ON THE ISOPERIMETRIC PROBLEM IN EUCLIDEAN SPACE WITH DENSITY

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ABSTRACT. We study the isoperimetric problem for Euclidean space endowed with a continuous density. In dimension one, we characterize isoperimetric regions for a unimodal density. In higher dimensions, we prove existence results and we derive stability conditions, which lead to the conjecture that for a radial log-convex density, balls about the origin are isoperimetric regions. Finally, we prove this conjecture and the uniqueness of minimizers for the density $\exp(|x|^2)$ by using symmetrization techniques.

1. INTRODUCTION

The *isoperimetric problem* inside a Riemannian manifold seeks regions of least perimeter enclosing a fixed amount of volume. This problem can also be studied in the more general setting of a *manifold with density*, where a given continuous positive function on the manifold is used to weight the Riemannian volume and boundary area. Such a density is not equivalent to scaling the metric conformally by a factor λ, since in that case volume and perimeter would scale by different powers of λ.

We shall consider the particular case of Euclidean space with a density $f = e^{\psi}$. For any Borel set Ω in $\mathbb{R}^{n+1}$, the (weighted) *volume* or *measure* of Ω, and the (weighted) *perimeter relative to an open set U* in $\mathbb{R}^{n+1}$ are given by

$$\mathrm{vol}(\Omega) = \int_{\Omega} f\,dv, \qquad P(\Omega, U) = \int_{\partial\Omega\cap U} f\,da,$$

where dv and da are elements of Euclidean volume and area, in general provided by Lebesgue measure and n-dimensional Hausdorff measure on $\mathbb{R}^{n+1}$. Let $P(\Omega) = P(\Omega, \mathbb{R}^{n+1})$.

Much of the information of the isoperimetric problem is contained in the *isoperimetric profile*, which is the function $I_f : (0, \mathrm{vol}(\mathbb{R}^{n+1})) \to \mathbb{R}$ given by

$$I_f(V) = \inf\{P(\Omega) : \Omega \text{ is a smooth open set with } \mathrm{vol}(\Omega) = V\}.$$

An *isoperimetric region* –or simply a *minimizer*– of volume V is an open set Ω such that $\mathrm{vol}(\Omega) = V$ and $P(\Omega) = I_f(V)$.

In the last years the study of isoperimetric problems in manifolds with density has increased. One of the first and most interesting examples, with applications in probability and statistics, is the *Gaussian density* $\exp(-\pi|x|^2)$. About 1975 C. Borell **[Bor1]** and V. N. Sudakov and B. S. Tirel'son **[ST]** independently proved that half-spaces minimize perimeter under a volume constraint for this density. In 1982 A. Ehrhard **[Eh1]** gave a new proof of the isoperimetric property of half-spaces by adapting symmetrization techniques to the

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Date: October 29, 2018.

2000 *Mathematics Subject Classification.* 49Q20, 53C17.

Key words and phrases. Manifolds with density, isoperimetric problem, generalized mean curvature, stability, symmetrization.

First and second authors are partially supported by MCyT-Feder research project MTM2004-01387, fourth author by the National Science Foundation.

Gaussian context. His proof can be simplified by following the generalization of Steiner symmetrization to product measures given by A Ros [**R**]. More recently, S. Bobkov and C. Houdré [**BoH**] considered "unimodal densities" with finite total measure in the real line. These authors explicitly computed the isoperimetric profile for such densities and found some of the isoperimetric solutions. M. Gromov [**Gr**] studied manifolds with density as "mm spaces" and mentioned the natural generalization of mean curvature obtained by the first variation of weighted area. V. Bayle [**Ba**] proved generalizations of the Lévy-Gromov isoperimetric inequality and other geometric comparisons depending on a lower bound on the generalized Ricci curvature of the manifold. For recent advances on manifolds with density we refer the reader to [**M4**], [**Ba**] and references therein.

In this paper we first prove some existence results of isoperimetric regions for densities in Euclidean space with infinite total measure (Theorems 2.2 and 2.6) and recall regularity properties of minimizers (Theorem 2.8). In Section 3 we use a variational approach to characterize stability of balls centered at the origin for radial densities (Theorem 3.10). This result leads us to Conjecture 3.12: *for radial log-convex densities in* $\mathbb{R}^{n+1}$, *balls about the origin provide minimizers of any given volume*. We will prove this conjecture in the one-dimensional case (Corollary 4.12) and for the radial density $\exp(|x|^2)$ in any dimension (Theorem 5.2).

In Section 4 we completely describe isoperimetric regions in the real line endowed with unimodal densities (Theorems 4.3 and 4.7). We use comparison arguments that provide at the same time existence and uniqueness of minimizers. As interesting consequences we solve the isoperimetric problem for log-concave and log-convex densities in the real line (Corollaries 4.8 and 4.11), improving previous results by S. Bobkov and C. Houdré [**BoH**]. We also treat the isoperimetric problem and the free boundary problem for the closed half-line $[0, +\infty)$ and for compact intervals.

In Section 5 we establish, in arbitrary dimension, the isoperimetric property of round balls about the origin for the density $\exp(c|x|^2)$, $c > 0$ (Theorem 5.2). A remarkable difference with respect to the Gaussian measure is that the density $\exp(c|x|^2)$ for $c > 0$ has infinite total volume and hence the existence of minimizers is a non-trivial question. The proof of Theorem 5.2 goes as follows. First, we apply our previous results in Section 2 to ensure existence of isoperimetric regions of any given volume. Second, we use the description of minimizers for log-convex densities on the real line (Corollary 4.12) and the symmetrization in spaces with product measures given by A. Ros [**R**], to construct a counterpart to Steiner symmetrization for the density $\exp(c|x|^2)$. Then we use this symmetrization in axis directions as employed by L. Bieberbach [**Bi**] to produce centrally symmetric minimizers with connected boundary. Finally we conclude by Hsiang symmetrization [**H**] that such a minimizer must be a round ball about the origin. As a corollary of Theorem 5.2 we deduce an eigenvalues comparison theorem for the density $\exp(c|x|^2)$, $c \geqslant 0$, generalizing the Faber-Krahn Inequality.

Usually the uniqueness of isoperimetric regions is difficult to prove. In the case of the Gaussian density, the complete characterization of equality cases in the isoperimetric inequality is due to E. A. Carlen and C. Kerce [**CK**], who proved that any perimeter minimizer for fixed volume is, up to a set of measure zero, a half-space. They obtained this result as consequence of the discussion of equality in a more general functional inequality due to S. Bobkov. Previous uniqueness results in the Gaussian setting involving a Brunn-Minkowski type inequality were given by A. Ehrhard [**Eh2**]. In Theorem 5.2 we also show the uniqueness of round balls centered at the origin as minimizers for the density $\exp(c|x|^2)$, $c > 0$. Since round balls appear as the result of finitely many symmetrizations, it suffices to see that if an axis symmetrization of a minimizer produces a ball, then the minimizer is a ball. We deduce this fact by standard arguments [**Ch2**, Lemma III.2.3].

An interesting consequence of our characterization of stable balls in Theorem 3.10 observed by K. Brakke is that any round ball about the origin is unstable in $\mathbb{R}^{n+1}$ endowed with a radial, strictly log-concave density. This fact, together with the isoperimetric property of half-spaces in the Gaussian space and our Corollary 4.9, where we proved the isoperimetric property of half-lines for densities on the real line, might suggest that half-spaces are isoperimetric regions for any radial, log-concave density on $\mathbb{R}^{n+1}$. In Corollary 3.13 we give an example showing that this is not true in general. We believe that this is a motivation to study in more detail the isoperimetric problem for these kind of densities where unexpected shapes appear.

After circulating this manuscript we heard from Franck Barthe and Michel Ledoux that the isoperimetric property of round balls about the origin for the density $\exp(c|x|^2)$, $c > 0$, was previously proved by C. Borell [**Bor2**, Theorem 4.1]. Borell's proof uses a Brunn-Minkowski inequality and does not yield uniqueness of minimizers.

Acknowledgements. This work began during Morgan's lecturers on "Geometric Measure Theory and Isoperimetric Problems" at the 2004 Summer School on Minimal Surfaces and Variational Problems held in the Institut de Mathématiques de Jussieu in Paris. We would like to thank the organizers: Pascal Romon, Marc Soret, Rabah Souam, Eric Toubiana, Frédéric Hélein, David Hoffman, Antonio Ros and Harold Rosenberg. We also thank Franck Barthe, Christer Borell and Michel Ledoux for bringing previous results on the isoperimetric problem for the density $\exp(|x|^2)$ to our attention.

2. Existence and regularity results

In this section we firstly deal existence of isoperimetric regions in Euclidean space with density. In general, for a Riemannian manifold with density, standard compactness arguments of Geometric Measure Theory (see [**Si**, 27.3 and 31.2] or [**M2**, 5.5 and 9.1], and [**M1**, 4.1] or [**RiR**, Thm. 2.1]) can be applied in order to provide isoperimetric regions, except that there can be loss of volume at infinity (by regularity Theorem 2.8, these are open sets with nice boundaries). In particular, if the total measure is finite, isoperimetric regions of any prescribed volume exist. We will prove some existence results for densities with infinite total volume. We begin with the following lemma:

Lemma 2.1. *Let f be a positive, nondecreasing function on $[0,+\infty)$ satisfying $f(r) \to +\infty$ when $r \to +\infty$. If the function $\psi = \log(f)$ satisfies*

$$\psi(r) \leqslant C\left(\frac{n+1}{n} - \varepsilon\right)^{r/2}$$

for some $n \in \mathbb{N}$, $C > 0$ and $\varepsilon \in (0,1)$, then the sequence

$$\zeta(m) = \frac{f(m)}{f(m+2)^{n/(n+1)}}$$

tends to infinity.

Conversely, if $\{\zeta(m)\} \to +\infty$, then there is $r_0 > 0$ and $C > 0$ such that

$$\psi(r) \leqslant C\left(\frac{n+1}{n}\right)^{r/2}, \qquad r \geqslant r_0.$$

Proof. We prove the first part of the statement by contradiction. Suppose that $\{\zeta(m)\}_{m\in\mathbb{N}}$ does not tend to infinity. We can assume, by passing to a subsequence if necessary, that there is $K > 0$ such that

$$f(m) \leqslant K\, f(m+2)^{n/(n+1)}, \qquad m \in \mathbb{N},$$

and therefore

$$\psi(m+2) \geqslant \frac{n+1}{n}\,(\psi(m)-\log(K)), \qquad m \in \mathbb{N}.$$

On the other hand, as $\{\psi(m)\} \to +\infty$ and $\varepsilon > 0$, we can find $m_0 \in \mathbb{N}$ such that

$$\psi(m+2) \geqslant \left(\frac{n+1}{n} - \frac{\varepsilon}{2}\right)\psi(m), \qquad m \geqslant m_0,$$

$$\psi(m_0+2k) \geqslant \left(\frac{n+1}{n} - \frac{\varepsilon}{2}\right)^k \psi(m_0).$$

Now take $r \geqslant m_0 + 2$ and $k \in \mathbb{N}$ such that $m_0 + 2k \leqslant r \leqslant m_0 + 2(k+1)$. By using that ψ is nondecreasing we have

$$\psi(r) \geqslant \psi(m_0+2k) \geqslant \left(\frac{n+1}{n} - \frac{\varepsilon}{2}\right)^k \psi(m_0) \geqslant \left(\frac{n+1}{n} - \frac{\varepsilon}{2}\right)^{r/2-m_0/2-1} \psi(m_0).$$

Hence, for $r \gg m_0 + 2$ we deduce

$$\psi(r) > C\,\left(\frac{n+1}{n} - \varepsilon\right)^{r/2},$$

and we get a contradiction.

Conversely, suppose that $\{\zeta(m)\} \to +\infty$. Then, we can find $m_0 \geqslant 2$ such that $\psi(m_0) > 0$ and $\zeta(m) \geqslant 1$ for $m \geqslant m_0$. As a consequence

$$\psi(m+2) \leqslant \left(\frac{n+1}{n}\right)\psi(m), \qquad m \geqslant m_0,$$

$$\psi(m_0+2k) \leqslant \left(\frac{n+1}{n}\right)^k \psi(m_0).$$

Finally, for $r \geqslant m_0$ there is $k \in \mathbb{N}$ such that $m_0 + 2(k-1) \leqslant r \leqslant m_0 + 2k$. Hence

$$\psi(r) \leqslant \psi(m_0+2k) \leqslant \left(\frac{n+1}{n}\right)^k \psi(m_0) \leqslant \left(\frac{n+1}{n}\right)^{r/2} \psi(m_0).$$

□

Now, we can prove our first existence result.

Theorem 2.2. *Let $f = e^\psi$ be a density on $\mathbb{R}^{n+1}$ such that $f(x) \to +\infty$ when $|x| \to +\infty$. Suppose that one of the following conditions holds:*

(i) *The sequence defined by*

$$\zeta(m) = \frac{\min\{f(x) : m \leqslant |x| \leqslant m+2\}}{\max\{f(x)^{n/(n+1)} : m \leqslant |x| \leqslant m+2\}}$$

tends to infinity.

(ii) *The density is radial, nondecreasing in $|x|$ and satisfies*

$$\psi(x) \leqslant C\,\left(\frac{n+1}{n} - \varepsilon\right)^{|x|/2},$$

for some constants $C > 0$ and $\varepsilon \in (0,1)$.

Then, minimizers of any given volume exist for this density and they are bounded subsets of $\mathbb{R}^{n+1}$.

Remark 2.3. The proof of the statement shows that it suffices to suppose that the ratio $\min f(x)/\max f(x)^{n/(n+1)}$ on lattice cubes goes to infinity.

Proof. By Lemma 2.1 we can assume that (i) holds. Denote by $v(\Omega)$ and $a(\partial\Omega)$ the Euclidean volume and boundary area of a set Ω. Partition $\mathbb{R}^{n+1}$ into lattice open cubes of diameter equal to 1 and Euclidean volume v_0. There is an isoperimetric constant $\alpha > 0$ such that any set Ω inside a cube C as above with $v(\Omega) \leqslant v_0/2$ satisfies

$$a(\partial\Omega \cap C) \geqslant \alpha\, v(\Omega)^{n/(n+1)}.$$

On the other hand, there is $m = m(C) \in \mathbb{N}$ such that the cube C is contained in the annulus $\{m \leqslant |x| \leqslant m+2\}$. Thus, the definition of weighted volume and perimeter, together with the definition of $\zeta(m)$, implies the inequality

$$P(\Omega, C) \geqslant \alpha\, \zeta(m)\, \mathrm{vol}(\Omega)^{n/(n+1)}, \tag{2.1}$$

for any $\Omega \subset C \subset \{m \leqslant |x| \leqslant m+2\}$ with $v(\Omega) \leqslant v_0/2$.

Fix $V > 0$, and consider a sequence of smooth open sets of volume V with perimeters approaching $I_f(V)$ and bounded from above by $I_f(V)+1$. By using the Compactness Theorem [**M3**, 9.1] we may assume that this sequence converges. Fix $\varepsilon > 0$. By hypothesis, there is $m_0 \in \mathbb{N}$ such that $\zeta(m) \geqslant (1/\varepsilon)^{n/(n+1)}$ for any $m \geqslant m_0$. On the other hand, as $f(x) \to +\infty$ when $|x| \to +\infty$, we can suppose that $v(\Omega) \leqslant v_0/2$ whenever $\Omega \subset C \subset \{|x| \geqslant m_0\}$. In particular, we can apply (2.1) to such an Ω, so that we obtain

$$\mathrm{vol}(\Omega) \leqslant \left(\frac{P(\Omega, C)}{\alpha\, \zeta(m)}\right)^{(n+1)/n} \leqslant \varepsilon\, \alpha' P(\Omega, C)^{(n+1)/n}.$$

By summing the previous inequality over the collection $\mathcal{C}_m$ of all cubes C contained in $\{|x| \geqslant m\}$ we deduce that for any set Ω of the given minimizing sequence and any $m \geqslant m_0$

$$\begin{aligned}\mathrm{vol}\left(\Omega \cap \Big(\bigcup_{C \in \mathcal{C}_m} C\Big)\right) &\leqslant \varepsilon\, \alpha' \left(\sum_{C \in \mathcal{C}_m} P(\Omega, C)\right)^{(n+1)/n} \\ &\leqslant \varepsilon\, \alpha' P(\Omega)^{(n+1)/n} \leqslant \varepsilon\, \alpha'\, (I_f(V)+1)^{(n+1)/n}.\end{aligned}$$

Hence, there is no loss of volume at infinity and the limit of our sequence is an isoperimetric region of volume V.

To prove that any minimizer Ω is a bounded subset of $\mathbb{R}^{n+1}$, we can proceed as in [**M2**, Lemma 13.6]. Consider any large cube $C_r = [-r, r]^{n+1}$ about the origin and partition almost all its complement into congruent open cubes of diameter at most 1. Denote $V(r) = \mathrm{vol}(\Omega - C_r)$ and $P(r) = P(\Omega, \mathbb{R}^{n+1} - C_r)$. As above, we have

$$V(r) \leqslant \varepsilon\, \alpha' P(r)^{(n+1)/n}, \qquad r \gg 0. \tag{2.2}$$

On the other hand, there is a constant $H > 0$ depending on $\partial\Omega$ such that small volume adjustments may be accomplished inside C_r at a cost

$$|\Delta P| \leqslant H\, |\Delta V|.$$

Thus replacing $\Omega - C_r$ costs at most $H|\Delta V| + |V'(r)|$ (due to the slice of ∂C_r) for almost all large r. By using that Ω is a minimizer, we get

$$P(r) \leqslant H\, V(r) + |V'(r)|, \qquad \text{for almost all } r \gg 0. \tag{2.3}$$

Since $V(r)$ is nonincreasing and tends to 0 when $r \to +\infty$, combining inequalities (2.2) and (2.3) yields for some $c > 0$,

$$c\, V(r)^{n/(n+1)} \leqslant -V'(r), \qquad \text{for almost all } r \gg 0.$$

If we suppose that Ω is unbounded, then $V(r) \neq 0$ and

$$(n+1)\, (V^{1/(n+1)})' = V^{-n/(n+1)}\, V' \leqslant -c < 0,$$

for almost all large $r > 0$, a contradiction since V is positive and nonincreasing. $\square$

Our next existence result is an improvement of Theorem 2.2 in dimension two. We need the following lemma:

Lemma 2.4. *Let f be a planar radial density nondecreasing on $[r_0, +\infty)$. Then, for any smooth, open set $\Omega \subset \mathbb{R}^2$ contained in $\{|x| \geqslant r_0\}$ and such that $P(\Omega) < 2\pi r_0 f(r_0)$, we have the isoperimetric inequality*

$$P(\Omega)^2 \geqslant 2\, f(r_0)\, \mathrm{vol}(\Omega).$$

Proof. First, we can assume that Ω is connected. Moreover, the hypothesis on the perimeter implies that Ω is bounded and the closure of Ω cannot contain a circle about the origin. Let r_1 and r_2 be the minimum and maximum distance from $\overline{\Omega}$ to the origin, respectively. The intersection Ω_t of Ω with the circle of radius $t \in (r_1, r_2)$ has Euclidean length strictly less than $2\pi t$, and the boundary $\partial\Omega_t$ has at least two points. Therefore, the coarea formula gives us

$$P(\Omega) \geqslant \int_{r_1}^{r_2} f(t)\, \mathrm{card}(\partial\Omega_t)\, dt \geqslant 2f(r_0)\,(r_2 - r_1), \tag{2.4}$$

where we have used that the density is nondecreasing on $[r_0, +\infty)$.

On the other hand, we consider the map $F : (r_1, r_2) \times \partial\Omega \to \mathbb{R}^2$ given by $F(t, x) = tx/|x|$. It is clear that $\Omega \subseteq F(A)$, where A is the open set of the pairs (t, x) where $t < |x|$ and $f(tx/|x|) < f(x)$. For any $(t, x) \in A$ the Jacobian of F is strictly less than 1. Thus, the definition of A, together with the coarea formula and Fubini's theorem implies

$$\mathrm{vol}(\Omega) \leqslant \mathrm{vol}(F(A)) \leqslant \int_A f\left(\frac{tx}{|x|}\right) d(t, x) \leqslant \int_A f(x)\, d(t, x) = (r_2 - r_1)\, P(\Omega). \tag{2.5}$$

Multiplying the estimates (2.4) and (2.5) we obtain the desired inequality. $\square$

Remark 2.5. We do not see how to generalize the previous lemma to $\mathbb{R}^{n+1}$. The analog of inequality (2.5) holds, but the estimation on the Euclidean boundary area $a(\partial\Omega_t)$ leading to (2.4) becomes

$$a(\partial\Omega_t) \geqslant \alpha\, f(t)^{1/n}\, v(\Omega_t)^{(n-1)/n} \geqslant C\, v(\Omega_t),$$

where $v(\Omega_t)$ denotes Euclidean volume. The last inequality can be integrated to deduce $P(\Omega) \geqslant C\ \mathrm{vol}(\Omega)$, which is inadequate to obtain an analog of Lemma 2.4.

Theorem 2.6. *Consider the plane endowed with a nondecreasing, radial density f such that $f(x) \to +\infty$ when $|x| \to +\infty$. Then, there are minimizers for this density of any given volume.*

Proof. Consider a sequence of smooth open sets of volume $V > 0$ with perimeters approaching $I_f(V)$. Applying the Compactness Theorem [**M3**, 9.1] we can assume that this sequence converges. We can also suppose that any set Ω of this sequence satisfies $P(\Omega) \leqslant I_f(V) + 1$. Moreover, as the density tends to $+\infty$, there is $m_0 \in \mathbb{N}$ such that $I_f(V) + 1 < 2\pi m f(m)$ for any $m \geqslant m_0$. In particular, we can apply Lemma 2.4 to the union Ω' of all connected components of Ω inside $\{|x| \geqslant m\}$. Hence, we get

$$\mathrm{vol}(\Omega') \leqslant \frac{P(\Omega')^2}{2f(m)} \leqslant \frac{(I_f(V) + 1)^2}{2f(m)}, \qquad m \geqslant m_0.$$

As $\lim_{m\to+\infty} f(m) = +\infty$ we conclude that there is no loss of volume at infinity and the limit of our sequence solves the isoperimetric problem for volume V. $\square$

Example 2.7. We illustrate here that Theorem 2.6 need not hold if we do not require the density to be nondecreasing. Consider in $\mathbb{R}^{n+1}$ $(n > 1)$ the density $f(x) = 1 + |x|^2$. Now, introduce bumps into the graph of f such that any volume V_k corresponding to a positive rational can be enclosed with perimeter $1/k$. Then, for any given volume we may find a sequence of sets enclosing this volume and with arbitrarily small perimeter, which implies that isoperimetric regions do not exist.

We finish this section by recalling regularity properties of the boundary of a minimizer in Euclidean space with density. The result is also valid for any smooth Riemannian manifold with density.

Theorem 2.8 ([**M3**, 3.10])**.** *Consider a smooth density on $\mathbb{R}^{n+1}$. If Ω is a minimizer, then the boundary $\Sigma = \partial\Omega$ is a real-analytic embedded hypersurface, up to a closed set of singularities with Euclidean Hausdorff dimension less than or equal to $n-7$.*

3. Variational formulae. Stable balls for radial densities

In this section we use a variational approach to derive some properties of sets minimizing perimeter up to second order for variations preserving volume.

Let $f = e^{\psi}$ be a smooth density on $\mathbb{R}^{n+1}$. Denote by $\Omega \subset \mathbb{R}^{n+1}$ a smooth open set with boundary Σ and inward unit normal vector N. We consider a one-parameter variation $\{\phi_t\}_{|t|<\varepsilon} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with associated infinitesimal vector field $X = d\phi_t/dt$ with normal component $u = \langle X, N\rangle$. Let $\Omega_t = \phi_t(\Omega)$ and $\Sigma_t = \phi_t(\Sigma)$. The volume and perimeter functions of the variation are $V(t) = \operatorname{vol}(\Omega_t)$ and $P(t) = P(\Omega_t)$, respectively. The first variation of volume and perimeter are computed in [**Ba**, Chapter 3]. We include here a proof for the sake of completeness.

Lemma 3.1. *The first variation of volume and perimeter of a smooth region Ω with boundary Σ in $\mathbb{R}^{n+1}$ endowed with smooth density $f = e^{\psi}$ for a flow with initial normal velocity u are given by*

$$V'(0) = -\int_\Sigma f u\, dv, \qquad P'(0) = -\int_\Sigma (nH - \langle \nabla\psi, N\rangle)\, f u\, da,$$

where H is the Euclidean mean curvature of Σ with respect to N (that is, the arithmetic mean of the principal curvatures of Σ) and $\nabla\psi$ is the Euclidean gradient of ψ.

Proof. Denote by $\operatorname{div} X$ (resp. $\operatorname{div}_\Sigma X$) the divergence of X in $\mathbb{R}^{n+1}$ (resp. relative to Σ). Let $X(f) = \langle \nabla f, X\rangle$. We have

$$\begin{aligned} V'(0) &= \int_\Omega X(f)\, dv + \int_\Omega f\, \frac{d}{dt}\bigg|_{t=0} (dv_t) \\ &= \int_\Omega (\langle \nabla f, X\rangle + f \operatorname{div} X)\, dv = \int_\Omega \operatorname{div}(fX)\, dv = -\int_\Sigma f u\, da. \end{aligned}$$

In the second equality we have used that $(d/dt)|_{t=0}\,(dv_t) = (\operatorname{div} X)\, dv$, see [**Si**, §16]. In the last one, we have applied the Gauss-Green theorem. For perimeter we have

$$\begin{aligned} P'(0) &= \int_\Sigma X(f)\, da + \int_\Sigma f\, \frac{d}{dt}\bigg|_{t=0} (da_t) \\ &= \int_\Sigma (\langle \nabla f, X\rangle + f \operatorname{div}_\Sigma X)\, da = \int_\Sigma (\langle \nabla f, uN\rangle + \operatorname{div}_\Sigma(fX))\, da \\ &= \int_\Sigma f u\, \langle \nabla\psi, N\rangle\, da - \int_\Sigma nHfu\, da = -\int_\Sigma (nH - \langle \nabla\psi, N\rangle)\, f u\, da. \end{aligned}$$

To obtain the fourth equality we have used that the integral over Σ of the divergence of the tangent part of fX vanishes by virtue of the divergence theorem. $\square$

We define, as in [**Ba**, Chapter 3], the (generalized) *mean curvature* of Σ with respect to N as the function

$$H_\psi = nH - \langle \nabla\psi, N\rangle, \tag{3.1}$$

so that the first variation of perimeter can be written as

$$P'(0) = -\int_\Sigma H_\psi f u \, da.$$

We say that a given variation $\{\phi_t\}_t$ *preserves volume* if $V(t)$ is constant for any small t. We say that Ω is *stationary* if $P'(0) = 0$ for any volume-preserving variation. It is clear that any isoperimetric region is also stationary. The following characterization of stationary sets is similar to the one established by J. L. Barbosa and M. do Carmo [**BdC**, Proposition 2.7] for the case of $\mathbb{R}^{n+1}$ with the standard density $f \equiv 1$. The proof is based on Lemma 3.1 and on the fact that any function u orthogonal to f in $L^2(\Sigma)$ is the normal component of a vector field associated to a volume-preserving variation of Ω, see [**BdC**, Lemma 2.2].

Proposition 3.2. *Consider a smooth density $f = e^\psi$ on $\mathbb{R}^{n+1}$. Then, for a smooth open set Ω, the following conditions are equivalent:*

(i) *Ω is stationary.*
(ii) *$\Sigma = \partial\Omega$ has (generalized) constant mean curvature H_0.*
(iii) *There is a constant H_0 such that $(P - H_0 V)'(0) = 0$ for any variation of Ω.*

Example 3.3. Let $f = e^\psi$ be a smooth density defined on the real line. Then, it is easy to show that a bounded interval (a, b) is stationary if and only if $\psi'(a) = -\psi'(b)$.

Now, we introduce some examples of hypersurfaces with constant mean curvature in $\mathbb{R}^{n+1}$ with a radial density.

Example 3.4. Suppose $f = e^\psi$, where $\psi(x) = \delta(|x|)$ for any $x \in \mathbb{R}^{n+1}$. The mean curvature H_ψ of a hypersurface Σ with respect to a unit normal vector N is given by

$$H_\psi(p) = nH(p) - \frac{\delta'(r)}{r}\,\langle p, N(p)\rangle, \quad r = |p|,$$

where H is the Euclidean mean curvature of Σ with respect to N. In particular, if Σ is a sphere of radius $r > 0$, then it has constant mean curvature if and only if Σ is centered at the origin. In this case, $H_\psi = n/r + \delta'(r)$ with respect to the inner normal vector.

On the other hand, if Σ is the hyperplane defined by $\{x \in \mathbb{R}^{n+1} : \langle x, u\rangle = c\}$, where $|u| = 1$, then the mean curvature of Σ with respect to $N = -u$ is

$$H_\psi(p) = -c\,\frac{\delta'(r)}{r}, \quad r = |p|. \tag{3.2}$$

It follows that any hyperplane passing through the origin is a minimal hypersurface of $\mathbb{R}^{n+1}$ with a radial density. In general, we cannot expect that any hyperplane has constant mean curvature for a radial density. In fact, a straightforward analysis of equation (3.2) leads us to the following:

Lemma 3.5. *Let $f = e^\psi$ be a smooth radial density on $\mathbb{R}^{n+1}$. Suppose that there is a hyperplane Σ which does not contain the origin and has constant mean curvature H_ψ. Then, there are constants $a, b \in \mathbb{R}$ and $r_0 > 0$, such that*

$$\psi(x) = e^{a|x|^2 + b}, \quad \text{whenever } |x| \geqslant r_0.$$

Now, we compute the second variation formula of the functional $P - H_\psi V$ for any variation of a stationary set.

Proposition 3.6 ([**Ba**, Section 3.4.6])**.** *Consider a stationary open set Ω in $\mathbb{R}^{n+1}$ endowed with a smooth density $f = e^\psi$. Let N be the inward unit normal vector to $\Sigma = \partial\Omega$, and H_ψ the constant mean curvature of Σ with respect to N. Consider a variation of Ω with associated vector field $X = uN$ on Σ. Then, we have*

$$(P - H_\psi V)''(0) = Q_\psi(u,u) := \int_\Sigma f\,(|\nabla_\Sigma u|^2 - |\sigma|^2 u^2)\,da + \int_\Sigma f u^2\,(\nabla^2\psi)\,(N,N)\,da, \tag{3.3}$$

where $\nabla_\Sigma u$ is the gradient of u relative to Σ, $|\sigma|^2$ is the squared sum of the principal curvatures of Σ, and $\nabla^2\psi$ is the Euclidean Hessian of ψ.

Proof. The first variation formula for volume and perimeter gives us

$$(P - H_\psi V)'(t) = -\int_{\Sigma_t} (H_\psi)_t\, f u_t\, da_t + H_\psi \int_{\Sigma_t} f u_t\, da_t,$$

where $(H_\psi)_t$ is the mean curvature of Σ_t. Hence

$$(P - H_\psi V)''(0) = -\int_\Sigma H'_\psi(0)\, f u\, da, \tag{3.4}$$

so that we have to compute the derivative of the generalized mean curvature along Σ_t. Denote by $D_U V$ the Levi-Civitá connection on $\mathbb{R}^{n+1}$. By (3.1), we get

$$\begin{aligned} H'_\psi(0) &= nH'(0) - \langle D_X \nabla\psi, N\rangle - \langle \nabla\psi, D_X N\rangle \\ &= nH'(0) - u\,(\nabla^2\psi)(N,N) + \langle \nabla\psi, \nabla_\Sigma u\rangle\,, \end{aligned}$$

where in the last equality we have used that $D_X N = -\nabla_\Sigma u$. On the other hand, it is well known [**Ro**] that

$$nH'(0) = \Delta_\Sigma u + |\sigma|^2 u, \tag{3.5}$$

where Δ_Σ is the Laplacian relative to Σ. Thus, we have obtained

$$H'_\psi(0) = \Delta_\Sigma u + |\sigma|^2 u - u\,(\nabla^2\psi)(N,N) + \langle \nabla_\Sigma\psi, \nabla_\Sigma u\rangle\,.$$

By substituting this information into (3.4) we conclude that

$$\begin{aligned} (P - H_\psi V)''(0) = &-\int_\Sigma f u\,(\Delta_\Sigma u + |\sigma|^2 u)\,da - \int_\Sigma f u\,\langle \nabla_\Sigma\psi, \nabla_\Sigma u\rangle\,da \\ &+ \int_\Sigma f u^2\,(\nabla^2\psi)\,(N,N)\,da. \end{aligned}$$

Finally, by using integration by parts, we deduce

$$-\int_\Sigma f u\,(\Delta_\Sigma u + |\sigma|^2 u)\,da - \int_\Sigma f u\,\langle \nabla_\Sigma\psi, \nabla_\Sigma u\rangle\,da = \int_\Sigma f\,(|\nabla_\Sigma u|^2 - |\sigma|^2 u^2)\,da,$$

and the result follows. □

Remark 3.7. In a smooth Riemannian manifold with density the second variation has an additional term depending on the Ricci curvature of the manifold in the normal direction N. This term comes from (3.5) and it is given by

$$-\int_\Sigma \mathrm{Ric}(N,N)\, f u^2\, da.$$

The expression (3.3) defines a quadratic form on $C_0^\infty(\Sigma)$ called the *index form* associated to Σ. We say that a smooth open set Ω is *stable* if it is stationary and $P''(0) \geqslant 0$ for any volume-preserving variation of Ω. Stability can be characterized in terms of the index form as in [**BdC**, Proposition 2.10]. More precisely, we have the following:

Lemma 3.8. *Let Ω be a smooth open set in $\mathbb{R}^{n+1}$ endowed with a smooth density $f = e^\psi$. Then, Ω is stable if and only if it is stationary and the index form* (3.3) *of $\Sigma = \partial\Omega$ satisfies*

$$Q_\psi(u,u) \geqslant 0 \text{ for any } u \in C_0^\infty(\Sigma) \text{ such that } \int_\Sigma f u \, da = 0.$$

Observe that the term in the index form containing $\nabla^2\psi$ indicates that the notion of stability is more restrictive when the density $f = e^\psi$ is log-concave. In fact, by inserting in (3.3) locally constant nowhere vanishing functions we easily deduce

Corollary 3.9. *If Ω is a smooth stable region in $\mathbb{R}^{n+1}$ with a smooth, log-concave density, then the hypersurface $\Sigma = \partial\Omega$ is connected or totally geodesic. Moreover, if the density is strictly log-concave, then Σ is connected.*

Our main result in this section characterizes the stability of round balls about the origin for radial densities.

Theorem 3.10. *Consider a smooth density $f = e^\psi$ on $\mathbb{R}^{n+1}$ such that $\psi(x) = \delta(|x|)$. Then, the round ball B about the origin of radius $r > 0$ is stable if and only if $\delta''(r) \geqslant 0$.*

Proof. We use Lemma 3.8. Denote by Σ the boundary of B, and by N the inward unit normal vector to Σ. Clearly the density is constant on Σ, so that a function $u \in C^\infty(\Sigma)$ is orthogonal to f in $L^2(\Sigma)$ if and only if it has mean zero on Σ. Moreover, $(\nabla^2\psi)(N,N) = \delta''(r)$ on Σ. As consequence, the index form (3.3) is given by

$$Q_\psi(u,u) = f(r) \int_\Sigma (|\nabla_\Sigma u|^2 - |\sigma|^2 u^2) \, da + f(r)\, \delta''(r) \int_\Sigma u^2 \, da.$$

Since Euclidean balls are stable regions in $\mathbb{R}^{n+1}$ with the standard density $f \equiv 1$, the first integral is nonnegative and vanishes for translations. Consequently, if $\delta''(r) \geqslant 0$, then $Q_\psi \geqslant 0$ and B is stable. Conversely, if B is stable under infinitesimal translations, then $\delta''(r) \geqslant 0$. $\square$

As an immediate consequence of Theorem 3.10 we obtain

Corollary 3.11. *In Euclidean space endowed with a smooth, radial, log-convex density, round balls centered at the origin are stable regions.*

The preceding corollary leads to the following conjecture inspired by Ken Brakke at Jussieu:

Conjecture 3.12. *In $\mathbb{R}^{n+1}$ with a smooth, radial, log-convex density, balls about the origin provide isoperimetric regions of any given volume.*

In Sections 4 and 5 we will prove some special cases of this conjecture. Another interesting consequence of Theorem 3.10 is the fact that for a strictly log-concave density on $\mathbb{R}^{n+1}$, round balls about the origin are unstable. This allows us to prove the following:

Corollary 3.13. *There are smooth, radial, log-concave densities with finite volume in $\mathbb{R}^2$ for which isoperimetric regions are neither half-planes nor round balls.*

Proof. Consider the density $f = e^{\psi}$, with $\psi(x) = -\sqrt{|x|^2+1}$. The total volume of this density is finite and hence minimizers of any given volume exist, as was indicated at the beginning of Section 2. The Hessian of ψ at any $(x,y) \in \mathbb{R}^2$ is given by

$$(\nabla^2\psi)_{(x,y)}(a,b) = \frac{-(bx-ay)^2-a^2-b^2}{(1+x^2+y^2)^{3/2}},$$

and hence f is strictly log-concave. It follows by Example 3.4 and Theorem 3.10 that any round disk is unstable for this density. On the other hand, by taking into account Example 3.4 and Lemma 3.5, we deduce that only planes passing through the origin have constant mean curvature H_ψ. As consequence, a minimizer with measure different from the half volume of $\mathbb{R}^2$ cannot be a disk nor a half-plane. □

4. ISOPERIMETRY IN THE REAL LINE WITH DENSITY

In this section we study the isoperimetric problem in the real line with a *unimodal density*: a density which is increasing (or decreasing) on $(-\infty, x_0)$ and decreasing (or increasing) on $(x_0, +\infty)$, for some $x_0 \in (-\infty, +\infty]$. S. Bobkov and C. Houdré [**BoH**, Section 13] previously considered this setting under the further assumption of finite total measure. They computed the isoperimetric profile and gave some examples of isoperimetric regions. We provide here a simple, more general approach, which leads us to the complete description of minimizers.

We begin by solving the isoperimetric problem for monotonic densities. We recall that, for a function f, an end $E = \pm\infty$ has *finite measure* if f is integrable in a neighborhood of E.

Proposition 4.1. *Let f be a monotonic density on $\mathbb{R}$ and denote by E the end where f attains its infimum. If E has finite measure, then for any given volume, a half-line containing E is the unique isoperimetric region. If E has infinite measure, then the isoperimetric profile coincides with $2\,f(E)$, and it is approached or attained by a bounded interval going off to E.*

Proof. If E has finite measure, then any candidate other than the half-line of the same measure has at least two boundary points and hence greater perimeter since at least one of them is beyond the half-line. If E has infinite measure, then any open set enclosing a given volume has at least two boundary points, so that the infimum perimeter is $2\,f(E)$, approached or attained as asserted. □

Example 4.2. For the density $f(x) = e^x$ the end $E = -\infty$ has finite measure. Then the half-lines $(-\infty, x)$ are the unique minimizers for fixed volume and the isoperimetric profile is given by $I_f(V) = V$, for any $V > 0$. For the density $f \equiv 1$ the profile is constant and isoperimetric regions are bounded intervals. Finally, the density $f(x) = e^x + 1$ is one for which the profile is constant while minimizers do not exist.

We say that a function f is *increasing-decreasing* if there is $x_0 \in \mathbb{R}$ such that f is increasing on $(-\infty, x_0)$ and decreasing on $(x_0, +\infty)$, not necessarily strictly.

Theorem 4.3. *Let f be an increasing-decreasing density on $\mathbb{R}$. Then, if a minimizer for given volume exists, it is a half-line, a bounded interval where f attains its maximum or equals its one-sided minimum, or the complement of one of these intervals. If it does not exist, the infimum perimeter is approached by a bounded interval going off to $\pm\infty$.*

Proof. Consider a smooth open set of the prescribed volume. If the closure contains a point x_0 where f attains its maximum, then we can replace the given set with an interval containing x_0. Otherwise, we can assume by Proposition 4.1 that the set consists of one interval on one side, or an interval on each side of the maxima of f. Among intervals containing x_0 there is one of least perimeter. Among intervals on one side, the infimum perimeter is $2\min\{f(-\infty), f(+\infty)\}$ or the unique minimizer is a half-line. The theorem follows. $\square$

In the following examples we illustrate that the different possibilities in Theorem 4.3 may occur.

Example 4.4. Consider the density $f(x) = e^{-|x|}$, which has finite total measure (see Figure 1). A straightforward computation shows that the isoperimetric candidates of volume $V = 1$ (half-lines, bounded intervals containing the origin, and complements) have the same perimeter. This illustrates that there is no uniqueness of minimizers for $V = 1$ since the different possibilities appear. Moreover, though the density is symmetric with respect to the origin, the bounded minimizers need not be symmetric.

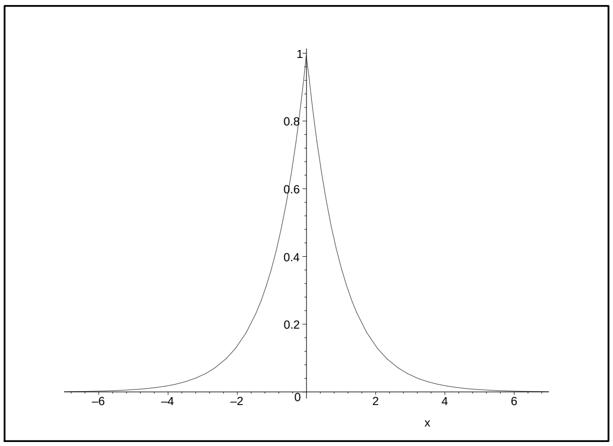

FIGURE 1. *For density $e^{-|x|}$ all types of minimizers occur.*

Example 4.5. Consider the density given by $f(x) = e^{-|x|}$ for $x \leqslant \log(6)$ and $f(x) = 1/6$ for $x \geqslant \log(6)$. The left end has finite measure while the right one has infinite measure (see Figure 2). Thus, only half-lines containing $-\infty$ and bounded intervals are possible minimizers for a fixed measure. For volume $V = 1/3$, it can be shown that the isoperimetric regions are the corresponding half-line containing $-\infty$ and any bounded interval inside the half-line $[1/6, +\infty)$. For a volume $V > 1/3$ only bounded intervals contained in $[1/6, +\infty)$ provide minimizers. This illustrates that bounded minimizers need not contain a point where the maximum of the density is achieved.

Example 4.6. Consider the density given by

$$f(x) = \begin{cases} e^{-|x|} & x \leqslant \log(6), \\ \frac{1}{9} + \frac{1}{x - \log(6) + 18} & x \geqslant \log(6), \end{cases}$$

which is depicted in Figure 3. As in the previous example, the ends $-\infty$ and $+\infty$ have finite and infinite measure, respectively. It is not difficult to prove that for small volumes, half-lines containing $-\infty$ are minimizers. However, for $V = 1/3$, we can consider a sequence of bounded intervals of volume V converging to $+\infty$ and whose perimeter tends to $2/9$. A direct computation shows that the half-line of volume V containing $-\infty$ and any bounded interval of volume V have strictly greater perimeter. As consequence, there are no isoperimetric regions of volume $1/3$ for this density.

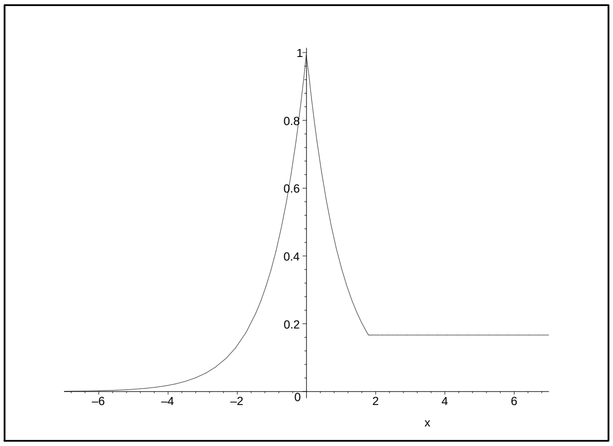

FIGURE 2. *A density for which minimizers are half-lines and bounded intervals.*

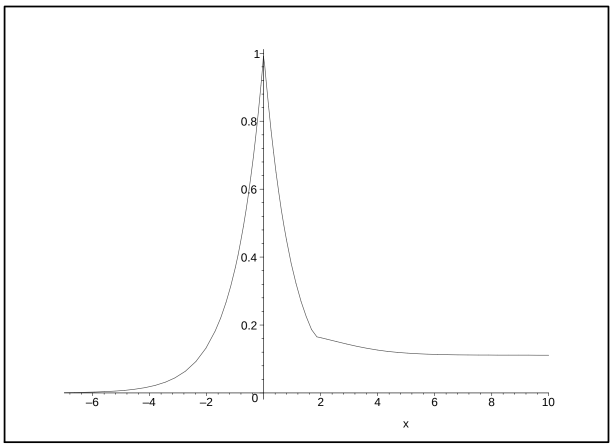

FIGURE 3. *A density for which minimizers do not always exist.*

We say that a density f is *decreasing-increasing* if $(-f)$ is increasing-decreasing. For these densities we have the following:

Theorem 4.7. *Let f be a decreasing-increasing density on $\mathbb{R}$. Then, isoperimetric regions exist for any given volume and they are bounded intervals in whose closure f attains its minimum.*

Proof. Take an open set Ω with finite volume and a point x_0 where f attains its minimum. It is easy to check that the bounded interval containing x_0 and with the same volume as Ω at both sides of x_0 has less perimeter than Ω. Finally, among intervals with fixed volume containing x_0 in its closure there is one of least perimeter. □

Now, we give some applications and improvements of the previous results for the particular cases of log-concave and log-convex densities. We begin with the following corollary, which is a direct consequence of Proposition 4.1, Theorem 4.3, and elementary properties of concave functions.

Corollary 4.8. *Let f be a log-concave density on $\mathbb{R}$. Then we have*

(i) *If the total measure is finite, then minimizers of any volume exist and they can be half-lines, unions of two disjoint half-lines, or bounded intervals where the maximum of the density is attained.*
(ii) *If both ends have infinite measure, then the density is constant and bounded intervals provide minimizers of any given volume.*
(iii) *If the density has infinite volume but one end has finite measure, then half-lines containing this end are the unique isoperimetric regions for fixed volume.*

Example 4.4 shows that all the different possibilities in Corollary 4.8 (i) can appear. C Borell ([**Bor3**, Corollary 2.2], see also [**BoH**, Corollary 13.8]) proved that half-lines are always minimizers for a log-concave density with finite total measure. In the next corollary we give a different proof of this fact showing also uniqueness of minimizers for strictly log-concave densities.

Corollary 4.9. *Let $f = e^{\psi}$ be a log-concave density on $\mathbb{R}$ with finite total measure. Then, half-lines are always isoperimetric regions of any given volume. Moreover, if the density is strictly log-concave, then half-lines are the unique minimizers.*

Proof. We have to compare the perimeter of the candidates provided by Corollary 4.8 (i). By taking complements we see that it is enough to compare the perimeter of bounded intervals and half-lines of the same measure. Fix an amount V of volume. Let x_V be the real number such that $\mathrm{vol}((x_V,+\infty)) = V$. For any $x \in (-\infty, x_V)$, let $y(x) > x$ be the unique value satisfying $\mathrm{vol}((x, y(x))) = V$. The perimeter of all bounded intervals enclosing volume V is given by the function $P(x) = f(x) + f(y(x))$. Clearly, $P(-\infty)$ and $P(x_V)$ represent the perimeter of the two half-lines of volume V. As $y(x)$ is increasing and the density is log-concave, we deduce that $P(x)$ is an absolutely continuous function with left and right derivatives at every point. In particular, the right derivative P'_r is given by

$$P'_r(x) = f(x)\,\{\psi'_r(x) + \psi'_r(y(x))\}, \qquad x \in (-\infty, x_V).$$

On the other hand, as ψ is concave, we get that ψ'_r is non-increasing and hence $P'_r(x)/f(x)$ is also non-increasing. Thus, $P(x)$ is monotonic or increasing–decreasing on $(-\infty, x_V)$. Anyway the infimum of $P(x)$ is achieved in a half-line of volume V. Moreover, if f is strictly log-concave, then the infimum of $P(x)$ is not attained on $(-\infty, x_V)$, so that the half-lines are the unique minimizers. □

Remark 4.10. Two relevant examples in probability and statistics where Corollary 4.9 is applied are the standard Gaussian density $f(x) = e^{-\pi x^2}$ and the logistic density $f(x) = e^{-x}\,(1+e^{-x})^{-2}$. As indicated in [**Bo**], for these densities it is also interesting to describe minimizers under a volume constraint of the functionals $\mathrm{vol}(\Omega + [-h,h])$ for any $h > 0$. In [**BoH**, Remark 13.9] it is pointed out that half-lines are solutions to this problem. In higher dimension, we can consider the same problem with the cube $[-h,h]^{n+1}$. It was shown in [**Bo**, Theorem 1.1] that half-spaces are minimizers for any product measure μ^{n+1} in $\mathbb{R}^{n+1}$ provided μ is log-concave with finite total volume (see also [**BoH**, Corollary 15.3] for the particular case of the logistic density).

Now we state a result similar to Corollary 4.8 where we completely describe isoperimetric regions for log-convex densities.

Corollary 4.11. *Let $f = e^{\psi}$ be a log-convex density on $\mathbb{R}$. Then we have*

(i) *If both ends have infinite measure and $f(-\infty) = f(+\infty) = +\infty$, then isoperimetric regions of any volume exist and they are bounded intervals in whose closure the density attains its minimum. Moreover, if f is strictly log-convex, then we have uniqueness of minimizers for given volume.*

(ii) *If both ends have infinite measure but there is one end E with $f(E) < +\infty$, then the isoperimetric profile is constant and it is approached or attained by a bounded interval going off to E.*
(iii) *If one end has finite measure, then the half-lines containing this end are the unique minimizers for given volume.*

Proof. The claim follows by using Proposition 4.1 and Theorem 4.7. The uniqueness in statement (i) follows from the argument in the proof of Corollary 4.9 since strict convexity of the density implies that the perimeter of bounded intervals with fixed volume achieves its minimum only at one point. □

As a direct consequence of the previous corollary and Example 3.3 we deduce the following result, which solves Conjecture 3.12 in dimension one.

Corollary 4.12. *Let f be a smooth, symmetric, strictly log-convex density on $\mathbb{R}$. Then, for a given volume, the symmetric interval of this volume is the unique minimizer.*

The comparison arguments in this section allow to study the isoperimetric problem in $[0, +\infty)$ or in a bounded interval $[a, b]$ with unimodal densities. In these settings we can prove similar results to Proposition 4.1, Theorem 4.3 and Theorem 4.7. The proofs are left to the reader.

Theorem 4.13. *Let f be a unimodal density on $[0, +\infty)$. Then we have*

(i) *If f is increasing, then the unique minimizers are the intervals $(0, x)$. If f is decreasing and $E = +\infty$ has finite measure, then the half-lines containing E are the unique minimizers. If f is decreasing and E has infinite measure, then the isoperimetric profile equals $2f(E)$ and it is approached or attained by a bounded interval going off to $+\infty$.*
(ii) *If f is increasing-decreasing and a minimizer of given volume exists, then it must coincide with an interval $(0, x)$, a half-line containing $+\infty$, a bounded interval where f attains its maximum or equals it one-sided minimum, or the complement of one of these intervals. If it does not exist, the infimum perimeter is approached by a bounded interval going off to $+\infty$.*
(iii) *If f is decreasing-increasing, then minimizers of any measure exist and they are bounded intervals in whose closure f attains its minimum.*

Now we shall state the corresponding result for the isoperimetric problem inside a bounded interval $[a, b]$. Observe that in this case the existence of minimizers is ensured by compactness.

Theorem 4.14. *Let f be a unimodal density on a bounded interval $[a, b]$. Then we have*

(i) *If f is monotonic, then any isoperimetric region is an interval whose closure contains the boundary point of $[a, b]$ where f attains its minimum.*
(ii) *If f is increasing-decreasing, then a minimizer must coincide with an interval whose closure contains a boundary point, an interval where f attains its maximum or equals its one-sided minimum, or the complement of one of these intervals.*
(iii) *If f is decreasing-increasing, then any minimizer is an open interval in whose closure f attains its minimum.*

The techniques in this section can also be applied to study the *free boundary problem* in $[0, +\infty)$ or $[a, b]$ which consists of finding global minimizers under a volume constraint of the *perimeter relative to* $(0, +\infty)$ or (a, b), respectively. This means that the boundary points of these intervals do not contribute to perimeter. For the case of $[0, +\infty)$ we have:

Theorem 4.15. *Let f be a unimodal density on $[0, +\infty)$. Then we have*

(i) *If f is increasing, then the unique minimizers for the free boundary problem in $[0,+\infty)$ are intervals of the form $(0,x)$. If f is decreasing and $E=+\infty$ has finite measure, then minimizers exist and they are half-lines containing E. If E has infinite measure and a minimizer exists, then it must coincide with an interval $(0,x)$ or a bounded interval where f equals its minimum. If a minimizer does not exist the infimum perimeter equals $2f(E)$.*
(ii) *If f is increasing-decreasing and a minimizer of given volume exists, then it is an interval $(0,x)$, a half-line containing $+\infty$, a bounded interval where f attains its maximum or equals its right-side minimum, or the complement of one of these intervals. If it does not exist, then the infimum perimeter is approached by a bounded interval going off to $+\infty$.*
(iii) *If f is decreasing-increasing then minimizers of any given volume are provided by intervals $(0,x)$ or bounded intervals in whose closure f attains its minimum.*

For the free boundary problem in $[a,b]$, existence of minimizers is assured by compactness. As to the description of isoperimetric regions, we can prove the following:

Theorem 4.16. *Let f be a unimodal density on $[a,b]$. Then*

(i) *If f is monotonic then the unique minimizers for the free boundary problem are intervals whose closure contains a boundary point.*
(ii) *If f is increasing-decreasing, then isoperimetric regions are provided by intervals where f attains its maximum, or whose closure contains a boundary point of $[a,b]$, or complements of these intervals.*
(iii) *If f is decreasing-increasing, then minimizers are intervals whose closure contains a boundary point of $[a,b]$ or a value where the minimum of f is attained, or complements of these intervals.*

5. Isoperimetric inequality for the density $\exp(|x|^2)$

In this last section of the paper we solve the isoperimetric problem in $\mathbb{R}^{n+1}$ with the radial log-convex density $f(x)=\exp(c|x|^2)$, where c is a positive constant. Precisely, we will prove that Conjecture 3.12 holds for this density: round balls about the origin provide isoperimetric regions of any given volume, like Euclidean space ($c=0$) and unlike Gauss space ($c<0$). As we pointed out in the Introduction, the proof combines Steiner symmetrization in axis directions as was employed by L. Bieberbach [**Bi**] together with Hsiang symmetrization [**H**]. We will also show uniqueness by a detailed analysis of the situation where an axis symmetrization of a minimizer produces a round ball.

The use of Steiner symmetrization in our setting is natural since the ambient density can be seen as a rotationally invariant product measure. Let us recall some facts about this construction; see [**R**, Section 3.2] for details. Let Ω be a compact set in $\mathbb{R}^{n+1}$. Consider a hyperplane π in $\mathbb{R}^{n+1}$ containing the origin. The restriction of the ambient density to any straight line orthogonal to π provides a smooth, symmetric, strictly log-convex density. We define the *symmetrization of Ω with respect to π* as the set Ω^* whose intersection with any straight line R orthogonal to π is the isoperimetric region in R of the same weighted length as $\Omega\cap R$. By Corollary 4.12 this will be an interval centered at $\pi\cap R$. It is clear that Ω^* is symmetric with respect to π. The main property of this construction is that it preserves volume (by Fubini's theorem) while decreasing perimeter.

Lemma 5.1 ([**R**, Proposition 8])**.** *For any hyperplane π through the origin in $\mathbb{R}^{n+1}$, the Steiner symmetrization Ω^* of a compact set Ω satisfies $\mathrm{vol}(\Omega^*)=\mathrm{vol}(\Omega)$ and $P(\Omega^*)\leqslant P(\Omega)$.*

Now, we will proceed to prove our main result in this section.

Theorem 5.2. *In $\mathbb{R}^{n+1}$ with the density $f(x) = \exp(c|x|^2)$, $c > 0$, round balls about the origin uniquely minimize perimeter for given volume.*

Proof. First observe that bounded minimizers of any given volume exist for this density by Theorem 2.2. Let us prove that round balls centered at the origin are isoperimetric regions. Take a minimizer Ω of volume $V > 0$. We apply Steiner symmetrization to $\overline{\Omega}$ with respect to any coordinate hyperplane in $\mathbb{R}^{n+1}$ so that we produce a minimizer Ω^* which is symmetric with respect to any of these hyperplanes and has connected boundary. In particular, Ω^* is centrally symmetric. Thus any hyperplane π through the origin divides Ω^* in two sets Ω_i^* contained in the corresponding open half-spaces π_i and with the same volume. Note that the reflection with respect to π preserves the perimeter relative to any π_i since the density is radial. It follows that $P(\Omega_1, \pi_1) = P(\Omega_2, \pi_2)$; otherwise, we would obtain by reflection a set with the same volume as Ω^* and strictly less perimeter. Therefore each Ω_i^* together with its reflection is a new minimizer of volume V. By the regularity properties in Theorem 2.8 and unique continuation for (real-analytic) generalized constant mean curvature surfaces, $\partial\Omega^*$ is symmetric across any hyperplane π through the origin. We conclude that Ω^* coincides with a ball centered at the origin.

To prove uniqueness, by induction it suffices to show that if symmetrization of a minimizer Ω with respect to a coordinate hyperplane π produces a ball B, then Ω is a ball. We can suppose that $\pi = \{x_{n+1} = 0\}$. Let $D \subset \pi$ be the projection of Ω. By Theorem 2.8 and Sard's theorem, for almost all $p \in D$ straight lines near p orthogonal to π intersect $\Sigma = \partial\Omega$ transversally at a fixed even number of points p_i, where Σ is the graph over $D_p \subset D$ of a smooth function h_i (if we did not know that Ω is bounded, we would allow p_i to be $\pm\infty$). Denote by $A \subseteq D$ the set of such points p. By the definition of Steiner symmetrization

$$\sum_{i\ odd} \int_{h_i}^{h_{i+1}} f(x)\, dx = 2 \int_0^{h^*} f(x)\, dx \qquad \text{on } D_p,$$

where h^* is the height function of ∂B with respect to π. As a consequence

$$\sum_{i\ odd} \big(f(h_{i+1})\, \nabla h_{i+1} - f(h_i)\, \nabla h_i\big) = 2 f(h^*)\, \nabla h^* \qquad \text{on } D_p,$$

so that we get

$$\sum_j f(h_j)\, |\nabla h_j| \geqslant 2 f(h^*)\, |\nabla h^*| \qquad \text{on } D_p.$$

On the other hand, by Corollary 4.12 we have

$$\sum_j f(h_j) \geqslant 2 f(h^*) \qquad \text{on } D_p,$$

and equality holds if and only if the corresponding slice of Ω is an interval centered at π.

Now we apply Lemma 5.3 below with $\alpha_j = f(h_j(p))$, $a_j = |\nabla h_j(p)|$, $\alpha = f(h^*(p))$ and $a = |\nabla h^*(p)|$. We get

$$\sum_j f(h_j(p)) \sqrt{1 + |\nabla h_j(p)|^2} \geqslant 2 f(h^*(p)) \sqrt{1 + |\nabla h^*(p)|^2}, \qquad p \in A, \tag{5.1}$$

with equality if and only if $|\nabla h_j(p)| = |\nabla h^*(p)|$ for any j, and the slice of Ω passing through p is an interval centered at π.

Finally we use the coarea formula and inequality (5.1) to obtain

$$P(\Omega) = \int_\Sigma f\, da \geqslant \int_A \Big(\sum_j f(h_j(p)) \sqrt{1 + |\nabla h_j(p)|^2} \Big)\, da$$
$$\geqslant \int_A 2f(h^*(p)) \sqrt{1 + |\nabla h^*(p)|^2}\, da = P(B),$$

where in the last equality we have used that $D - A$ does not contribute to the perimeter of B. As Ω is a minimizer we have equality above and hence in (5.1) too. It follows that for every $p \in A$ the slice of Ω passing through p is a symmetric interval of the same length as the corresponding slice for B. Thus, up to a set of measure zero, Ω coincides with a round ball about the origin. □

Lemma 5.3. *Suppose that we have finitely many nonnegative real numbers with $\sum_j \alpha_j\, a_j \geqslant 2\alpha\, a$ and $\sum_j \alpha_j \geqslant 2\,\alpha$. Then the following inequality holds*

$$\sum_j \alpha_j \sqrt{1 + a_j^2} \geqslant 2\alpha \sqrt{1 + a^2},$$

with equality if and only if $a_j = a$ for every j and $\sum_j \alpha_j = 2\alpha$.

Proof. The function $g(x) = \sqrt{1 + x^2}$ is strictly convex with $0 < g'(x) < 1$ for any $x > 0$. Let $\alpha_0 = \sum_j \alpha_j$. We claim that

$$\sum_j \frac{\alpha_j}{\alpha_0}\, g(a_j) \geqslant g\big(\sum_j \frac{\alpha_j}{\alpha_0}\, a_j \big) \geqslant g\big(\frac{2\alpha}{\alpha_0}\, a \big) \geqslant \frac{2\alpha}{\alpha_0}\, g(a).$$

The first inequality holds because g is convex. The second and third inequalities come from the fact that $0 < g'(x) < 1$ for $x > 0$. If equality holds in the second inequality, then $\sum_j \alpha_j\, a_j = 2\alpha\, a$. If equality holds in the third inequality too, then $2\alpha = \alpha_0 = \sum_j \alpha_j$. If equality holds in the first inequality as well, then $a_j = a$ for every j. □

We finish the paper with an eigenvalues comparison theorem obtained as a consequence of the isoperimetric inequality in Theorem 5.2. For a smooth bounded domain Ω in $\mathbb{R}^{n+1}$, we consider the second order differential operator L on $C_0^\infty(\Omega)$ whose invariant measure has density $f(x) = \exp(c|x|^2)$ $(c \geqslant 0)$, namely

$$(Lu)(x) = (\Delta u)(x) - 2c\, \langle x, (\nabla u)(x) \rangle, \qquad u \in C_0^\infty(\Omega), \quad x \in \Omega, \tag{5.2}$$

where Δ denotes the Euclidean Laplace operator on Ω.

Corollary 5.4. *Let Ω be a smooth bounded domain in $\mathbb{R}^{n+1}$. Then, the lowest non-zero eigenvalue $\lambda_1(\Omega)$ for the second order differential operator* (5.2) *with Dirichlet boundary condition on $\partial\Omega$ satisfies*

$$\lambda_1(\Omega) \geqslant \lambda_1(B),$$

where B is the round ball centered at the origin with the same volume as Ω for the density $f(x) = \exp(c|x|^2)$, $c > 0$. Moreover, equality holds if and only if $\Omega = B$.

Proof. The comparison is an adaptation of the symmetrization technique used to prove the Faber-Krahn Inequality in $\mathbb{R}^{n+1}$ (see [**Ch1**, p. 87]), which corresponds to the desired inequality for the case $c = 0$. □

REFERENCES

[BdC] João Lucas Barbosa and Manfredo do Carmo, *Stability of hypersurfaces with constant mean curvature*, Math. Z. **185** (1984), no. 3, 339–353.

[Ba] Vincent Bayle, *Propriétés de concavité du profil isopérimétrique et applications*, Thèse de Doctorat. 2003.

[Bi] Ludwig Bieberbach, *Über eine Extremaleigenschaft des Kreises*, J.-ber. Deutsch. Math.-Verein. **24** (1915), 247–250.

[Bo] Sergei Bobkov, *Extremal properties of half-spaces for log-concave distributions*, Ann. Probab. **24**, 1996, 35–48.

[BoH] Sergei Bobkov and Christian Houdré, *Some connections between isoperimetric and Sobolev-type inequalities*, Mem. Amer. Math. Soc. **129**, 1997.

[Bor1] Christer Borell, *The Brunn-Minkoski inequality in Gauss space*, Invent. Math. **30** (1975), no. 2, 207–216.

[Bor2] ———, *The Orntein-Uhlenbeck velocity process in backward time and isoperimetry*, preprint Chalmers University of Technology 1986-03/ISSN 0347-2809.

[Bor3] ———, *Intrinsic bounds for some real-valued stationary random functions*, Lecture Notes in Math. 1153, 72–95, Springer, Berlin, 1985.

[CK] E. A. Carlen and C. Kerce, *On the cases of equality in Bobkov's inequality and Gaussian rearrangement*, Calc. Var. **13** (2001), 1–18.

[Ch1] Isaac Chavel, *Eigenvalues in Riemannian geometry*, Pure and Applied Mathematics, vol. 115, Academic Press Inc., Orlando, FL, 1984.

[Ch2] ———, *Isoperimetric Inequalities. Differential Geometric and Analytic Perspectives*, Cambridge Tracts in Mathematics, no. 145, Cambridge University Press, Cambridge, 2001.

[Eh1] Antoine Ehrhard, *Symétrisation dans l'espace de Gauss*, Math. Scand. **53** (1982), 281–301.

[Eh2] ———, *Éléments extrémaux pour les inégalités de Brunn-Minkowski gaussiennes*, Ann. Inst. H. Poincaré Probab. Statist. **22** (1986), no. 2, 149–168.

[Gr] Misha Gromov, *Isoperimetry of waists and concentration of maps*, Geom. Funct. Anal. **13** (2003), 178–215.

[H] W. Y. Hiang, *A symmetry theorem on isoperimetric regions*, PAM-409 (1988), UC Berkeley.

[M1] Frank Morgan, *Clusters minimizing area plus length of singular curves*, Math. Ann. **299** (1994), 697–714.

[M2] ———, *Geometric measure theory: a beginner's guide*, third ed., Academic Press Inc., San Diego, CA, 2000.

[M3] ———, *Regularity of isoperimetric hypersurfaces in Riemannian manifolds*, Trans. Amer. Math. Soc. **355** (2003), no. 12, 5041–5052.

[M4] ———, *Manifolds with density*, Notices Amer. Math. Soc. **52** (2005), 853–858.

[RiR] Manuel Ritoré and César Rosales, *Existence and characterization of regions minimizing perimeter under a volume constraint inside Euclidean cones*, Trans. Amer. Math. Soc. **356** (2004), no. 11, 4601–4622.

[R] Antonio Ros, *The isoperimetric problem*, Global Theory of Minimal Surfaces (Proc. Clay Math. Inst. 2001 Summer School, MSRI, David Hoffman, ed.), Amer. Math. Soc., 2005, 175–209.

[Ro] Harold Rosenberg, *Hypersurfaces of constant curvature in space forms*, Bull. Sc. Math. **117** (1993), 211–239.

[Si] Leon Simon, *Lectures on geometric measure theory*, Proceedings of the Centre for Mathematical Analysis, Australian National University, vol. 3, Australian National University Centre for Mathematical Analysis, Canberra, 1983.

[ST] V. N. Sudakov and B. S. Tsirel'son, *Extremal properties of half-spaces for spherically invariant measures*, J. Soviet Math. (1978), 9–18.

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