A Sufficient Degree Condition for a Graph to Contain All Trees of Size k

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Abstract The Erdős–Sós conjecture says that a graph G on n vertices and number of edges $e(G) > n(k-1)/2$ contains all trees of size k. In this paper we prove a sufficient condition for a graph to contain every tree of size k formulated in terms of the minimum edge degree $\xi(G)$ of a graph G defined as $\xi(G) = \min\{d(u) + d(v) - 2 : uv \in E(G)\}$. More precisely, we show that a connected graph G with maximum degree $\Delta(G) \geq k$ and minimum edge degree $\xi(G) \geq 2k-4$ contains every tree of k edges if $d_G(x) + d_G(y) \geq 2k-4$ for all pairs x, y of nonadjacent neighbors of a vertex u of $d_G(u) \geq k$.

Keywords Erdős–Sós conjecture

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1 Introduction

Erdős and Gallai [1] proved that if the size of graph G on n vertices is at least $e(G) > n(k-1)/2$, then G contains a path of size k. This fact leads Erdős and Sós to formulate the following conjecture.

Conjecture 1 (Erdős–Sós [2]) *If G is a graph on n vertices and the number of edges of G is $e(G) > n(k-1)/2$, then G contains all trees of size k.*

The Erdős–Sós conjecture is clearly true for stars of size k, because if $e(G) > n(k-1)/2$, then some vertex in G must have degree at least k. Furthermore, the conjecture is true for the trees collected in the following theorem.

Theorem A *Let G be a graph of order n and size $e(G) > n(k-1)/2$, where $k \geq 1$ is an integer. Then G contains the following trees of size k:*

(i) *Paths* [1].

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(ii) *Comets* (*trees obtained from a star and a path by identifying one leaf of the star with one leaf of the path*) [3].

(iii) *Caterpillars. It is mentioned in* [4] *that this fact was proved by Perles in* 1990.

(iv) *Trees with a vertex adjacent to t leaves, where $t \geq (k-1)/2$* [5].

(v) *Trees with a vertex adjacent to t leaves, where $t \geq (k-3)/2$* [6].

(vi) *Spiders* (*trees with a unique vertex of degree greater than two*) *with three legs or spiders with legs of length at most three* [7].

(vii) *Trees with diameter four* [8].

(viii) *Trees with size $k \leq 7$* [5, 6].

The items (i)–(iv) were mentioned by Woźniak in [3]. Moreover, the Erdős–Sós conjecture has been proved for both certain values of k and certain graphs gathered in the theorem below.

Theorem B *Let G be a graph of order n and size $e(G) > n(k-1)/2$, where $k \geq 1$ is an integer. Then G contains all trees of size k if one of the following assertions holds*:

(i) $k = n-1$ [9, 10]; $k = n-2$ [11]; $k = n-3$ [3]; $k = n-4$ [12].

(ii) *The girth of G is at least* 5 [13].

(iii) *The graph G does not contain the cycle C_4* [14].

(iv) *The graphs whose complements contain no C_4* [15].

(v) *The graph G does not contain $K_{2,\lfloor k/18 \rfloor}$ as a subgraph* [16, 17].

(vi) *The girth of the complement graph of G is at least* 5 [18].

Other recent variations on the Erdős–Sós conjecture can be seen in [19].

1.1 Our Result

It is also known that if the minimum degree of G is $\delta(G) \geq k$, then G contains all trees of size k (see Chartrand and Lesniak [20, Theorem 3.8, p. 60]). Clearly if the requirement on the number of edges of Erdős–Sós conjecture is true, then the maximum degree is $\Delta(G) \geq k$. In this note we state that if the minimum degree of a graph G is $\delta(G) \geq k-1$ and the maximum degree is $\Delta(G) \geq k$, then G contains all trees of size k.

This is derived as a direct consequence of the following stronger result, formulated in terms of the *minimum edge degree* $\xi(G)$ of a graph G, defined as $\xi(G) = \min\{d(u)+d(v)-2 : uv \in E(G)\}$.

Theorem 1.1 *Let $k \geq 4$ be an integer and let G be a connected graph with maximum degree $\Delta(G) \geq k$ and minimum edge degree $\xi(G) \geq 2k-4$. Then G contains every tree of k edges if $d_G(x) + d_G(y) \geq 2k-4$ for all pair x, y of nonadjacent neighbors of a vertex u of $d_G(u) \geq k$.*

As a consequence of Theorem 1.1 the following result is now apparent.

Corollary 1.2 *Any graph G with a connected component of minimum degree $\delta(G) \geq k-1$ and maximum degree $\Delta(G) \geq k$ contains every tree of size k.*

Observe that any pair of adjacent vertices x, y of $K_{k,k-2}$ satisfies $d_G(x)+d_G(y) \geq k+k-2 = 2k-2$ and any pair of vertices of the same partite set of cardinality k are neighbors of a vertex of degree k and this pair satisfies $d_G(x) + d_G(y) \geq 2k-4$. Hence the conditions of Theorem 1.1

hold and so the following result holds.

Corollary 1.3 *Let $k \geq 4$ be an integer. Any graph G containing a complete bipartite $K_{k,k-2}$ as a subgraph contains every tree of size k.*

Theorem 1.1 is in a certain sense stronger than that established by Erdős–Sós conjecture as shown by Corollary 1.3. Indeed, $K_{k,k-2}$ is a graph having fewer edges than required by Erdős–Sós hypothesis, but applying Theorem 1.1 we obtain that any graph G containing a complete bipartite $K_{k,k-2}$ as a subgraph contains every tree of size k.

In the following section we provide the proof of Theorem 1.1.

2 Proof

In the proof of Theorem 1.1 we follow the terminology introduced in [13, 14] which we recall next. A vertex of degree 1 is a *leaf*, and a *penultimate vertex* in a tree is a leaf in the subtree of T obtained by deleting all leaves of T. In other words, all the neighbors except one of a penultimate vertex are leaves.

Proof of Theorem 1.1 Note that a star of 4 vertices satisfies the conditions of the theorem for $k = 3$, but obviously this graph does not contain a path of length 3. This is the reason for assuming $k \geq 4$. The proof is by induction on k. First let us see that the result is true for $k = 4$. Notice that G contains a star of size 4, since $\Delta(G) \geq k = 4$; say $s \in V(G)$ adjacent to $s_1, s_2, s_3, s_4 \in V(G)$. If the induced subgraph $G[s_1, s_2, s_3, s_4]$ of G has at least two edges, clearly G contains all trees of size 4. Then we may assume that $G[s_1, s_2, s_3, s_4]$ has at most one edge. Since all pairs x, y of nonadjacent neighbors of a vertex of degree at least 4 satisfy $d_G(x) + d_G(y) \geq 4$, it follows that G contains a spider with at least two legs of length 2. This implies that G contains every tree of size $k = 4$. Therefore, from now on suppose that $k \geq 5$ and the result is true for all integers less than k. Let T be a tree with k edges and G a graph satisfying the hypothesis of the theorem.

Let us choose a penultimate vertex u of T adjacent (in T) to r leaves with r as small as possible. Denote by u^- the unique non-leaf vertex of T such that $u^-u \in E(T)$. Let T' be the tree obtained from T by removing the r leaf-neighbors of vertex u, and T'' the tree obtained from T' by removing vertex u. Thus, by induction, both trees T' and T'' are subgraphs of G. Observe also that $|V(T'')| = k - r$. We denote $U = N_G(u^-) \setminus V(T'')$, clearly $u \in U$. Suppose that there exists some vertex $u' \in U$ having at least r neighbors in G outside of T''. Then by replacing in T' vertex u with this vertex u' we would obtain a tree subgraph of G isomorphic to T and the result is true. Therefore every vertex $u' \in U$ has at most $r - 1$ neighbors in G outside of T''. It follows that $d_G(u') \leq r - 1 + |V(T'')| = k - 1$ for all $u' \in U$; so each u' misses (i.e., is not adjacent to) at most one vertex in T''. Let us distinguish the following cases.

Case $|U| \leq r$ For all $u' \in U$ we have

$$\begin{aligned}\xi(G) &\leq d_G(u') + d_G(u^-) - 2 \leq k - 1 + |V(T'')| - 1 + |U| - 2 \\ &= 2k + |U| - r - 4 \leq 2k - 4.\end{aligned}$$

Since by hypothesis $\xi(G) \geq 2k-4$ it follows that all the above inequalities are equalities, that is, $\xi(G) = 2k-4$, $|U| = r$, $d_G(u') = k-1$ for all $u' \in U$ and $d_G(u^-) = |V(T'')| - 1 + |U| = k-1$. Therefore $(V(T'') - \{u^-\}) \cup U = N_G(u^-)$ and also $V(T'') \subseteq N_G(u')$ for all $u' \in U$, thus each vertex from $U \cup \{u^-\}$ is adjacent in G to all the vertices of T''.

Observe that $|V(T'') \cup U| = k$, so the maximum degree of the induced subgraph $G[V(T'') \cup U]$ of G is $k-1$. Since by hypothesis G is connected and has maximum degree at least k, it follows that $|V(G)| \geq 1+k$ so that some vertex of $G[V(T'') \cup U]$ must be adjacent to some external vertex.

If some u_0 of U has $y \in N_G(u_0) \setminus (V(T'') \cup U)$, then we obtain a tree subgraph of G isomorphic to T by changing in T'' a penultimate vertex w for u_0 and one of its leaves w_i for y (we can do that because u_0 is adjacent in G to every vertex of T''); by joining vertex u^- of T'' with w by an edge (because $u^-w \in E(G)$) and by joining w with w_i and the $r-1$ vertices of $U - u_0$, because every vertex of $U - u_0$ is adjacent in G with $w \in V(T'')$.

If $N_G(U) \subset V(T'') \cup U$ then some t of $T'' - u^-$ has $z \in N_G(t) \setminus (V(T'') \cup U)$. We obtain a tree subgraph of G isomorphic to T by changing in T' vertex t for u, by joining vertex u^- with vertex t by an edge because $u^-t \in E(G)$, and by joining t with z and with $r-1$ vertices of U different from u.

In any case, G contains every tree of size k and the theorem holds.

Case $|U| \geq r+1$ Suppose that there exists some $u_0 \in U$ with $d_G(u_0) = k-1$, then $V(T'') \subseteq N_G(u_0)$. Thus by replacing in T'' vertex u^- with this vertex u_0 and by replacing in T' vertex u with u^-, we obtain a tree subgraph of G isomorphic to T because u^- is adjacent to at least $|U - u_0| \geq r$ vertices outside of T''. Hence, we may assume that $d_G(u') \leq k-2$ for all $u' \in U$. Moreover, since $\xi(G) \geq 2k-4$ it follows that $d_G(u^-) \geq k$ because for all $u' \in U$, $\xi(G) \leq d_G(u') + d_G(u^-) - 2$. Moreover, the vertices of U are independent because otherwise we would have $\xi(G) \leq d_G(u') + d_G(u'') - 2 \leq 2(k-2) - 2$, which is a contradiction. Therefore by the hypothesis of the theorem we have for all $u', u'' \in U$ that $d_G(u') + d_G(u'') \geq 2k-4$, implying that $d_G(u') = k-2$ for all $u' \in U$.

Let us denote $L_1 = \{t \in V(T'') : u^-t \in E(T'')\} \subseteq N_G(u^-)$. Note that if $L_1 \subseteq N_G(u')$ for some $u' \in U$, then we obtain again a tree subgraph of G isomorphic to T by replacing vertex u^- with u' in T'' and u with u^- in T', so assume this is not the case. Then

$$\text{for each } u' \in U, \text{ there exists } t_{u'} \in L_1 \text{ such that } V(T'' - t_{u'}) \subseteq N_G(u'). \tag{2.1}$$

Suppose that there is some penultimate w of T'' such that every vertex of U is adjacent to w and all its leaves, $w_1, \dots, w_{r'}$ with $r' \geq r$. Let us consider the tree H obtained from T' by replacing $w_1, \dots, w_r$ with r vertices of U different from u. Thus we obtain a tree subgraph of G isomorphic to T by joining vertex u with $w_1, \dots, w_r$. Consequently, by (2.1) it only remains to study the case in which every penultimate vertex w of T'' satisfies $w \in L_1$ and $w = t_{u'}$ for some $u' \in U$.

Let $w \in L_1$ be a penultimate vertex in T. If there exists some $u_0 \in U$ such that $wu_0 \in E(G)$, then $d_G(w) \geq k$ because $d_G(u_0) = k-2$ and $\xi(G) \geq 2k-4$. In this case let us consider the tree H obtained by interchanging in T' vertex u_0 for w, because u_0 is adjacent to all the leaves of w by (2.1). Since $d_G(w) \geq k$, then there are at least r neighbors of w outside of this tree H, and hence G contains a tree subgraph isomorphic to T. Therefore we may assume that w is independent of every vertex $u' \in U$, which implies that $t_{u'} = w$ for all $u' \in U$, and moreover w is the unique penultimate in L_1. In this case the other vertex of L_1, if any, must be a leaf adjacent to every vertex of U. Suppose there is a leaf $\ell \in L_1$. Then $d_G(\ell) \geq k$ (because $u\ell \in E(G)$, $d_G(u) = k-2$ and $\xi(G) \geq 2k-4$) and we obtain a tree subgraph isomorphic to T by changing in T' vertex u for ℓ and ℓ for u. Consequently we may suppose that $L_1 = \{w\}$, yielding that tree T consists of a path u, u^-, w of length 2 with u adjacent to r leaves and w adjacent to $k-r-2$ leaves.

Let us consider the tree R formed by $u_1, \ldots, u_r \in U$ neighbors of u^- and the path u^-, w, w_1, where w_1 is a leaf of w. We have $d_G(w_1) \geq k$ because $\xi(G) \geq 2k-4$ and by (2.1), w_1 is adjacent to any vertex $u' \in U$ of $d_G(u') = k-2$. Therefore w_1 must be adjacent to at least $k-r-2$ vertices outside of R. Hence the tree R along with these external neighbors of w_1 is a tree subgraph isomorphic to T, and the theorem holds. □

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