Snell's law in an isoperimetric setting

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Abstract. In this work we focus on the isoperimetric problem in $\mathbb{R}^2$ endowed with a piecewise constant density. We will see that the boundary of an isoperimetric solution is not a smooth curve in general, since some corners may appear according to a rule analogous to the Snell refraction law from Optics.

Keywords: Snell's law, isoperimetric problem, manifolds with density
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INTRODUCTION

Given a surface M, the isoperimetric problem looks for the least-perimeter set in M enclosing a prescribed quantity of area. A priori, the existence of such a set is not assured, and it will be called *isoperimetric region* if it exists. In the literature we can find several papers classifying the isoperimetric regions for different surfaces ([9], [5], [2]).

In the last years, this problem has been studied considering a *density function* on the plane, which is just a positive function that weights the area and perimeter functionals (see [1], [4], [8]). More precisely, if $\mathbb{R}^2$ is endowed with a density $f : \mathbb{R}^2 \to \mathbb{R}^+$, the area and the perimeter of a set $\Omega \subset \mathbb{R}^2$ will be given by

$$A(\Omega) = \int_{\Omega} f\, da, \quad P(\Omega) = \int_{\partial\Omega} f\, dx, \tag{1}$$

where da and dx are the area and perimeter elements [8]. Observe that when $f = 1$, we obtain the standard Euclidean definitions of area and perimeter, but in general we will have that the area and the perimeter of a set Ω will depend on the values of the density on the points of Ω and $\partial\Omega$. We remark that this new setting, apart from being a generalization of the classical isoperimetric problem, corresponds to a change of the measure in $\mathbb{R}^2$ (see [7] for further details).

In particular, in this work we will focus on *piecewise constant densities* defined on $\mathbb{R}^2$, summarizing some interesting results we have obtained in [3]. We note that this kind of densities will always have a set of discontinuities, and up to [3], the isoperimetric problem in a discontinuous density setting had not been treated in literature. Also observe that, due to definitions (1), each different density will yield a different isoperimetric problem.

Example 1 *Fix $\mu \in \mathbb{R}$, $\mu > 1$, and consider the function in $\mathbb{R}^2$ defined by taking value one in the lower half-plane $\{x_2 \leq 0\}$, and taking value μ in the upper half-plane $\{x_2 > 0\}$ (see Figure 1). This is a piecewise constant positive function, and so it gives a piecewise*

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constant density called the half-plane density. For instance, observe that a ball of radius $r = 1/2$ has different area and perimeter depending whether it is contained in the lower or in the upper half-plane, from the definitions (1) given above (if it is contained in the lower half-plane, the area is πr^2 and the perimeter is $2\pi r$, whereas if contained in the upper half-plane, the area equals $\mu\pi r^2$ and the perimeter is $2\mu\pi r$). We will describe the isoperimetric regions for this density in Subsection 2.1.

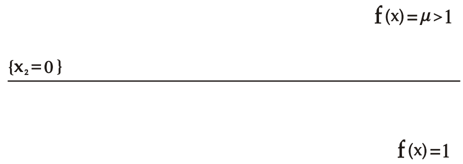

FIGURE 1. The half-plane density in $\mathbb{R}^2$

Example 2 *Let B be the closed unit ball in the plane, and fix $\lambda \in (0,1)$. We define the ball density by the function in the plane taking value λ in B, and taking value one outside B (see Figure 2). As this function is positive and piecewise constant, it yields a piecewise constant density on the plane. We will also show the corresponding isoperimetric regions for this density in Subsection 2.2.*

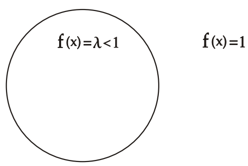

FIGURE 2. The ball density in $\mathbb{R}^2$

In this setting, an interesting phenomenon arises due to the discontinuity of the density function. In the case that the boundary curve of an isoperimetric region crosses the set of discontinuities, a change of direction will happen and a corner will appear on the crossing point. This behaviour is analogous to the one described in Optics by the *Snell refraction law* (for instant, see [10]), which explains the change of direction of a ray of ligth when passing through two different media. In that situation, the ray of light looks for the least-time path, which is determined by means of the Snell law expression. In some sense, this fact agrees with the general isoperimetric approach, which tries to minimize the perimeter under an area constraint (that is, in both cases the aim is *minimizing the energy*). We shall see in Theorem 1.2 that a similiar rule is satisfied in our setting, with the constant values of the density playing the role of the refraction's coeficients.

We remark that, in this piecewise constant density setting, this last property constitutes the main difference with respect to the classical isoperimetric problem (without considering a density), where the isoperimetric boundaries are always smooth curves [6].

1. PROPERTIES OF THE ISOPERIMETRIC BOUNDARIES

Let $\{\Omega_1, \ldots, \Omega_k\}$ be a family of closed sets partitioning $\mathbb{R}^2$, such that the interiors $\mathring{\Omega}_1, \ldots, \mathring{\Omega}_k$ are disjoint sets, and call

$$\Gamma = \bigcup_{i=1}^{k} \partial \Omega_i.$$

Consider a piecewise constant density $f : \mathbb{R}^2 \to \mathbb{R}^+$, that is, a function defined in the plane as

$$f(p) = \begin{cases} f_i, & p \in \mathring{\Omega}_i, \\ \min\{f_i : p \in \partial\Omega_i\}, & p \in \Gamma, \end{cases}$$

where $f_1, \ldots, f_k$ are positive real numbers. Note that Γ is the set of discontinuities of f.

Let E be an isoperimetric region in the plane with density f, and call $\Sigma = \partial E$ its boundary. In this setting, Σ satisfies an interesting property: the geodesic curvature of $\Sigma - \Gamma$ is constant (when considering the Euclidean metric), and so it will be composed of arcs of a circle (all of them with the same radius), or of line segments. We point out that this kind of density functions defined on the plane will not affect the value of the geodesic curvature, since their derivatives vanish in each set $\mathring{\Omega}_i$ (the definition of the geodesic curvature for general densities can be found in [8, §. 3]). On the other hand, we remark that some pieces of Γ may bound our isoperimetric region (that is, Σ may contain pieces of Γ).

The main property in this setting is the following. It may happen that the isoperimetric boundary Σ crosses transversally Γ, passing through regions of the plane with different values of density. If this is the case, a similar rule to the Snell law must be satisfied, as we will see in Theorem 1.2. In order to prove this fact, we need the first variations of area and perimeter (see [3, Prop. 2.11 and eq. (11)]). Following Proposition 1.1 gives these expressions in a more general way, for piecewise regular densities. These densities are piecewise defined in each set $\mathring{\Omega}_i$ by means of a smooth positive function f_i. It is clear that when f_i are positive real numbers, we will have a piecewise constant density of our family. A detailed proof of this Proposition can be found in [3].

Proposition 1.1 *Let f be a piecewise regular density in $\mathbb{R}^2$, and Γ its set of discontinuities. Consider $E \subset \mathbb{R}^2$ and call $\Sigma = \partial E$ its boundary. Assume $\Sigma \cap \Gamma \neq \emptyset$, and consider $p \in \Sigma \cap \Gamma$ with $p \in \partial\Omega_i \cap \partial\Omega_j$. For a smooth one-parameter variation $\{\Phi_t : \mathbb{R}^2 \to \mathbb{R}^2\}_{t \geq 0}$ with compact support in a neighborhood of p, set $A(t) = A(\Phi_t(E))$ and $P(t) = P(\Phi_t(E))$. Then, if we denote by X the associated vector field of the variation, and by ν_Σ the inward unit normal vector to Σ, we have that the first variation of* *area and perimeter of E are given by*

$$A'(0) = -\int_{\Sigma} f u da, \tag{2}$$

and

$$P'(0) = \int_{\Sigma} \langle \nabla \psi, \nu_{\Sigma} \rangle f u dx - \int_{\Sigma} H f u dx + f_i \langle X, \nu_{\Sigma_i} \rangle (p) + f_j \langle X, \nu_{\Sigma_j} \rangle (p), \tag{3}$$

where $u =: \langle X, \nu_{\Sigma} \rangle$, $f = e^{\psi}$, *H is the geodesic curvature of* Σ *and* $\Sigma_i = \Sigma \cap \Omega_i$.

When above Proposition 1.1 is applied for a piecewise constant density, we obtained the following consequence, which constitutes the main result of our work:

Theorem 1.2 *Let f be a piecewise constant density in* $\mathbb{R}^2$, *with* Γ *the set of discontinuities. Let E be an isoperimetric region in the plane with density f, and* $\Sigma = \partial E$. *Assume that* Σ *crosses transversally* Γ *at a point* $p \in \partial\Omega_i \cap \partial\Omega_j$. *Then*

$$f_i \cos \alpha_i = f_j \cos \alpha_j, \tag{4}$$

where α_i, α_j *are the angles at p between* Σ *and* Γ *(see Figure 3).*

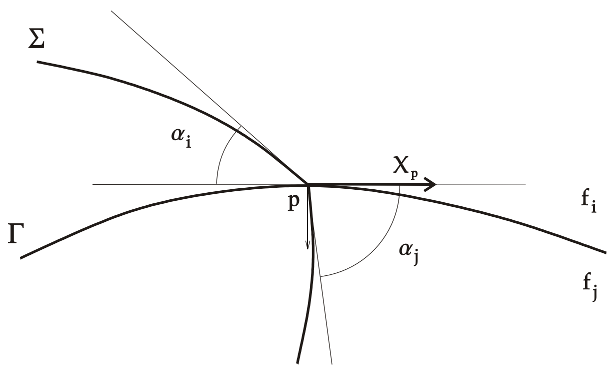

FIGURE 3. Snell's law in our isoperimetric setting

Proof. We give an sketch of the proof of (4). Consider a variation of E with compact support contained in a neighborhood of p preserving the area enclosed, that is, $A'(0) = 0$, with $X(p)$ tangent to Γ (intuitively, you can construct such variation deforming E far away from p in order to balance the area change, see [3, Prop. 2.13] or [8, §. 3] for further details). As E is an isoperimetric region, we have that its boundary Σ is a stationary set, and so $P'(0) = 0$. Observe that the first term in (3) vanishes since $\nabla\psi = 0$ (recall that f is piecewise constant, and so ψ is). Moreover, taking into account that the geodesic curvature H of Σ is constant, the second term in (3) also vanishes due to the area-preserving condition (2). Hence we have

$$f_i \langle X, \nu_{\Sigma_i} \rangle (p) + f_j \langle X, \nu_{\Sigma_j} \rangle (p) = 0,$$

which easily yields Snell's law (4) by applying the standard definition of the scalar product, taking into account that

$$\cos\alpha_i = \cos(-X(p), \nu_{\Sigma_i}(p)), \quad \cos\alpha_j = \cos(X(p), \nu_{\Sigma_j}(p)),$$

from Figure 3.

Remark 1.3 *We note that Theorem 1.2 follows easily from Proposition 1.1, since the density is piecewise constant. In a more general case, when considering piecewise regular densities, this Theorem also holds (even in $\mathbb{R}^n$), although the proof is more elaborate. For interested readers, it can be found in [3, Prop. 2.13].*

2. SOME PARTICULAR EXAMPLES

As particular situations, in this Section we will describe the isoperimetric regions for the piecewise constant densities described in Examples 1 and 2 in the Introduction, the *half-plane density*, and the *ball density* with $\lambda \in (0,1)$. We shall check that the isoperimetric regions satisfy the Snell law (as stated in Theorem 1.2) when the boundaries cross transversally the set of discontinuities of the density, and that different kinds of isoperimetric regions may appear for a given density, depending on the prescribed quantity of area. We shall omit the proofs of Theorems 2.1 and 2.4, which can be found in [3, §. 3].

2.1. The half-plane density

Consider $\mathbb{R}^2$ endowed with the half-plane density from Example 1, taking value one in the lower half-plane $\{x_2 \leq 0\}$ and value $\mu > 1$ in the upper half-plane $\{x_2 > 0\}$. First, we point out that the existence of isoperimetric regions for this density is assured for any value of the area (essentially because any minimizing sequence is convergent, see [3, Th. 3.2]). The key result in this case is that any isoperimetric solution must be contained in the lower half-plane [3, Prop. 3.1]. Hence, from the standard classical isoperimetric problem, we deduce the following consequence.

Theorem 2.1 *([3, Th. 3.2]) The isoperimetric region of area v in the plane for the half-plane density with $\mu > 1$ is a round ball contained in $\{x_2 \leq 0\}$.*

Remark 2.2 *Observe that the boundary of any isoperimetric region for this density does not cross the corresponding set of discontinuities $\{x_2 = 0\}$, and so the Snell law cannot be applied in this case.*

Remark 2.3 *We note that the case $0 < \mu < 1$ in the half-plane density is analogous to the above one, with the isoperimetric regions consisting of round balls contained in the upper half-plane, where the density takes its minimum value.*

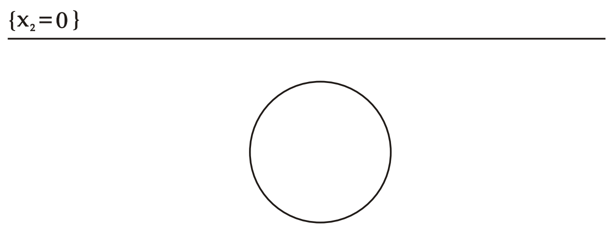

FIGURE 4. Isoperimetric regions in $\mathbb{R}^2$ for the half-plane density

2.2. The ball density

We now focus on the isoperimetric problem in $\mathbb{R}^2$ when considering the density defined in the Example 2, the ball density with $\lambda \in (0,1)$. Recall that this density takes value λ in the closed unit ball B, and value one outside B, being the corresponding set of discontinuities Γ equal to ∂B. As for the previous density, the existence of isoperimetric solutions is guaranteed (since any minimizing sequence is convergent [3, Th. 3.18]). In this case, after classifying the different isoperimetric candidates by using the properties from Section 1, and discarding non-possible solutions, we finally have the following result.

Theorem 2.4 *([3, Th. 3.23]) The isoperimetric region of area v in the plane for the ball density with $\lambda \in (0,1)$ is (see Figure 5):*

i) a ball of type a)*, entirely contained in B, if $v \leq \lambda\pi$;*
ii) a set of type b), bounded by a piece of Γ and an arc of circle outside B, if $\lambda\pi \leq v \leq v_1$;
iii) a set of type b) or c), if $v_1 < v < v_2$;
iv) a ball of type c), crossing orthogonally ∂B, if $v \geq v_2$.

where v_1, v_2 are certain real numbers depending on λ, $v_1, v_2 > \lambda\pi$.

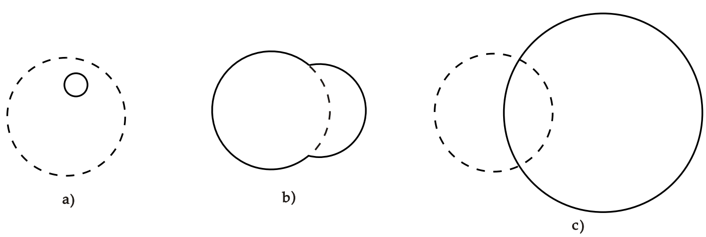

FIGURE 5. Isoperimetric regions in $\mathbb{R}^2$ for the ball density, $\lambda \in (0,1)$

Remark 2.5 *We point out that Snell's law (4) must be satisfied in the two vertices of sets of type b), and therefore we have some isoperimetric boundaries which are not smooth curves. On the other hand, notice that orthogonal crossing with the set of discontinuities* Γ *is allowed by Snell's law (4), and it actually occurs for sets of type c) below.*

Remark 2.6 *Regarding Theorem 2.4, we conjecture that* $v_1 = v_2$*, and so the possibility iii) from the statement will not occur in any case, due to some numerical computations we have done.*

Remark 2.7 *The description of the isoperimetric regions in the plane for others particular piecewise constant densities can be found in [3, §. 3], as well as some related open questions.*

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