\begin{array}{l}
\qquad\quad 0 \\
\qquad\quad \downarrow \\
\mathrm{Coker}\,[\varprojlim\{\pi_{n+1}(U_j)\} \longrightarrow \pi_{n+1}(X)] \\
\qquad\quad \downarrow \\
0 \longrightarrow \mathrm{Ext}(\mathcal{S}; \Pi_{n+1}(X)) \longrightarrow \pi_n(X; \mathcal{S}) \longrightarrow \mathrm{Ker}\,[\varprojlim\{\pi_n(U_j)\} \longrightarrow \pi_n(X)] \longrightarrow 0 \\
\qquad\quad \downarrow \\
\varprojlim{}^1\{\pi_{n+1}(U_j)\} \\
\qquad\quad \downarrow \\
\qquad\quad 0
\end{array}$$

The exact row appears in [3].

b) If $\mathcal{S}\equiv\{\boldsymbol{Z} \xleftarrow{\text{id}} \boldsymbol{Z} \xleftarrow{\text{id}} \boldsymbol{Z} \leftarrow \cdots\}$ then $R(\mathcal{S}, n)=S^n\times J$ and $\Pi_n(X; \mathcal{S})$ is the n-th group defined in [8]. Proposition 2.14 yields the known sequence (see [22])

$$0 \longrightarrow \varprojlim{}^1\{\pi_{n+1}(U_j)\} \longrightarrow \pi_n(X; \mathcal{S}) \longrightarrow \varprojlim\{\pi^u(U_j)\} \longrightarrow 0$$

c) Finally, if $\mathcal{S}\equiv\{\boldsymbol{Q} \xleftarrow{b_1} \boldsymbol{Q} \xleftarrow{b_2} \boldsymbol{Q} \xleftarrow{b_3} \cdots\}$ with $b_k(1)=1/k+1$ one can readily check that g.p.d. $\mathcal{S}=1$, and $R(\mathcal{S}, n)$ is the rational sphere described in [13; p. 21]. That is, the infinite telescope constructed from the diagram $\{S^n \xleftarrow{f_1} S^n \xleftarrow{f_2} S^n \xleftarrow{f_3} \cdots\}$ where f_k is a map of degree $k+1$. Moreover, Proposition 2.14 yields an exact sequence

$$0 \longrightarrow \varprojlim{}^1\{\pi_{n+1}(U_j)\}^{\boldsymbol{Q}} \longrightarrow \pi_n(X; \mathcal{S}) \longrightarrow \varprojlim\{\pi_n(U_j)\}^{\boldsymbol{Q}} \longrightarrow 0$$

Here $\mathcal{A}^Q$ denotes the tower constructed from $\mathcal{A}$ by replacing the k-th bonding morphism γ_k of $\mathcal{A}$ by $(k+1)\gamma_k$ $(k\geqq 1)$.

Notice that $\mathcal{S}$ is isomorphic to the constant tower $\boldsymbol{Q}$ in tow-$\mathcal{Ab}$.

3. Uniqueness of proper Moore spaces.

The uniqueness of Moore spaces in ordinary homotopy is a consequence of the fact that the category $\mathcal{Ab}$ has projective dimension 1. As was pointed out in § 1, (tow-$\mathcal{Ab}$, $\mathcal{Ab}$) has projective dimension 2. That is, the second derived functor Ext^2 does not always vanish. In this section we shall define an obstruction in certain Ext^2-term to the uniqueness of proper Moore spaces.

In order to define that obstruction, we shall use a natural action of (tow-$\mathcal{Ab}$, $\mathcal{Ab}$)$(C_{n+1}(X)\,;\, \Pi_{n+1}(Y^{n+1}))$ on $[X^{n+1}, Y^{n+1}]_p^J$ for two strongly locally finite properly $(n-1)$-connected CW-complexes X and Y $(n\geqq 2)$. Here $\boldsymbol{C}_{n+1}(X)=\Pi_{n+1}(X^{n+1}, X^n)$. The action is described as follows.

Let $D^n\subseteqq B^n$ be a copy of B^n such that $D^n\cap S^{n-1}=\partial D^n\cap S^{n-1}=\{*\}$. When we shrink ∂D^n to $\{*\}$ we get the wedge $B^n\vee S^n$.

According to Remark 2.7, we can assume $X^{n+1}=X^n\cup_f B^{n+1}$ for certain proper map $f:\tilde{S}^n\to X^n$. The quotient space obtained from X^{n+1} by identifying a copy of ∂D^n to point inside each $(n+1)$-cell of X^{n+1} is clearly homeomorphic to $X^{n+1}\vee_p\tilde{S}^{n+1}$, where $\tilde{S}^{n+1}$ is now the string of spheres $X^{n+1}/_pX^n$. Let $\mu: X^{n+1}\to X^{n+1}\vee_p\tilde{S}^{n+1}$ denote the quotient map.

Given $f: X^{n+1}\to Y^{n+1}$ and $\alpha: \boldsymbol{C}_{n+1}(X^{n+1})\to\Pi_{n+1}(Y^{n+1})$, the operation is defined just as in ordinary homotopy. That is, the action of α on $[f]$, $[f]+\alpha$, is represented by the composition

$$X^{n+1}\xrightarrow{\ \mu\ } X^{n+1}\vee\tilde{S}^{n+1}\xrightarrow{\ f\vee g\ } Y^{n+1}$$

where g is a representative of α by the identification (tow-$\mathcal{Ab}$, $\mathcal{Ab}$)$(\boldsymbol{C}_{n+1}(X^{n+1}),$ $\Pi_{n+1}(Y^{n+1}))\overset{(1)}{\cong}[\tilde{S}^{n+1}, Y^{n+1}]_p^J$ provided by Lemma 2.6.

Let $\boldsymbol{\Gamma}_{n+1}(Y)$ be the Whitehead tower

$$\mathrm{Im}\,[i_*: \Pi_{n+1}(Y^n)\longrightarrow \Pi_{n+1}(Y^{n+1})]\,.$$

Since $\boldsymbol{\Gamma}_{n+1}(Y)$ is a subtower of $\Pi_{n+1}(Y^{n+1})$ we may consider the restriction of the above action to (tow-$\mathcal{Ab}$, $\mathcal{Ab}$)$(\boldsymbol{C}_{n+1}(X), \boldsymbol{\Gamma}_{n+1}(Y))$.

3.1. Lemma—a) *$(f+\alpha)_*: \Pi_{n+1}(X^{n+1})\to\Pi_{n+1}(Y^{n+1})$ is $f_*+\alpha\circ j_*$, where $j_*: \Pi_{n+1}(X^{n+1})\to\Pi_{n+1}(X^{n+1}, X^n)=\boldsymbol{C}_{n+1}(X)$.*

b) $(f+\alpha)_*: \boldsymbol{C}_{n+1}(X)\to\boldsymbol{C}_{n+1}(Y)$ *is* $f_*+k_*\alpha$, *where* $k_*: \boldsymbol{\Pi}_{n+1}(Y^{n+1})\to\boldsymbol{C}_{n+1}(Y)$. *In particular,* $f_*=(f+\alpha)_*$ *if* $\alpha: \boldsymbol{C}_{n+1}(X)\to\boldsymbol{\Gamma}_{n+1}(Y)$.

PROOF. a) We have the commutative diagram

$$\begin{tikzcd}
\boldsymbol{\Pi}_{n=1}(X^{n+1})\oplus\boldsymbol{\Pi}_{n+1}(\tilde{S}^{n+1}) \arrow[r, "i_{1*}+i_{2*}", "\cong(1)"'] \arrow[dr, "f_*\oplus g_*"'] & \boldsymbol{\Pi}_{n+1}(X^{n+1}\vee\tilde{S}^{n+1}) \arrow[d, "(f\vee g)_*"] \\
& \boldsymbol{\Pi}_{n+1}(Y^{n+1})
\end{tikzcd}$$

where "$\oplus$" denotes the coproduct or sum in the abelian category (tow-$\mathcal{A}b$, $\mathcal{A}b$) and (1) is a levelwise isomorphism with inverse the morphism (p_{1*}, p_{2*}) induced by the proper projections

$$p_1: X^{n+1}\vee\tilde{S}^{n+1} \longrightarrow X^{n+1}\vee J=X^{n+1}$$

$$p_2: X^{n+1}\vee\tilde{S}^{n+1} \longrightarrow J\vee\tilde{S}^{n+1}=\tilde{S}^{n+1}$$

(we recall that X^{n+1} is properly 1-connected). Thus

$$(f+\alpha)_*=f_*\circ p_{1*}\circ\mu_*+g_*\circ p_{2*}\circ\mu_*$$

And one can check that $p_1\circ\mu=q_1\cong_p \mathrm{id}|X^{n+1}$, where q_1 shrinks the balls D^n in the $(n+1)$-cells of X^{n+1} to point, and $p_2\circ\mu\cong_p q_2$ where $q_2: X^{n+1}\xrightarrow{\pi} X^{n+1}/_pX^n\overset{k}{\cong}_p\tilde{S}^{n+1}$ with π the canonical projection and k the natural proper homotopy equivalence which carries the complement of each D^n to point.

Finally, the identification (1) given in the above definition of $f+\alpha$ is induced by q'_* in the following commutative diagram

$$\begin{tikzcd}
\boldsymbol{\Pi}_{n+1}(X^{n+1}) \arrow[r, "q_{2*}"] \arrow[rr, bend left, "j_*"] & \boldsymbol{\Pi}_{n+1}(\tilde{S}^{n+1}) \arrow[r, "q'_{2*}"] \arrow[d, "g_*"] & \boldsymbol{\Pi}_{n+1}(X^{n+1}, X^n) \arrow[dl, "\alpha"] \\
& \boldsymbol{\Pi}_{n+1}(Y^{n+1}) &
\end{tikzcd}$$

Then, it follows that $\alpha\circ j_*=g_*\circ q_{2*}$.

Part b) is essentially proven in the same way.

Now we can prove

3.2. THEOREM.—*Let Y be a properly $(n-1)$-connected, strongly locally finite*

CW-complex (we may assume that $Y^{n-1}=J$, *by Proposition* 1.1). *Given a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & F(\mathcal{L}_3) \arrow[r, "d_{n+2}"] \arrow[d, "\varphi_{n+2}"] & F(\mathcal{L}_2) \arrow[r, "d_{n+1}"] \arrow[d, "\varphi_{n+1}"] & F(\mathcal{L}_1) \arrow[r] \arrow[d, "\varphi_n"] & \mathcal{S} \arrow[r] \arrow[d, "\varphi"] & 0 & (*) \\
\arrow[r] & \boldsymbol{C}_{n+2}(Y) \arrow[r, "d'_{n+2}"] & \boldsymbol{C}_{n+1}(Y) \arrow[r, "d'_{n+1}"] & \boldsymbol{C}_n(Y) \arrow[r] & \boldsymbol{H}_n(Y) \arrow[r] & 0
\end{tikzcd}$$

where the upper row is a free resolution of $\mathcal{S}$, *if* X *is a proper Moore space of type* $(\mathcal{S}\,;n)$ *there is a well defined obstruction* $c(\varphi)\in \boldsymbol{H}^{n+2}(X\,;\boldsymbol{\Gamma}_{n+1}(Y))$ *such that* $c(\varphi)=0$ *if and only if there is a p-map* $f:X\to Y$ *realizing the diagram* (*).

REMARK.—As $F(\mathcal{L}_i)$ is projective, when $\boldsymbol{H}_{n+1}(Y)=0$ the morphism φ always yields morphisms φ_i $(i=n,\ n+1,\ n+2)$ such that the diagram (*) commutes.

PROOF. From Lemma 2.6 we may realize φ_n by a p-map $f_n: X^n=\tilde{S}^n\to Y^n=\tilde{S}^n$. We also have the commutative diagram of unbroken arrows

$$\begin{tikzcd}
\boldsymbol{\Pi}_n(\tilde{S}^n) \arrow[rrr, bend left, "h_*"] \arrow[ddd, dashed, "\zeta_*"] & \boldsymbol{\Pi}_{n+1}(\tilde{B}^{n+1}, \tilde{S}^n) \arrow[l, "\cong"'] \arrow[r, "\cong"] & \boldsymbol{\Pi}_{n+1}(X^{n+1}, X^n) \arrow[r] \arrow[d, equal] & \boldsymbol{\Pi}_n(X^n) \arrow[d, equal] \arrow[ddd, bend left] \\
 & & F(\mathcal{L}_2) \arrow[r] \arrow[d, "\varphi_{n+1}"'] & F(\mathcal{L}_1) \arrow[d, "\varphi_n=f_{n*}"] \\
 & (1) & \boldsymbol{C}_{n+1}(Y) \arrow[r] \arrow[d, equal] & \boldsymbol{C}_n(Y) \arrow[d, equal] \\
\boldsymbol{\Pi}_n(\tilde{S}^n) \arrow[rrr, bend right, "g_*"'] & \boldsymbol{\Pi}_{n+1}(\tilde{B}^{n+1}, \tilde{S}^n) \arrow[l, "\cong"'] \arrow[r, "\cong"] & \boldsymbol{\Pi}_{n+1}(Y^{n+1}, Y^n) \arrow[r] & \boldsymbol{\Pi}_n(Y^n)
\end{tikzcd}$$

where we assume that $Y^{n+1}=\tilde{B}^{n+1}\cup_g Y^n$ and $X^{n+1}=\tilde{B}^{n+1}\cup X^n$ by Remark 2.7. Then by Lemma 2.6 we may find a p-map $\zeta:\tilde{S}^n\to\tilde{S}^n$ making (1) commutative. Therefore $f_n\circ h\cong_p g\circ\zeta$ and by using the Proper Homotopy Extension Property we may find a p-map $\gamma:(\tilde{B}^{n+1},\tilde{S}^n)\to(Y^{n+1},Y^n)$ extending $f_n\circ h$ and p-homotopic to $\tilde{g}\circ\zeta'$, where ζ' is the cone extension of ζ and $\tilde{g}:\tilde{B}^{n+1}\to Y^{n+1}$ is the natural p-map defined by the characteristic maps of the $(n+1)$-cells of Y.

We define $f_{n+1}: X^{n+1}\to Y^{n+1}$ by $f_{n+1}|X^n=f_n$ and $f_{n+1}(\tilde{h}(x))=\gamma(x)$ where $\tilde{h}$

is the p-map defined by the characteristic maps of the $(n+1)$-cells of X. We easily check that $f_{n+1*}=\varphi_{n+1}$.

When one tries to go further an obstruction appears as follows. In the diagram

$$\begin{array}{ccccc} \Pi_{n+2}(X^{n+2}, X^{n+1}) & \xrightarrow{\partial_{n+2}} & \Pi_{n+1}(X^{n+1}) & \longrightarrow & \Pi_{n+1}(X^{n+1}, X^{n}) \\ \varphi_{n+2}\downarrow & (1) & \downarrow f_{n+1*} & & \downarrow f_{n+1*}=\varphi_{n+1} \\ \Pi_{n+2}(Y^{n+2}, Y^{n+1}) & \xrightarrow{\partial'_{n+2}} & \Pi_{n+1}(Y^{n+1}) & \longrightarrow & \Pi_{n+1}(Y^{n+1}, Y^{n}) \end{array}$$

the square (1) needs not be commutative. One defines an element in (tow-$\mathcal{A}b$, $\mathcal{A}b$) $(F(\mathcal{L}_3); \Pi_{n+1}(Y^{n+1}))$ by the difference

$$\beta(f_{n+1})=f_{n+1*}\partial_{n+2}-\partial'_{n+2}\varphi_{n+2}.$$

Since the other square is commutative we have, by definition of Ker j_* in (tow-$\mathcal{A}b$, $\mathcal{A}b$) that β can be regarded as a morphism $\beta(f_{n+1}): F(\mathcal{L}_3)\to\operatorname{Ker} j_*=\boldsymbol{\Gamma}_{n+1}Y$. Obviously β is a cocycle and defines a class $c(\varphi)\in\boldsymbol{H}^{n+2}(X, \boldsymbol{\Gamma}_{n+1}Y)$. The next lemma shows that $c(\varphi)$ is a well defined obstruction.

3.3. LEMMA.—1) *$c(\varphi)$ does not depend on the morphisms φ_i $(i=n, n+1, n+2)$.*
2) *$c(\varphi)$ is an obstruction to realizing φ.*

PROOF. 1) Let $\{\varphi'_i\}$ be another morphism such that the diagram (*) commutes and let $f'_{n+1}: X^{n+1}\to Y^{n+1}$ be a p-map realizing φ'_{n+1}. It is a well-known fact from Homological Algebra in abelian categories that $\{\varphi_i\}$ and $\{\varphi'_i\}$ are homotopic chain morphisms. Thus, there exist morphisms $\{\alpha_i: \boldsymbol{C}_i(X)\to\boldsymbol{C}_{i+1}(Y)\}$ $(i=n, n+1, n+2)$ such that the following equalities hold

a) $\varphi'_n-\varphi_n=d'_{n+1}\circ\alpha_n$; b) $\varphi'_{n+1}-\varphi_{n+1}=\alpha_n\circ d_{n+1}+d'_{n+2}\circ\alpha_{n+1}$; and

c) $\varphi'_{n+2}-\varphi_{n+2}=d'_{n+3}\circ\alpha_{n+2}+\alpha_{n+1}\circ d_{n+2}$.

By c) and the definition of $\beta(f_{n+1})$ we have

(I) $\beta(f'_{n+1})-\beta(f_{n+1})=(f'_{n+1*}-f_{n+1*}-\partial'_{n+2}\circ\alpha_{n+1}\circ j_*)\circ\partial_{n+2}$

where $j_*: \Pi_{n+1}(Y^{n+1})\to\boldsymbol{C}_{n+1}(Y)$. Now, a) provides a p-homotopy $H: X^n\times I\to Y^{n+1}$ between f_n and $f'_n=f'_{n+1}|_{X^n}$. As in ordinary homotopy theory, H yields a "difference" morphism

$$\Delta=d(f'_{n+1}, H, f_{n+1}): \boldsymbol{C}_{n+1}(X)\longrightarrow\Pi_{n+1}(Y^{n+1})$$

(see [15]). Moreover, H can be chosen in such a way that $j_*\circ\Delta=f'_{n+1*}-f_{n+1*}-\alpha_n\circ d_{n+1}$. Take $\beta=\Delta-\partial'_{n+2}\circ\alpha_{n+1}$. By b), $j_*(\beta)=0$. And the projectiveness of $\boldsymbol{C}_{n+1}(X)$ allows us to regard β as an element in (tow-$\mathcal{A}b$, $\mathcal{A}b$)$(\boldsymbol{C}_{n+1}(X); \boldsymbol{\Gamma}_{n+1}(Y))$ since Ker $j_*=\boldsymbol{\Gamma}_{n+1}(Y)$.

Finally, one can readily check from the definitions that $f'_{n+1} \simeq_p f_{n+1} + (\beta + \partial'_{n+2} \circ \alpha_{n+1})$ and then, by Lemma 3.1 the right side in the equality (I) is $\beta \circ j_* \circ \partial_{n+2} = \beta \circ d_{n+2} = \delta\beta$. This proves $[\beta(f_{n+1})] = [\beta(f'_{n+1})] \in \boldsymbol{H}^{n+2}(X\,;\boldsymbol{\Gamma}_{n+1}(Y))$.

2) If $c(\varphi)=0$, let $w \in (\text{tow-}\mathcal{Ab}, \mathcal{Ab})(\boldsymbol{C}_{n+1}(X), \boldsymbol{\Gamma}_{n+1}(Y))$ be such that $\beta(f_{n+1}) = \delta w = w \circ d_{n+2} = w \circ j_* \circ \partial_{n+2}$. Take $\bar{f}_{n+1} = f_{n+1} + w$. By Lemma 3.1, $\beta(\bar{f}_{n+1}) = (f_{n+1*} - w \circ j_*)\partial_{n+2} - \partial'_{n+2} \circ \varphi_{n+2} = 0$, and $\bar{f}_{n+1}$ extends to a p-map $f_{n+2} : X \to Y$ with $f_{n+2*} = \varphi : \boldsymbol{H}_n(X) \to \boldsymbol{H}_n(Y)$.

3.4. REMARKS.—a) The obstruction $c(\varphi)$ was already considered by J. H. C. Whitehead in ordinary homotopy (see [25; § 6]) and it can be defined within the general setting of cofibration categories (see [5; VII. 1. 13]).

b) For any $n \geqq 3$ there are two non properly equivalent Moore spaces of type $(\mathcal{S}\,; n)$. The examples are given in Appendix A.

As an immediate consequence of Theorem 3.2 we have

3.5. COROLLARY.—*If* $\mathcal{S}$ *is a tower with g.p.d.* $\mathcal{S}=1$ *then there exists (up to p-homotopy) a unique proper Moore space of type* $(\mathcal{S}, n)$ $(n \geqq 2)$.

More generally, we can state

3.6. COROLLARY.—*If* $\mathcal{S}$ *is a geometrically admissible tower with* $\mathrm{Ext}^2(\mathcal{S}\,; \boldsymbol{\Gamma}_n(\mathcal{S}))=0$, *then there exists a unique proper Moore space of type* $(\mathcal{S}, n)$ $(n \geqq 2)$.

Here $\boldsymbol{\Gamma}_n(\mathcal{S})$ denotes the tower obtained from $\mathcal{S}$ by applying levelwise the algebraic Whitehead Γ_n-functor (see [26; Ch. II] or [5; IX. 4]). It is known that $\Gamma_n = - \otimes \boldsymbol{Z}_2$ when $n \geqq 3$. So, (3.6) yields

3.7. COROLLARY.—*If* $\mathcal{S}$ *is a geometrically admissible tower with* $\mathrm{Ext}^2(\mathcal{S}\,; \mathcal{S} \otimes \boldsymbol{Z}_2)=0$ *then there exists a unique proper Moore space of type* $(\mathcal{S}, n)$ *for all* $n \geqq 3$.

PROOF OF (3.6). Let Y be a proper Moore space of type (S, n) constructed as in the proof of Theorem 2.9. Let $Y \supseteqq U_1 \supseteqq \cdots \supseteqq U_n \cdots$ be a system of ∞-neighbourhoods such that each U_j is a subcomplex. Moreover, each U_j is $(n-1)$-connected by construction. Thus by [19; VIII. 2. 4] for $n \geqq 3$ and [26; III. 14] for $n=2$ we have $\Gamma_{n+1} U_j \cong \Gamma_n(H_n(U_j))$ and therefore $\boldsymbol{\Gamma}_{n+1} Y \cong \boldsymbol{\Gamma}_n(\mathcal{S})$.

If X is another proper Moore space of type $(\mathcal{S}\,; n)$ we can realize $\mathrm{id} : \mathcal{S} \to \mathcal{S}$ by a p-map $f : X \to Y$ by Theorem 3.2 since $\boldsymbol{H}^{n+2}(X\,; \boldsymbol{\Gamma}_{n+1} Y) = \mathrm{Ext}^2(\mathcal{S}\,; \boldsymbol{\Gamma}_{n+1} Y) = 0$. By Theorem 1.6 f is actually a p-homotopy equivalence.

3.8. REMARK.—The tower $\mathcal{S}=\{\boldsymbol{Z}_2 \xleftarrow{p_1} \boldsymbol{Z}_4 \xleftarrow{p_2} \boldsymbol{Z}_8 \leftarrow \cdots\}$ where $p_i(1)=1$, has geometrical projective dimension 2 since $\mathcal{S}$ is the n-th homology tower of the CW-complex given in Remark 2.4. Nevertheless $\boldsymbol{\Gamma}_n\mathcal{S}$ is the constant tower $\boldsymbol{Z}_2$ when $n \geqq 3$, and $\boldsymbol{\Gamma}_2\mathcal{S}$ is isomorphic to $\mathcal{S}$ since $\Gamma_2\boldsymbol{Z}_{2n}=\boldsymbol{Z}_{4n}$ according to [26; II. (B)]. Then, one can check as in Appendix A that $\mathrm{Ext}^2(\mathcal{S}\,;\boldsymbol{\Gamma}_n\mathcal{S})=0$. So, there is a unique proper Moore space of type $(\mathcal{S}\,;n)$ $(n \geqq 3)$ by Corollary 3.7. The same result holds for $n=2$.

Another sufficient algebraic condition on $\mathcal{S}$ for the uniqueness of proper Moore spaces of type $(\mathcal{S}\,;n)$ $n \geqq 3)$ is the following.

3.9. PROPOSITION.—*Let $\mathcal{S}$ be a geometrically admissible tower such that* $\mathrm{Tor}^1(\mathcal{S}\,;\boldsymbol{Z}_2)=0$. *Then there is a unique Moore space of type $(\mathcal{S}\,;n)$ $(n \geqq 3)$, unique up to p-homotopy.*

Before starting the proof of Proposition 3.9 we shall fix notation and prove a lemma whose proof is similar to the proof of [14; Lemma 2].

Let (tow-$\boldsymbol{Z}_2$, $\boldsymbol{Z}_2$) denote the abelian category defined in the same way as (tow-$\mathcal{Ab}$, $\mathcal{Ab}$) by using $\boldsymbol{Z}_2$-vector spaces instead of abelian groups. Given a free tower $F(\mathcal{L})$, let $\boldsymbol{Z}_2(L_i)$ denote $F(L_i)\otimes\boldsymbol{Z}_2$. Then

3.10. LEMMA.—*Any tower $\{V_0 \leftarrow V_1 \leftarrow V_2 \cdots\}$ with $V_i \subseteqq \boldsymbol{Z}_2(L_i)$ and with bonding morphisms the corresponding restrictions is projective in (tow-$\boldsymbol{Z}_2$, $\boldsymbol{Z}_2$).*

PROOF. We may find a basis T_i of V_i/V_{i+1} such that any element of T_i is represented by a linear combination of elements in $L_i - L_{i+1}$. Let B_i be the union $\cup\{T_j\,;\,j \geqq \mathrm{I}\}$. It is easy to check that B_i is a basis for V_i. Since $B_0 \supseteqq B_1 \cdots$ and $\cap B_i=\emptyset$ it is straightforwardly shown that $\{V_0 \leftarrow V_1 \leftarrow V_2 \cdots\}$ is projective.

PROOF OF PROPOSITION 3.9. Let

$$0 \longrightarrow C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_2} C_0 \xrightarrow{d_0} \mathcal{S} \longrightarrow 0$$

be a free resolution of $\mathcal{S}$. Since $\mathcal{S}\otimes\boldsymbol{Z}_2$ is a tower of groups of order 2 we have an isomorphism

$$(\text{tow-}\mathcal{Ab},\ \mathcal{Ab})(C_i\,;\mathcal{S}\otimes\boldsymbol{Z}_2)\cong(\text{tow-}\boldsymbol{Z}_2,\ \boldsymbol{Z}_2)(C_i\otimes\boldsymbol{Z}_2\,;\mathcal{S}\otimes\boldsymbol{Z}_2)$$

Therefore,

$$\mathrm{Ext}^2(\mathcal{S}\,;\mathcal{S}\otimes\boldsymbol{Z}_2)\cong\mathrm{Ext}^1(\mathrm{Im}\ d_1\,;\mathcal{S}\otimes\boldsymbol{Z}_2)\cong\mathrm{Coker}\ ((d_2\otimes 1)^*)$$

where "*" stands for the dual morphism.

Now from the exact sequence

$$0 \longrightarrow C_2 \longrightarrow C_1 \longrightarrow \mathrm{Im}\, d_1 \longrightarrow 0$$

we get the exact sequence

$$0 \longrightarrow C_2 \otimes \boldsymbol{Z}_2 \xrightarrow{d_2 \otimes 1} C_1 \otimes \boldsymbol{Z}_2 \xrightarrow{d_1 \otimes 1} \mathrm{Im}\, \mathrm{d}_1 \otimes \boldsymbol{Z}_2 \longrightarrow 0$$

since the tower $Ker(d_2 \otimes 1)$ is isomorphic to the tower $\{\mathrm{Tor}(\mathrm{Im}\, d_1^i\,;\, \boldsymbol{Z}_2)\}$ which is trivial because each component $\mathrm{Im}\,[d_1^i\colon C_1^{k(i)} \to C_0^i]$ of the tower $\mathrm{Im}\, d_1$ is a free abelian group. Thus,

$$\mathrm{Coker}\,((d_2 \otimes 1)^*) \cong \mathrm{Ext}^1_{\boldsymbol{Z}_2}(\mathrm{Im}\, d_1 \otimes \boldsymbol{Z}_2\,;\, \mathcal{S} \otimes \boldsymbol{Z}_2)$$

where the right side is the Ext^1 functor in the category (tow-$\boldsymbol{Z}_2$, $\boldsymbol{Z}_2$).

On the other hand we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_2 \otimes \boldsymbol{Z}_2 & \xrightarrow{d_2 \otimes 1} & C_1 \otimes \boldsymbol{Z}_2 & \xrightarrow{d_1 \otimes 1} & \mathrm{Im}\, d_1 \otimes \boldsymbol{Z}_2 & \longrightarrow & 0 \\ & & \| & & \| & & & & \\ 0 & \longrightarrow & C_2 \otimes \boldsymbol{Z}_2 & \xrightarrow{d_2 \otimes 1} & C_1 \otimes \boldsymbol{Z}_2 & \xrightarrow{d_1 \otimes 1} & \mathrm{Im}\,(d_1 \otimes 1) & \longrightarrow & 0 \end{array}$$

where the upper row is exact as it was proven above. And the lower row is also exact since $\mathrm{Tor}^1(\mathcal{S}\,;\, \boldsymbol{Z}_2) = \mathrm{Ker}\,(d_1 \otimes 1)/\mathrm{Im}\,(d_2 \otimes 1) = 0$ by hypothesis. Thus $\mathrm{Im}\, d_1 \otimes \boldsymbol{Z}_2 \cong \mathrm{Im}\,(d_1 \otimes 1)$ and hence $\mathrm{Ext}^1_{\boldsymbol{Z}_2}(\mathrm{Im}\, d_1 \otimes \boldsymbol{Z}_2\,;\, \mathcal{S} \otimes \boldsymbol{Z}_2) \cong \mathrm{Ext}^1_{\boldsymbol{Z}_2}(\mathrm{Im}(d_1 \otimes 1)\,;\, \mathcal{S} \otimes \boldsymbol{Z}_2)$. Now the former term vanishes because $\mathrm{Im}\,(d_1 \otimes 1)$ is projective by Lemma 3.10. This yields $\mathrm{Ext}^2(\mathcal{S}\,;\, \mathcal{S} \otimes \boldsymbol{Z}_2) = 0$ and the uniqueness follows from Corollary 3.7.

Final Remark. The category of trees of abelian groups (see [12]) seems to be the right algebraic framework for a generalization of the results of this paper to spaces with many Freudenthal ends.

Appendix A.

Two non properly equivalent proper More spaces of type $(\mathcal{S}\,;\, n)$, $n \geqq 3$.

Let $\mathcal{S}$ be the tower

$$\left\{ \bigoplus_1^{\infty} \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \xrightarrow{k_1 \oplus 1} \bigoplus_2^{\infty} \boldsymbol{Z}_2 \boldsymbol{Z}_2 \xrightarrow{k_2 \oplus 1} \cdots \right\}$$

in (tow-$\mathcal{A}b$, $\mathcal{A}b$), with k_j standing for the natural inclusion morphism. A free resolution for $\mathcal{S}$ is

$$0 \longrightarrow F(\mathcal{L}_3) \xrightarrow{\partial_2} F(\mathcal{L}_2) \xrightarrow{\partial_2} F(\mathcal{L}_1) \longrightarrow \mathcal{S} \longrightarrow 0$$

where $\mathcal{L}_3$ is $L_1^3 \supseteq L_2^3 \supseteq \cdots$ with $L_j^3 = \{\alpha_i ; i \geqq j\}$. $\mathcal{L}_2$ is $L_1^2 \supseteq L_2^2 \supseteq \cdots$ with $L_j^2 = \{\rho_i, \mu_i, \sigma_i ; i \geqq j\}$ and $\mathcal{L}_1$ is $L_1^1 \supseteq L_2^1 \supseteq \cdots$ with $L_j^1 = \{\varepsilon_i, \gamma_i ; i \geqq j\}$. And the morphisms are given by $\partial_2(\alpha_i) = \mu_{i+1} - 2\sigma_i - \mu_i$; $\partial_1(\rho_i) = 2\varepsilon_i$, $\theta_1(\mu_i) = 2\gamma_i$ and $\partial_1(\sigma_i) = \gamma_{i+1} - \gamma_i$.

Clearly, $\mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2$, where $\mathcal{S}_1$ is the tower

$$\left\{ \bigoplus_1^\infty \boldsymbol{Z}_2 \xrightarrow{k_1} \bigoplus_2^\infty \boldsymbol{Z}_2 \xrightarrow{k_2} \cdots \right\}$$

and $\mathcal{S}_2$ is the constant tower $\{\boldsymbol{Z}_2 = \boldsymbol{Z}_2 = \cdots\}$.

A.1. LEMMA.—$\mathrm{Ext}^2(\mathcal{S} ; \mathcal{S}) \neq 0$, $\mathrm{Ext}^2(\mathcal{S}_1 ; \mathcal{S}_1) = \mathrm{Ext}^2(\mathcal{S}_2 ; \mathcal{S}_2) = 0$.

PROOF. By the naturality of "Ext" and "$\oplus$" we have $\mathrm{Ext}^2(\mathcal{S} ; \mathcal{S}) = \oplus \{\mathrm{Ext}^2(\mathcal{S}_i ; \mathcal{S}_j) ; i, j \leqq 2\}$.

On the other hand, it is easy to check that g. p. d. $\mathcal{S}_1 = 1$, and so $\mathrm{Ext}^2(\mathcal{S}_1 ; \mathcal{S}_j) = 0$. With the above notations, $\mathcal{S}_2$ admits the free resolution

$$0 \longrightarrow F(\{\alpha_i\}) \xrightarrow{\partial_2} F(\{\mu_i, \sigma_i\}) \xrightarrow{\partial_1} F(\{\gamma_i\}) \longrightarrow \mathcal{S}_2 \longrightarrow 0$$

and by the standard Hom-Ext exact sequence we get

$$\mathrm{Ext}^2(\mathcal{S}_2 ; \mathcal{S}_2) = \mathrm{Ext}^1(\mathrm{Im}\, \partial_1 ; \mathcal{S}_2) = 0 .$$

Indeed, for any $\varphi \in (\text{tow-}\mathcal{A}b, \mathcal{A}b)(F(\{\alpha_i\}), \mathcal{S}_2)$ we may define $\bar{\varphi} \in (\text{tow-}\mathcal{A}b, \mathcal{A}b)(F(\{\mu_i, \sigma_i\}, \mathcal{S})$ by $\bar{\varphi}(\sigma_i) = 0$, $\bar{\varphi}(\mu_1) = 0$ and $\bar{\varphi}(\mu_j) = \sum \{\varphi(\alpha_i) ; i \leqq j-1\}$. Then $\bar{\varphi} \circ \partial_2 = \varphi$.

Finally $\mathrm{Ext}^2(\mathcal{S}_2, \mathcal{S}_1) = \mathrm{Ext}^1(\mathrm{Im}\, \partial_1 ; \mathcal{S}_1) \neq 0$ since $\xi : F(\{\alpha_i\}) \to \mathcal{S}_1$ given by $\xi(\alpha_i) = \varepsilon_i \otimes 1$ defines a non-trivial element. Otherwise, $\xi = \tau \circ \partial_2$ for some $\varepsilon : F(\{\mu_i, \sigma_i\}) \to \mathcal{S}_1$ and τ yields the equalities $\varepsilon_i \otimes 1 = \tau(\mu_{i+1}) - \tau(\mu_i)$ $(i \geqq 1)$. As τ is a pro-morphism one can inductively prove that $\tau(\mu_i) \in \bigoplus_{k \geq 1} \boldsymbol{Z}_2$ and the sequence $\{\varepsilon_i \otimes 1\}$ would represent the trivial element in $\lim^1 \mathcal{S}_1$ and it is a well-known fact that it does not.

Lemma A.1 and Corollary 3.7 yield that $R(\mathcal{S}_1 ; n)$ and $R(\mathcal{S}_2 ; n)$ are uniquely determined up to p-homotopy $(n \geqq 3)$. Actually these types are represented by $\widetilde{W}$ and $W \times [0, \infty)$, where $\widetilde{W}$ is obtained by attaching one copy of W at each natural coordinate of $[0, \infty)$ and $W = S^n \cup_2 e^{n+1}$ is the n-sphere with an $(n+1)$-cell attached by a map of degree 2. Thus, $X = R(\mathcal{S}_1 ; n) \vee_p R(\mathcal{S}_2 ; n)$ is a representative of $R(\mathcal{S} ; n)$ by Corollary 2.10.

A.2. LEMMA. *The natural map* $[X ; X]_p^{\mathcal{J}} \to (\text{tow-}\mathcal{A}b, \mathcal{A}b)(\mathcal{S} ; \mathcal{S})$ *is onto.*

PROOF. By Proposition 2.14 $[R(\mathcal{S}_1\,;n)\,;X]_p^J \to [\text{tow-}\mathcal{A}b, \mathcal{A}b)(\mathcal{S}_1\,;\mathcal{S})$ is onto. On the other hand, $(\text{tow-}\mathcal{A}b, \mathcal{A}b)(\mathcal{S}_2\,;\mathcal{S}_1)=\varprojlim \mathcal{S}_1=0$ and $[R(\mathcal{S}_2\,;n)\,;X]_p^J \twoheadrightarrow$

$$[R(\mathcal{S}_2\,;n)\,;R(\mathcal{S}_2\,;n)]_p^J \cong [W\,;W]^J \twoheadrightarrow \mathcal{A}b(\boldsymbol{Z}_2\,;\boldsymbol{Z}_2)\cong(\text{tow-}\mathcal{A}b, \mathcal{A}b)(\mathcal{S}_2\,;\mathcal{S}_2).$$

where the first bijection is given by the Edwards-Hastings embedding Theorem ([10; 6.27]). Finally the natural bijection $[X\,;X]_p^J \cong [R(\mathcal{S}_1\,;n)\,;X]_p^J \times [R(\mathcal{S}_2\,;n)\,;X]_p^J$ completes the proof.

Now we choose a non-trivial element $\alpha \in H^{n+2}(X\,;\boldsymbol{\Gamma}_{n+1}X)\cong \text{Ext}^2(\mathcal{S}\,;\mathcal{S})\neq 0$. By Remark 2.7 we may assume $X=\tilde{B}^{n+2}\cup_{h_0} X^{n+1}$. Now, the commutative diagram

$$\begin{array}{ccc}
\boldsymbol{C}_{n+2}(X)=\Pi_{n+2}(X, X^{n+1}) & \xrightarrow[h^*]{\cong} & \Pi_{n+2}(\tilde{B}^{n+1}, \tilde{S}^{n+1}) \\
\downarrow d_{n+2} & & \cong \downarrow d_{n+2} \\
\Pi_{n+1}(X^{n+1}) & \xrightarrow{h_{0*}} & \Pi_{n+1}(\tilde{S}^{n+1})
\end{array}$$

allows us to identify the boundary operator d_{n+2} with the morphism h_{0*} (h is the characteristic map $h: \tilde{B}^{n+2}\to X$).

The isomorphism $d_{n+2}\circ h_*^{-1}$ also gives the identification

(I) $(\text{tow-}\mathcal{A}b, \mathcal{A}b)(\boldsymbol{C}_{n+2}(X)\,;\Pi_{n+1}(X^{n+1}))$

$$\cong(\text{tow-}\mathcal{A}b, \mathcal{A}b)(\Pi_{n+1}(\tilde{S}^{n+1})\,;\Pi_{n+1}(X^{n+1}))\cong[\tilde{S}^{n+1}\,;X^{n+1}]_p^J$$

where the second isomorphism is given by Lemma 2.6. Thus, if $\alpha=[a]$, a can be regarded as a p-map $g: \tilde{S}^{n+1}\to X^n \subseteq X^{n+1}$. Let $\bar{h}_0$ be a representative of $[h_0]+[g]\in[\tilde{S}^{n+1}\,;X]_p^J$ and let Y be the proper cone of $\bar{h}_0$. Since $\operatorname{Im} g\subseteq X^n$, the complexes towers of cellular chains of X and Y are the same. But

A.3. LEMMA.—*The obstruction* $c(\text{id})\in \boldsymbol{H}^{n+2}(X\,;\boldsymbol{\Gamma}_{n+1}Y)$ *given in Theorem* 3.2 *for* $\text{id}: \boldsymbol{H}_n(X)=\mathcal{S}\to\mathcal{S}=\boldsymbol{H}_n(Y)$ *is non-trivial.*

PROOF. Since $c(\text{id})$ does not depend on the morphisms $\varphi_i: \boldsymbol{C}_i(X)\to \boldsymbol{C}_i(Y)$ inducing $\text{id}: \mathcal{S}\to\mathcal{S}$ (see Lemma 3.3(1)), one can consider $\varphi_i=\text{id}$ for each $i=n$, $n+1$, $n+2$. So, $c(\text{id})$ is represented by $\beta(\text{id})=d_{n+2}-d'_{n+2}$, where d_{n+2} is given in the above diagram for X. Similarly d'_{n+2} for Y. Bearing in mind the identification (I) $\beta(\text{id})$ is regarded as $h_*-h_*-g_*=-g_*$. Then $\beta(\text{id})$ is actually $-a$ and $c(\text{id})=-\alpha\neq 0$.

Finally we get,

A.4. PROPOSITION.—*X and Y are not p-homotopically equivalent.*

PROOF. If $h: X \to Y$ is a p-homotopy equivalence, let h' be a p-homotopic inverse of h. The morphism $h'_*: \boldsymbol{H}_n(Y)=\mathcal{S} \to \boldsymbol{H}_n(X)=\mathcal{S}$ can be realized by a h-map $f: X \to X$ according to Lemma A.2. Then $h \circ f: X \to Y$ is a p-map with $(h \circ f)_* = \mathrm{id}: \boldsymbol{H}_n(X)=\mathcal{S} \to \mathcal{S}=\boldsymbol{H}_n(Y)$, and this cannot happen by Lemma A.3.

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