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Least-perimeter partitions of the disk

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Abstract.

In this work we study the isoperimetric problem of partitioning a planar disk into n regions of prescribed areas using the least-possible perimeter. We obtain the regularity conditions that must be satisfied by the solutions, and solve completely the problem in the cases of two and three regions.

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1. Introduction

In the last years, the study of isoperimetric problems has been of great interest not only for its own nature but also for its relation with multitude of physical phenomena. Among them, the ones relating to isoperimetric partitions can model several natural situations.

In this work we treat the following partitioning problem: consider a planar disk D, and n positive numbers $a_1, \ldots, a_n$ whose sum is the total area of the disk. Then we want to find the way of dividing the disk into n regions R_i, each one of area a_i, with the least possible perimeter.

This question explains properly many situations in Nature: the first steps in the process of division of a cell, the shape of the interfaces separating different fluids in a round dish, and other ones (see [4, Ch. VII and VIII]).

For this problem, the main trouble is that regions may have several components, since we do not assume them to be connected. Therefore there are many ways of dividing the disk into n regions of the given areas.

From [3] we obtain the existence of a solution, for any number of regions we consider:

Theorem 1. (Existence Theorem) *Given a planar disk D and n positive numbers $a_1, \ldots, a_n$ such that $\sum_{i=1}^n a_i = area\,(D)$, there exists a least-perimeter way of dividing the disk into n regions of areas $a_1, \ldots, a_n$, consisting of smooth curves meeting in threes in the interior of the disk, and meeting ∂D only one curve at each time.*

2. Regularity Conditions

Let $D \subset \mathbb{R}^2$ be a closed disk, centered at the origin. Along this work, a *graph* C will consist of a finite number of vertices and edges in D such that at every interior vertex (a vertex in the interior of the disk) three edges meet, and at every vertex in ∂D, only one edge arrives. Note that, in view of Theorem 1, this is a natural definition.

We shall assume that a graph C decomposes the open disk into n regions R_i, possibly nonconnected as commented before, and we shall denote by $C_{ij} \subset C$ the curve separating two adjacent regions R_i and R_j (this curve may not be connected), by N_{ij} the normal vector to C_{ij} pointing into R_i, and by h_{ij} the geodesic curvature of C_{ij} with respect to N_{ij} (as we are working in the plane, the geodesic curvature coincides with the usual curvature of a curve).

We will call *minimizing graph* to the least-perimeter graph dividing the disk into n regions of the given areas.

Let $\varphi_t : C \to D$ be a smooth variation of a graph C, for t small, such that $\varphi_t(C \cap \partial D) \subset \partial D$. Denoting the associated vector field by $X = d\varphi_t/dt|_{t=0}$, with normal components $u_{ij} = X \cdot N_{ij}$ on C_{ij}, then the derivative of the area A_i enclosed by a region R_i at $t = 0$ is equal to

$$\left.\frac{dA_i}{dt}\right|_{t=0} = -\sum_{j\in I(i)} \int_{C_{ij}} u_{ij},$$

where $I(i) = \{j \neq i; R_j \text{ touches } R_i\}$.

Proposition 2. (First variation of length) *Given a graph C and a smooth variation $\varphi_t : C \to D$, the first derivative of the length functional of $\varphi_t(C)$ at $t = 0$ is equal to*

$$\left.\frac{dL}{dt}\right|_{t=0} = -\frac{1}{2} \sum_{\substack{i\in\{1,\ldots,n\}\\ j\in I(i)}} \left\{ \int_{C_{ij}} h_{ij}u_{ij} + \sum_{p\in\partial C_{ij}} X(p)\cdot\nu_{ij}(p) \right\}, \qquad (1)$$

where $\nu_{ij}(p)$ is the inner conormal to C_{ij} in p.

A graph will be said *stationary* if $\frac{dL}{dt}|_{t=0} = 0$, for any variation preserving the areas. Observe that stationary graphs are critical points for the length functional when the areas A_i are fixed. Since we want to minimize such a functional, it follows that a minimizing graph must be stationary.

From Proposition 2 we obtain the regularity conditions that stationary graphs (and then minimizing graphs) must verify:

Theorem 3. (Regularity Conditions) *Given a stationary graph C, the following regularity conditions must be satisfied:*

i) The curvature h_{ij} is constant on C_{ij}, and accordingly the edges will be circular arcs or line segments.

ii) The edges of C meet in threes at 120-degree angles in interior vertices.

iii) Three edges C_{ij}, C_{jk}, C_{ki} meeting in an interior vertex satisfy

$$h_{ij} + h_{jk} + h_{ki} = 0.$$

iv) The edges of C meet ∂D orthogonally.

Proof. By considering appropriate area-preserving variations in (1), the conditions above easily follow.

3. Minimizing graph for two regions

The problem for two regions can now be solved from Theorem 3. In this case no triple interior vertices can appear in the minimizing graph (any interior vertex would be surrounded by three different regions), and so any edge will be a circular arc or a line segment meeting ∂D orthogonally.

Let us assume that the minimizing graph has more than one edge. Then, by rotating one of them about the origin until touching another one, we will obtain a new minimizing graph (since the perimeter and the areas enclosed are preserved) with a non-allowed vertex in ∂D, which is contradictory.

Hence we have the following

Theorem 4. *Let $C \subset D$ be a minimizing graph for two given areas. Then C consists of a circular arc or a line segment meeting orthogonally ∂D.*

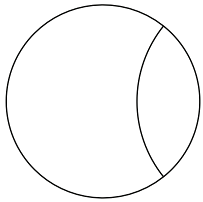

Figure 1: The least-perimeter partition of the disk into two given areas

4. Minimizing graph for three regions

Now we treat the problem for three regions. Straightforward calculations give the next proposition.

Proposition 5. (Second variation of length) *For a stationary graph C and a variation $\{\varphi_t\}$ that preserves the areas, the second derivative of the length functional at $t = 0$ is given by*

$$-\frac{1}{2}\sum_{\substack{i=1,\dots,n\\ j\in I(i)}}\left\{\int_{C_{ij}}(u''_{ij}+h^2_{ij}u_{ij})\,u_{ij}+\sum_{\substack{p\in\partial C_{ij}\\ p\in \mathrm{int}(D)}}\left(-q_{ij}u^2_{ij}+u_{ij}\frac{\partial u_{ij}}{\partial \nu_{ij}}\right)(p)\right. \quad (2)$$

$$\left.+\sum_{\substack{p\in\partial C_{ij}\\ p\in\partial D}}\left(u^2_{ij}+u_{ij}\frac{\partial u_{ij}}{\partial\nu_{ij}}\right)(p)\right\},$$

where $q_{ij}(p) = (h_{ki}+h_{kj})(p)/\sqrt{3}$, and R_k is the third region touching the vertex p.

Let us introduce an important concept in this work: for a region R_i, it is possible to define its *pressure* p_i as a real number such that, for any edge $C_{ij}\subset C$,

$$h_{ij} = p_i - p_j.$$

Then, for area-preserving variations *by stationary graphs* (that is, at each instant of the deformation we obtain stationary graphs, satisfying the conditions of Theorem 3), the second variation of length (2), expressed in terms of the pressures, turns

$$\frac{d^2L}{dt^2} = \sum_\alpha \frac{dp_\alpha}{dt}\,\frac{dA_\alpha}{dt}, \quad (3)$$

where α labels the *components* of the graph (recall that regions may have various components).

We will say that a stationary graph C is *stable* if $\frac{d^2L}{dt^2}\big|_{t=0} \geqslant 0$, for any area-preserving variation. This means that stable graphs are second order local minima for the length when the areas A_i are preserved. Then it is clear that a minimizing graph must be stable.

A *hexagonal component* of a region is a component bounded by six edges. Fix R_1 as the *region of largest pressure.* From Proposition 5 we get the following result:

Proposition 6. *Consider a stable graph dividing the disk into n regions. Then, R_1 has at most $n-1$ nonhexagonal components.*

In the case $n = 3$, this result allows us to discuss the possible configurations for a minimizing graph, once we have checked, using Gauss-Bonnet Theorem, that hexagonal components cannot occur:

Proposition 7. *Let $C \subset D$ be a minimizing graph dividing D into three regions. Then C is one of the graphs in Figure 2.*

Configuration 2(**10**) will be called *standard graph.* Now we will show how to discard the non-standard possibilities, to conclude that the minimizing graph for three regions is the standard graph.

Let us check that configurations 2(**1**) and 2(**2**) are unstable, and hence not minimizing. The motive is that both of them have a region with two *triangles* (components of three edges) touching ∂D.

Proposition 8. *Given a stationary graph with a triangle Ω touching ∂D, there exists a variation by stationary graphs such that*

(i) increases the area of Ω,

(ii) decreases the pressure of Ω, keeping the other pressures unchanged, and

(iii) leaves invariant the edges of the graph not placed in $\partial\Omega$.

Remark. For this variation, we have that $\frac{dp}{dt}\frac{dA}{dt} < 0$ in Ω.

Proposition 9. *Any stationary graph with a region with two triangles touching the boundary of the disk is unstable.*

Sketch of the proof. First, consider in each triangle Ω_i the variation of Proposition 8. By combining both variations, we can construct another variation by stationary graphs too, preserving the areas. Finally, by using Equation (3) to compute the second variation of length we have

$$\frac{d^2L}{dt^2}\bigg|_{t=0} = \sum_{\alpha} \frac{dp_\alpha}{dt}\frac{dA_\alpha}{dt} = \frac{dp_1}{dt}\frac{dA_1}{dt} + \frac{dp_2}{dt}\frac{dA_2}{dt} < 0,$$

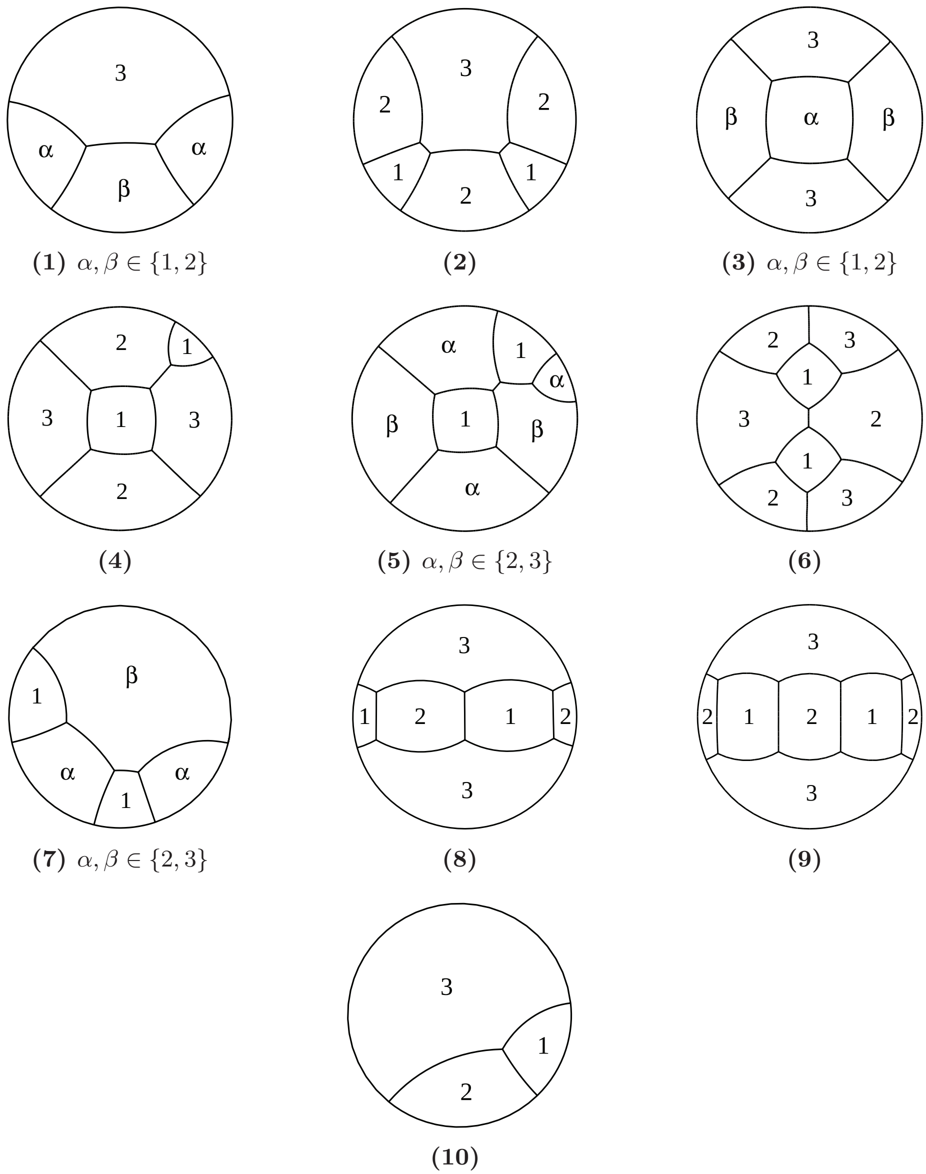

Figure 2: The ten possible configurations for a minimizing graph

since the only pressures that change along the deformation are the ones of the triangles Ω_1 and Ω_2.

Therefore, Proposition 9 yields instability of configurations 2(**1**) and 2(**2**).

A function $u : C = \bigcup_{i,j} C_{ij} \to \mathbb{R}$ is said a *Jacobi function* if the restrictions to C_{ij} satisfy

$$u''_{ij} + h^2_{ij}\, u_{ij} = 0.$$

The following result will show that configurations 2(**3**) to 2(**7**) of Figure 2 are unstable:

Proposition 10. ([2]) *Let C be a stationary graph separating the disk into three regions. If there exists a Jacobi function defined on C with at least four nodal domains, then C is unstable.*

In this case, the normal components of the rotations vector field about the origin constitute a suitable Jacobi function (recall that a nodal domain is a domain in C where that function does not vanish). Then, because of some symmetries we find in these configurations, we can apply Proposition 10 and obtain the instability of all of them.

Finally let us see that configuration 2(**8**) cannot be a minimizing graph. It is possible to construct a new configuration with the same perimeter and enclosing the same areas by this geometric transformation:

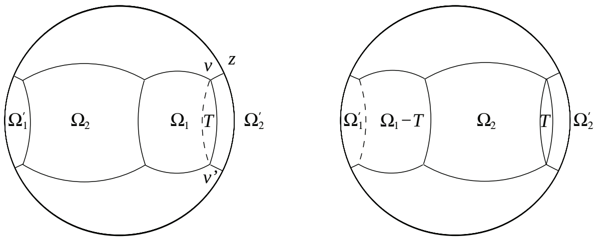

Figure 3: Geometric transformation creating two non-allowed vertices

As the new configuration has two non-allowed vertices, it cannot be minimizing, and so, neither configuration 2(**8**). The same argument can be applied for configuration 2(**9**).

Then, the nine non-standard possibilities have been discarded, so we obtain the following theorem:

Theorem 11. ([1]) *Let $C \subset D$ be a minimizing graph for three given areas. Then C is a standard graph.*

Remark. For two and three regions, the solutions are *unique* up to rigid motions of the disk.

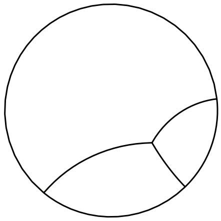

Figure 4: The least-perimeter partition of the disk into three given areas

Acknowledgments

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