The moduli space of three-qutrit states

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We study the invariant theory of trilinear forms over a three-dimensional complex vector space, and apply it to investigate the behavior of pure entangled three-partite qutrit states and their normal forms under local filtering operations (SLOCC). We describe the orbit space of the SLOCC group $\mathrm{SL}(3,\mathbb{C})^{\times 3}$ both in its affine and projective versions in terms of a very symmetric normal form parametrized by three complex numbers. The parameters of the possible normal forms of a given state are roots of an algebraic equation, which is proved to be solvable by radicals. The structure of the sets of equivalent normal forms is related to the geometry of certain regular complex polytopes.

I. INTRODUCTION

The invariant theory of trilinear forms over a three-dimensional complex vector space is an old subject with a long history, which, as we shall see, appears even longer if we take into account certain indirect but highly relevant contributions.[1–4] This question has been recently revived in the field of Quantum Information Theory as the problem of classifying entanglement patterns of three-qutrit states.

Indeed, since the advent of quantum computation and quantum cryptography, entanglement has been promoted to a resource that allows quantum physics to perform tasks that are classically impossible. Quantum cryptography[5,6] proved that this gap even exists with small systems of two entangled qubits. Furthermore, it is expected that the study of higher dimensional systems and of multipartite (e.g., 3-partite) states would lead to more applications. A seminal example is the so-called 3-qutrit Aharonov-state, which "*is so elegant it had to be useful*":[7] Fitzi, Gisin, and Maurer[7] found out that the classically impossible Byzantine agreement problem[8] can be solved using 3-partite qutrit states. From a more fundamental point a view, the Aharonov state led to nontrivial counterexamples of the conjectures on additivity of the relative entropy of entanglement[9] and of the output purity of quantum channels.[10] Obviously, these results provide a strong motivation for studying 3-partite qutrit states. Furthermore, interesting families of higher-dimensional states are perfectly suited to address questions concerning local realism and Bell inequalities (see, e.g., Ref. 11 for a study of three-qutrit correlations).

It is therefore of interest to find some classification scheme for three-qutrit states. A possible direction is to look for classes of equivalent states, in the sense that they are equivalent up to local unitary transformations[12–14] or local filtering operations (also called SLOCC operations).[14–19] In the case of three qubits, especially the last classification proved to yield a lot of insights (the

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classification up to local unitaries has too many parameters left); the reason for that is that in the closure of each generic orbit induced by SLOCC operations, there is a unique state (up to local unitary transformations) with maximal entanglement.[14,17]

In Ref. 19, a numerical method converging to such a maximally entangled state has been described. It has been experimentally observed that, when applied to a three-qutrit state, this method converged to a very special normal form. We shall provide a formal proof of this property, and then study in some detail the geometry of those normal forms. Precise statements of the results are summarized in the forthcoming section.

II. RESULTS

Let $V=\mathbb{C}^3$ and $\mathcal{H}=V\otimes V\otimes V$ regarded as a representation of the group $G=\mathrm{SL}(3,\mathbb{C})^{\times 3}$. The elements of $\mathcal{H}$ will be interpreted either as three-qutrit states

$$|\psi\rangle = \sum_{i,j,k=0}^{2} A_{ijk}|i,j,k\rangle \tag{1}$$

or as trilinear forms

$$f = f(\mathbf{x},\mathbf{y},\mathbf{z}) = \sum_{i,j,k=0}^{2} A_{ijk}x_i y_j z_k, \tag{2}$$

that is, we identify the basis state $|ijk\rangle$ with the monomial $x_i y_j z_k$. If $g=(g^{(1)},g^{(2)},g^{(3)})\in G$ is a triple of matrices, we define $x_i'=\Sigma_p g_{ip}^{(1)}x_p$, $y_j'=\Sigma_q\, g_{jq}^{(2)}y_q$, $z_k'=\Sigma_r\, g_{kr}^{(3)}z_r$, and the coefficients A_{ijk}' by the condition

$$\sum A_{ijk}'x_i'y_j'z_k' = \sum A_{ijk}x_i y_j z_k, \tag{3}$$

the action of G on $\mathcal{H}$ being defined by

$$g\cdot A = \sum A_{ijk}'x_i y_j z_k, \tag{4}$$

It has been shown by Vinberg[20] that a generic state can be reduced to the normal form

$$A_{ijk}' = u\delta_{ijk} + \frac{w-v}{2}\epsilon_{ijk} + \frac{w+v}{2}|\epsilon_{ijk}| \tag{5}$$

(where δ_{ijk} is the Kronecker symbol and ϵ_{ijk} the completely antisymmetric tensor) by an appropriate choice of $g\in G$.

Our first result is as follows.

Theorem II.1: *When applied to a generic 3-qutrit state (1) the numerical algorithm of Ref. 19 converges to a state which is a Vinberg normal form, generically in the same G-orbit as* $|\psi\rangle$.

As proved in Refs. 14 and 19, the normal form $|\psi'\rangle$ is unique up to local unitary transformations. More precisely, we have the following.

Theorem II.2: *A generic state has exactly* 648 *different normal forms. For special states, this number can be reduced to* 216, 72, 27 *or* 1. *Moreover, the coefficients* u, v, w *of the normal form can be computed algebraically*.

Theorem II.3: *The coefficients of the normal forms are determined, up to a sign, by an algebraic equation of degree* 1296, *which is explicitly solvable by radicals.*

To form this equation, we need some notions of invariant theory.

A polynomial $P(A)$ in the coefficients A_{ijk} is an *invariant* of the action of G on $\mathcal{H}$ if $P(A')=P(A)$ for all $g\in G$. These invariants form a graded algebra R (any invariant P is a sum of homogeneous invariants) and the first issue is to determine the dimension of the space R_d of homogeneous invariants of degree d. The Hilbert series

$$h(t) = \sum_{d \geq 0} \dim R_d t^d \tag{6}$$

is known[20]

$$h(t) = \frac{1}{(1-t^6)(1-t^9)(1-t^{12})} \tag{7}$$

and in fact, one can prove that R is a polynomial algebra generated by three algebraically independent invariants of respective degree 6, 9, and 12.

The modern way to prove this result is due to Vinberg, who obtained it from his notion of Weyl group of a graded Lie algebra, applied to a Z_3-grading of the exceptional Lie algebra E_6.[21]

In Sec. III, we shall explain how it can be deduced from the work of Chanler.[22] We prove that certain invariants I_6, I_9, and I_{12} introduced in Ref. 22 are indeed algebraic generators of R and explain how to compute them from the numerical values of the coefficients A_{ijk}, by expressing them in terms of *transvectants*, that is, by means of certain differential polynomials in the form f, rather than in terms of the classical symbolic notation. Given the values of the invariants for a particular state, we show how to form and solve the system of algebraic equations determining the coefficients, u, v, w of the normal form.

Let $a=I_6$, $b=I_{12}$, and $c=I_{18}$ (a certain polynomial in the fundamental invariants). Then, the symmetric functions of u^3, v^3, and w^3

$$\psi = u^3 + v^3 + w^3, \quad \chi = u^3v^3 + u^3w^3 + v^3w^3, \quad \lambda = 216u^3v^3w^3 \tag{8}$$

satisfy

$$\begin{aligned} &\psi^2 - 12\chi - a = 0, \\ &\psi^4 + \lambda\psi - b = 0, \\ &\psi^6 - \tfrac{5}{2}\lambda\psi^3 - \tfrac{1}{8}\lambda^2 - c = 0. \end{aligned} \tag{9}$$

Theorem II.4: *The system (9) has generically* 1296 *solutions* (u,v,w), *which can be obtained by solving a chain of algebraic equations of degree at most* 4. *Only* 648 *of them give the correct sign for* I_9. *The number of solutions (with the correct sign for* I_9*) can be reduced only to* 216, 72, 27 *or* 1. *Moreover, the isotropy groups of these degenerate orbits can be determined, and the configuration of the points* (u,v,w) *in* $\mathbb{C}^3$ *can be interpreted in terms of the geometry of regular complex polyhedra.*

The details are given in Sec. VII.

III. THE FUNDAMENTAL INVARIANTS

In this section, we describe the fundamental invariants, as well as the other concomitants obtained by Chanler,[22] in a form suitable for calculations, in particular for their numerical evaluation (see also Refs. 23 and 24).

As already mentioned, we shall identify a three-qutrit state $|\psi\rangle \in \mathcal{H}$ with a trilinear form

$$f(\mathbf{x},\mathbf{y},\mathbf{z}) = \sum_{1 \leq i,j,k \leq 3} A_{ijk} x_i y_j z_k \tag{10}$$

in three ternary variables. To construct its fundamental invariants, we shall need the notion of a transvectant, which is defined by means of Cayley's omega process (see, e.g., Ref. 25).

Let f_1, f_2, and f_3 be three forms in a ternary variable $\mathbf{x}=(x_1,x_2,x_3)$. Their tensor product $f_1 \otimes f_2 \otimes f_3$ is identified with the polynomial $f_1(\mathbf{x}^{(1)})f_2(\mathbf{x}^{(2)})f_3(\mathbf{x}^{(3)})$ in the three independent ternary variables $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$. We use the "trace" notation of Olver[26] to denote the multiplication map $f_1 \otimes f_2 \otimes f_3 \to f_1 f_2 f_3$, that is,

$$\operatorname{tr} f_1(\mathbf{x}^{(1)}) f_2(\mathbf{x}^{(2)}) f_3(\mathbf{x}^{(3)}) = f_1(\mathbf{x}) f_2(\mathbf{x}) f_3(\mathbf{x}). \tag{11}$$

Cayley's operator $\Omega_{\mathbf{x}}$ is the differential operator

$$\Omega_{\mathbf{x}} = \begin{vmatrix} \frac{\partial}{\partial x_1^{(1)}} & \frac{\partial}{\partial x_1^{(2)}} & \frac{\partial}{\partial x_1^{(3)}} \\ \frac{\partial}{\partial x_2^{(1)}} & \frac{\partial}{\partial x_2^{(2)}} & \frac{\partial}{\partial x_2^{(3)}} \\ \frac{\partial}{\partial x_3^{(1)}} & \frac{\partial}{\partial x_3^{(2)}} & \frac{\partial}{\partial x_3^{(3)}} \end{vmatrix}. \tag{12}$$

Now, we consider three independent ternary variables $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ together with the associated dual (contravariant) variables $\xi=(\xi_1,\xi_2,\xi_3)$, $\eta=(\eta_1,\eta_2,\eta_3)$, $\zeta=(\zeta_1,\zeta_2,\zeta_3)$ [that is, ξ_i is the linear form on the $\mathbf{x}$ space such that $\xi_i(x_j)=\delta_{ij}$].

A *concomitant* of f is, by definition, a polynomial F in the A_{ijk}, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, ξ, η, ζ, such that if $g=(g_1,g_2,g_3)\in \mathrm{SL}(3,\mathbb{C})^3$, then, with A', $\mathbf{x}'$, etc., as above,

$$F(A';\mathbf{x}',\mathbf{y}',\mathbf{z}';\xi',\eta',\zeta') = F(A;\mathbf{x},\mathbf{y},\mathbf{z};\xi,\eta,\zeta). \tag{13}$$

The algebra of concomitants admits only one generator of degree 1 in the A_{ijk}, which is the form f itself. Other concomitants can be deduced from f and the three absolute invariants $P_\alpha = \Sigma \xi_i x_i$, $P_\beta = \Sigma \eta_j y_j$, and $P_\gamma = \Sigma \zeta_k z_k$, using transvectants. If F_1, F_2, and F_3 are three 6-tuple forms in the independent ternary variables $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, ξ, η, and ζ, one defines for any $(n_1,n_2,n_3)\times(m_1,m_2,m_3)\in\mathbb{N}^3\times\mathbb{N}^3$ the multiple transvectant of F_1, F_2, and F_3 by

$$(F_1,F_2,F_3)^{n_1n_2n_3}_{m_1m_2m_3} = \operatorname{tr} \quad \Omega_{\mathbf{x}}^{n_1}\Omega_{\mathbf{y}}^{n_2}\Omega_{\mathbf{z}}^{n_3}\Omega_{\xi}^{m_1}\Omega_{\eta}^{m_2}\Omega_{\zeta}^{m_3}\prod_{i=1}^{3} F_i(\mathbf{x}^{(i)},\mathbf{y}^{(i)},\mathbf{z}^{(i)};\xi^{(i)},\eta^{(i)},\zeta^{(i)}). \tag{14}$$

For convenience, we will set $(F_1,F_2,F_3)^{n_1n_2n_3}=(F_1,F_2,F_3)^{n_1n_2n_3}_{000}$. The concomitants of degree 2 given by Chanler[22] can be obtained using these operations,

$$Q_\alpha = (f,f,P_\beta P_\gamma)^{011}, \tag{15}$$

$$Q_\beta = (f,f,P_\alpha P_\gamma)^{101}, \tag{16}$$

$$Q_\gamma = (f,f,P_\alpha P_\beta)^{110}. \tag{17}$$

The invariant I_6 is then

$$I_6 = \tfrac{1}{96}(Q_\alpha,Q_\alpha,Q_\alpha)^{200}_{011} = \tfrac{1}{96}(Q_\beta,Q_\beta,Q_\beta)^{020}_{101} = \tfrac{1}{96}(Q_\gamma,Q_\gamma,Q_\gamma)^{002}_{110}. \tag{18}$$

There is an alternative expression using only the ground form f,

$$I_6 = \frac{1}{1152}(f^2,f^2,f^2)^{222}. \tag{19}$$

Now, in degree 3 the covariants B_α, B_β, and B_γ of Ref. 22 are

$$B_\alpha = (f,f,f)^{011}, \tag{20}$$

$$B_\beta = (f,f,f)^{101}, \tag{21}$$

$$B_\gamma = (f,f,f)^{110}. \tag{22}$$

The other concomitants found by Chanler can be written in a similar way,

$$C_{\alpha\beta} = \tfrac{1}{4}(f, f, fP_\beta)^{110}, \tag{23}$$

$$C_{\beta\alpha} = \tfrac{1}{4}(f, f, fP_\alpha)^{110}, \tag{24}$$

$$C_{\alpha\gamma} = \tfrac{1}{4}(f, f, fP_\gamma)^{101}, \tag{25}$$

$$C_{\gamma\alpha} = \tfrac{1}{4}(f, f, fP_\alpha)^{101}, \tag{26}$$

$$C_{\beta\gamma} = \tfrac{1}{4}(f, f, fP_\gamma)^{011}, \tag{27}$$

$$C_{\gamma\beta} = \tfrac{1}{4}(f, f, fP_\beta)^{011}, \tag{28}$$

$$D_\alpha = -2(fP_\beta, fP_\gamma, f)^{111}, \tag{29}$$

$$D_\beta = 2(fP_\alpha, fP_\gamma, f)^{111}, \tag{30}$$

$$D_\gamma = -2(fP_\alpha, fP_\beta, f)^{111}, \tag{31}$$

$$E_\alpha = (Q_\alpha, f, P_\alpha)^{100}, \tag{32}$$

$$E_\beta = (Q_\beta, f, P_\beta)^{010}, \tag{33}$$

$$E_\gamma = (Q_\gamma, f, P_\gamma)^{001}, \tag{34}$$

$$G_\alpha = -\tfrac{3}{8}(fP_\beta, fP_\gamma, f)^{011} + \tfrac{5}{16}(fP_\beta P_\gamma, f, f)^{011}, \tag{35}$$

$$G_\beta = -\tfrac{3}{8}(fP_\alpha, fP_\gamma, f)^{101} + \tfrac{5}{16}(fP_\alpha P_\gamma, f, f)^{101}, \tag{36}$$

$$G_\gamma = -\tfrac{3}{8}(fP_\alpha, fP_\beta, f)^{110} + \tfrac{5}{16}(fP_\alpha P_\beta, f, f)^{110}, \tag{37}$$

$$H = \tfrac{1}{2}(fP_\alpha, fP_\beta, fP_\gamma)^{111}. \tag{38}$$

Here, we have combined the concomitants of degrees 0, 1, and 2 into independent concomitants of degree 3. Next, we have chosen the scalar factors so that the syzygies given by Chanler[22] hold in the form

$$H + E_\alpha - E_\gamma + D_\beta P_\beta = 0, \tag{39}$$

$$H + E_\beta - E_\alpha + D_\gamma P_\gamma = 0, \tag{40}$$

$$H + E_\gamma - E_\beta + D_\alpha P_\alpha = 0, \tag{41}$$

$$3C_{\alpha\beta} - B_\gamma P_\beta = 0, \tag{42}$$

$$3C_{\beta\alpha} - B_\gamma P_\alpha = 0, \tag{43}$$

$$3C_{\alpha\gamma} - B_\beta P_\gamma = 0, \tag{44}$$

$$3C_{\gamma\alpha} - B_\beta P_\alpha = 0, \tag{45}$$

$$3C_{\beta\gamma} - B_\alpha P_\gamma = 0, \tag{46}$$

$$3C_{\gamma\beta} - B_\alpha P_\beta = 0, \tag{47}$$

$$6G_\alpha - 3Q_\alpha f + B_\alpha P_\beta P_\gamma = 0, \tag{48}$$

$$6G_\beta - 3Q_\beta f + B_\beta P_\alpha P_\gamma = 0, \tag{49}$$

$$6G_\gamma - 3Q_\gamma f + B_\gamma P_\alpha P_\beta = 0. \tag{50}$$

One can remark that a basis of the space of the concomitants of degree 3 found by Chanler can be constructed using only transvections and products from smaller degrees,

$$f^3, Q_\alpha f, Q_\beta f, Q_\gamma f, B_\alpha, B_\beta, B_\gamma, D_\alpha, D_\beta, D_\gamma, E_\alpha. \tag{51}$$

The knowledge of these concomitants allows one to construct the invariants I_9 and I_{12},

$$I_9 = \tfrac{1}{576}(E_\alpha, E_\beta, E_\beta)^{111}_{111}, \tag{52}$$

$$I_{12} = \frac{1}{124\,416}(B_\alpha f, B_\alpha f, B_\alpha f)^{411}. \tag{53}$$

These expressions, which can be easily implemented in any computer algebra system, will prove convenient to compute the specializations discussed in the sequel.

IV. NORMAL FORM AND INVARIANTS

It will now be shown that a generic state can be reduced to the normal form

$$A_{ijk} = u\delta_{ijk} + \frac{w-v}{2}\epsilon_{ijk} + \frac{w+v}{2}|\epsilon_{ijk}|, \tag{54}$$

where ϵ_{ijk} is the alternating tensor, or, otherwise said, that the generic trilinear form $f(\mathbf{x},\mathbf{y},\mathbf{z})$ is equivalent to some

$$\begin{aligned} N_{uvw}(\mathbf{x},\mathbf{y},\mathbf{z}) = {} & u(x_1y_1z_1 + x_2y_2z_2 + x_3y_3z_3) + v(x_1y_3z_2 + x_2y_1z_3 + x_3y_2z_1) \\ & + w(x_1y_2z_3 + x_2y_3z_1 + x_3y_1z_2). \end{aligned} \tag{55}$$

For such a state, the local density operators are all proportional to the identity. This property is automatically satistfied by the limiting state obtained from the numerical method of Ref. 19, and implies maximal entanglement as well. Since this algorithm amounts to an infinite sequence of invertible local filtering operations, the genericity of Vinberg's normal form, together with the previously mentioned properties, implies convergence to a Vinberg normal form for a generic input state, that is, our Theorem II.1 (see also Refs. 27 and 28).

This normal form is in general not unique, and the relations between the various N_{uvw} in a given orbit is an interesting question, which will be addressed in the sequel.

Although, the validity of this normal form follows from Vinberg's theory,[21] it can also be proved in other ways, some of them being particularly instructive. We shall detail one of these possibilities, which will give us the opportunity to introduce some important polynomials, playing a role in the algebraic calculation of the normal form and in the geometric discussion of the orbits.

The shortest possibility, although not the most elementary, relies on the results of Ref. 22, and starts with computing the invariants of N_{uvw}. We then use a few results of algebraic geometry, which can be found in Ref. 29. Let us denote by $C_k \equiv C_k(u,v,w)$ $(k=6,9,12)$ the values of the I_k on N_{uvw}. Direct calculation gives, denoting by m_{pqr} the monomial symmetric functions of u,v,w (sum of all distinct permutations of the monomial $u^p v^q w^r$),

$$C_6 = m_{(6)} - 10m_{(3,3)}, \tag{56}$$

$$C_9 = (u^3 - v^3)(u^3 - w^3)(v^3 - w^3), \tag{57}$$

$$C_{12} = m_{(12)} + 4m_{(9,3)} + 6m_{(6,6)} + 228m_{(6,3,3)}. \tag{58}$$

It is easily checked by direct calculation that the Jacobian of these three functions is nonzero for generic values of (u,v,w). Actually, its zero set consists of 12 planes, whose geometric significance will be discussed below.

Let us denote by $\varphi:\mathcal{H} \xrightarrow{(I_6,I_9,I_{12})} \mathbb{C}^3$, the map sending a trilinear form to its three invariants, so that $(C_6,C_9,C_{12})=\varphi(N_{uvw})$. Let $S=\{N_{uvw} | (u,v,w) \in \mathbb{C}^3\}$ be the three-dimensional space of normal forms. The nonvanishing of the Jacobian proves that φ induces a dominant mapping from S to $\mathbb{C}^3$ (that is, the direct image of any nonempty open subset of S contains a nonempty open subset of $\mathbb{C}^3$). Note that the independence of C_6,C_9,C_{12} implies the independence of I_6,I_9,I_{12}. Now, Chanler[22] has shown that I_6,I_9,I_{12} separate the orbits in general position. This proves that the field of rational invariants of G is freely generated by I_6,I_9,I_{12} (Ref. 29, Lemma 2.1). As a consequence, φ is a *rational quotient* (Ref. 29, Sec. 2.4) for the action of G on $\mathcal{H}$ (actually, this also implies that φ is a categorical quotient, by Ref. 29, Proposition 2.5 and Theorem 4.12, using that $\varphi_{|S}$ is surjective, whence also φ).

There exists a nonempty open subset Y_0 of $\mathbb{C}^3$ such that the fiber of φ over each of its points is the closure of an orbit (Ref. 29, Proposition 2.5). Let then $U_0=\varphi^{-1}(Y_0)$. This set cuts S since $\varphi_{|S}$ is dominant. Let U_1 be the union of all orbits having maximal dimension (a nonempty open set, the function *dimension of the orbit* being lower semicontinuous). It is easy to see that U_1 intersects S (for instance at $u=1$, $v=1$, $w=-1$, whose orbit has dimension $24=\dim G$, as may be checked by direct calculation). Let $S_0=U_1\cap S$, a dense open subset of S. The set $\varphi^{-1}\varphi(S_0)$ thus contains a dense open subset U_2 of $\mathcal{H}$. One then checks that $U_0\cap U_1\cap U_2$ (a dense open subset, as an intersection of dense open subsets of an irreducible space) is contained in GS. This proves $\overline{GS}=\mathcal{H}$, that is, the normal form N_{uvw} is generic.

Let us remark that the above discussion also proves, thanks to Igusa's theorem (Ref. 29, Theorem 4.12) that $\mathbb{C}[\mathcal{H}]^G=\mathbb{C}[I_6,I_9,I_{12}]$, that is, the algebra of invariants is freely generated by Chanler's invariants.

Is is also possible to give a direct proof of the normal form by using the same technique as in Ref. 22. Chanler's method relies on the geometry of plane cubics, which will play a prominent role in the sequel.

V. THE FUNDAMENTAL CUBICS

The trilinear form $f(\mathbf{x},\mathbf{y},\mathbf{z})$ can be encoded in three ways by a 3×3 matrix of linear forms $M_x(\mathbf{x})$, $M_y(\mathbf{y})$, and $M_z(\mathbf{z})$, defined by

$$f(\mathbf{x},\mathbf{y},\mathbf{z}) = {}^t\mathbf{y}M_x(\mathbf{x})\mathbf{z} = {}^t\mathbf{x}M_y(\mathbf{y})\mathbf{z} = {}^t\mathbf{x}M_z(\mathbf{z})\mathbf{y} \tag{59}$$

and the classification of trilinear forms amounts to the classification of one of these matrices, say $M_x(\mathbf{x})$ up to left and right multiplication by elements of $\mathrm{SL}(3,\mathbb{C})$ and action of $\mathrm{SL}(3,\mathbb{C})$ on the variable $\mathbf{x}$.

The most immediate covariants of f are the determinants of these matrices

$$X(\mathbf{x}) = \det M_x(\mathbf{x}) = \tfrac{1}{6}B_\alpha, \tag{60}$$

$$Y(\mathbf{y}) = \det M_y(\mathbf{y}) = \tfrac{1}{6}B_\beta, \tag{61}$$

$$Z(\mathbf{z}) = \det M_z(\mathbf{z}) = \tfrac{1}{6}B_\gamma. \tag{62}$$

These are ternary cubic forms, and for generic f the equations $X(\mathbf{x})=0$, etc., will define nonsingular cubics (elliptic curves) in $\mathbb{P}^2$. It is shown in Ref. 24 that whenever one of these curves is elliptic, so are the other two ones, and moreover, all three are projectively equivalent. Actually, one can check by direct calculation that they have the same invariants. When $f=N_{uvw}$, these three cubics have even the same equation and are in the Hesse canonical form[30]

$$X(\mathbf{x}) = -\phi(x_1^3 + x_2^3 + x_3^3) + \psi x_1 x_2 x_3 = Y(\mathbf{x}) = Z(\mathbf{x}), \tag{63}$$

where we introduced, following the notation of Ref. 2,

$$\phi = uvw, \quad \psi = u^3 + v^3 + w^3. \tag{64}$$

The Aronhold invariants of the cubics (63) are given by

$$6^4 S = -\phi(\psi^3 + (6\phi)^3), \tag{65}$$

$$6^6 T = (6\phi)^6 + 20(6\phi^3)\psi^3 - 8\psi^6. \tag{66}$$

These are of course invariants of f. We recognize that $6^4S=-C_{12}$, and we introduce an invariant I_{18} such that $C_{18}=I_{18}(N_{uvw})=6^6T$. The three cubics have the same discriminant $64S^3+T^2$, known to be proportional to the hyperdeterminant of f (see Refs. 31 and 32), which we normalize as

$$\Delta = 27(64S^3 + T^2). \tag{67}$$

Then $\Delta=C'^3_{12}$, where C'_{12} is the product of 12 linear forms

$$\begin{aligned} C'_{12} = {} & uvw(u+v+w)(\varepsilon u+v+w)(u+\varepsilon v+w)(\varepsilon^2 u+\varepsilon v+w)(u+\varepsilon^2 v+w) \\ & \times (\varepsilon u+\varepsilon v+w)(\varepsilon^2 u+v+w)(\varepsilon u+\varepsilon^2 v+w)(\varepsilon^2 u+\varepsilon^2 v+w), \end{aligned} \tag{68}$$

where $\varepsilon=e^{2i\pi/3}$, so that $C'_{12}=0$ is the equation in $\mathbb{P}^2$ of the twelve lines containing 3×3 the nine inflection points of the pencil of cubics,

$$u^3 + v^3 + w^3 + 6m\, uvw = 0, \tag{69}$$

obtained from X, Y, Z by treating the original variables as parameters. We note also that the Jacobian of C_6, C_9, C_{12} is proportional to C'^2_{12}.

VI. SYMMETRIES OF THE NORMAL FORMS

In this section, we will prove Theorem II.2. That is, a generic f has 648 different normal forms [the points (u,v,w) for which this number is reduced will be studied in Sec. VII].

To prove the theorem, we remark that the Hilbert series (7) is also the one of the ring of invariants of G_{25}, the group number 25 in the classification of irreducible complex reflection groups of Shephard and Todd.[4] This group, which we will denote for short by K, has order 648. It is one of the groups considered by Maschke[2] in his determination of the invariants of the symmetry group of the 27 lines of a general cubic surface in $\mathbb{P}^3$ (a group with 51 840 elements, which is related to the exceptional root system E_6). To define K, we first have to introduce Maschke's group H, a group of order 1296, which is generated by the matrices of the linear transformations on $\mathbb{C}^3$ given in Table I.

TABLE I. The generators of H.

	A	B	C	D	E
u'	v	u	u	u	$1/i\sqrt{3}(u+v+w)$
v'	w	w	εv	εv	$1/i\sqrt{3}(u+\varepsilon v+\varepsilon^2 w)$
w'	u	v	$\varepsilon^2 w$	εw	$1/i\sqrt{3}(u+\varepsilon^2 v+\varepsilon w)$

This group contains in particular the permutation matrices, and simultaneous multiplication by $\pm\varepsilon^k$, since $E^2=-B$. The subgroup K is the one in which odd permutations can appear only with a minus sign. It is generated by A,C,D,E.

Then, as proved by Maschke, the algebra of invariants of K in $\mathbb{C}[u,v,w]$ is precisely $\mathbb{C}[C_6,C_9,C_{12}]$.

Hence, we can conclude that K is the symmetry group of the normal forms N_{uvw}. There was another, equally natural possibility leading to the same Hilbert series. The symmetry group L of the equianharmonic cubic surface $\Sigma: z_0^3+z_1^3+z_2^3+z_3^3=0$ acting on the homogeneous coordinate ring $\mathbb{C}[\Sigma]$ has as fundamental invariants the elementary symmetric functions of the z_i^3, the first one being 0 by definition, so that the Hilbert series of $\mathbb{C}[\Sigma]^L$ coincides with (7). Moreover, L is also of order 648, but it is known that it is not isomorphic to K.

Taking into account the results of Sec. IV, we see that

$$S=\{N_{uvw}|(u,v,w)\in\mathbb{C}^3\} \tag{70}$$

is what is usually called a Chevalley section of the action of G on $\mathcal{H}$, with Weyl group K (see Ref. 29, p. 174).

VII. THE FORM PROBLEM

This section contains the proofs of Theorems II.3 and II.4. Klein (see Ref. 33) has introduced and investigated the notion of "Formenproblem" associated to a finite group action. This is the following: given the numerical values of the invariants, compute the coordinates of a point of the corresponding orbit.

In our case, we shall see that the problem of finding the parameters (u,v,w) of the normal form of a given generic f, given the values of the invariants, can be reduced to a chain of algebraic equations of degree at most 4, hence sovable by radicals.

Let $a=I_6$, $b=I_{12}$ and $c=I_{18}$ (we start with I_{18}, because C_{18} is a symmetric function of u^3,v^3,w^3, and at the end of the calculation, select the solutions which give the correct sign for C_9, which is alternating).

What we have to do is to determine the elementary symmetric functions $e_1=\psi, e_2=\chi, e_3=\phi^3$ of u^3,v^3,w^3. Let $\lambda=216\phi^3$. Then,

$$\psi^4+\lambda\psi-b=0, \tag{71}$$

$$\psi^6-\tfrac{5}{2}\lambda\psi^3-\tfrac{1}{8}\lambda^2-c=0. \tag{72}$$

Eliminating λ from these equations, we get a quartic equation for ψ^2,

$$27\psi^8-18b\psi^4-8c\psi^2-b^2=0. \tag{73}$$

The discriminant (with respect to ψ) of this polynomial is proportional to $D=b^2(b^3-c^2)^4$. When it is nonzero, we get eight values for e_1, each of which determines univocally e_2 and e_3. Hence, we obtain eight cubic equations whose roots are the possible values of u^3,v^3,w^3. This gives eight sets, whence $8\times 6=48$ triples, each of which providing generically 27 values of (u,v,w), in all $48\times 27=1296$ triples corresponding to the given values of a,b,c, among which exactly $1296/2=648$ give the correct sign for I_9. The common discriminant of the eight cubics is $\delta=a^3-3ab$

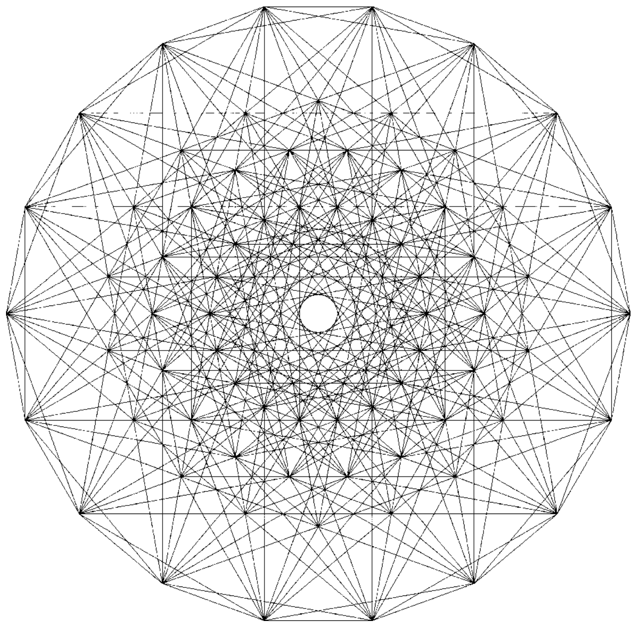

FIG. 1. The polyhedron 2{4}3{3}3.

$+2c$. Clearly, when $\delta \neq 0$, we will have 648 triples. If $\delta = 0$, one can check that the cubics cannot have a triple root, and that no root is zero. Hence, in this case, we obtain again 648 triples.

If $D=0$, we can have $b^3=c^2$ or $b=0$. In the first case, setting $b=q^2, c=q^3$, the equation becomes

$$(\psi^2 - q)^3(\psi^2 + 2q)^3 = 0. \tag{74}$$

In this case, we get only four quartics for ψ^2. If $C_9 \neq 0$, we obtain 216 triples. If $C_9=0$ and $b=a^2/4$, $c=-a^3/8$ we obtain again 216 triples which form the centers of the edges of a complex polyhedron of type 2{4}3{3}3 in $\mathbb{C}^3$ (see Fig. 1), in the notation of Ref. 34. The vertices of this polyhedron are the vertices of two reciprocal Hessian polyhedra (see Fig. 2) and its edges join each vertex of one Hessian polyhedron to the eight closest vertices of the other one. In Fig. 2, the edges of the Hessian polyhedron, which are complex lines, are represented by real equilateral triangles, so that the figure can as well be interpreted as a two-dimensional projection of a six-dimensional Gosset polytope 2_{21}. If $C_9=0$ and $b=a^2$, $c=a^3$, we obtain only 72 triples which

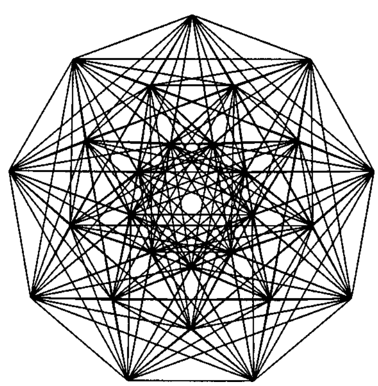

FIG. 2. The Hessian polyhedron.

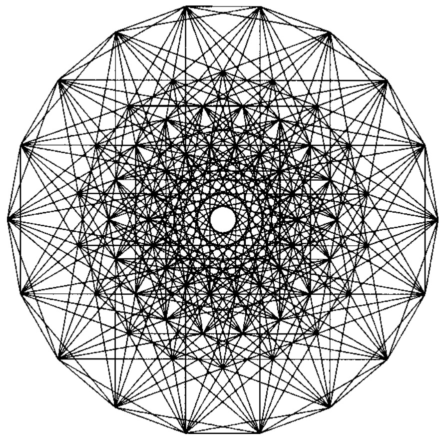

FIG. 3. The polyhedron 3{3}3{4}2.

are the centers of the edges of a Hessian polyhedron and the vertices of a complex polytope of type 3{3}3{4}2 (see Fig. 3).

In the case where $b=0$, we have to distinguish between the cases $c\neq 0$ and $c=0$. If $c\neq 0$, we find 648 triples, whatever the value of a. If $c=0$, we obtain 27 triples if $a\neq 0$, and only one if $a=0$.

Indeed, for $b=c=0$, the ψ-equation reduces to $\psi^8=0$, and all the cubics collapse to $12U^3 - aU=0$. For $a\neq 0$ we obtain precisely 27 triples (u,v,w) which form the vertices of a Hessian polyhedron in $\mathbb{C}^3$ (see Ref. 1).

From the results of Ref. 3 about the arrangement of 12 planes formed by the mirrors of the pseudoreflections of $K=G_{25}$, we can determine the structure of the stabilizers of the normal forms. The only nontrivial cases are as follows:

(i) the orbits with 216 elements, for which the stabilizer is the cyclic group C_3;
(ii) the orbits with 72 elements, for which it is $C_3\times C_3$;
(iii) the Hessian orbits with 27 elements, for which it is the group G_4 of the Shephard–Todd classification.

These results can be regarded as a complete description of the moduli space of three-qutrit states. To see what this means, let us recall some definitions from geometric invariant theory.

It is well known that it in general, the orbits of a group action on an algebraic variety cannot be regarded as the points of an algebraic variety. To remedy this situation, one has to discard certain degenerate orbits. It is then possible to construct a *categorical quotient* and a *moduli space*, which describe the geometry of sufficiently generic orbits, respectively, in the affine and projective situation.

The *categorical quotient* $Y=\mathcal{H}//G$ is defined as the affine variety whose affine coordinate ring is the ring of polynomial invariants $R=\mathbb{C}[\mathcal{H}]^G$. The moduli space is the projective variety $\mathcal{M}=\mathrm{Proj}(R)$ of which R is the homogeneous coordinate ring. It is the quotient of the set $\mathbb{P}(\mathcal{H})^{\mathrm{ss}}$ of *semistable* points by the action of G (by definition, a point is semistable iff at least one of its algebraic invariants is nonzero, see Ref. 29).

Now, since in our case the algebra of invariants is a polynomial algebra, we see that the categorical quotient is just the affine space $\mathbb{C}^3$.

The moduli space is more interesting. The projective variety whose homogeneous coordinate ring is a polynomial algebra over generators of respective degrees $d_1,\ldots,d_m$ is called a *weighted projective space* $\mathbb{P}(d_1,\ldots,d_m)$. Hence, by definition, our moduli space $\mathcal{M}$ is the weighted projective space $\mathbb{P}(6,9,12)\simeq\mathbb{P}(2,3,4)$. It is known that this space is isomorphic to $\mathbb{P}(1,2,3)$,[35] which in turn can be embedded as a sextic surface in $\mathbb{P}^6$, the so-called *del Pezzo surface* F^6 (see Ref. 36). The del Pezzo surfaces are very interesting objects, known to be related to the exceptional root systems (see, e.g., Ref. 37).

The above results can then be interpreted as a description of the singularities of $\mathcal{M}$, since one can view it as the quotient of the projective plane $\mathbb{P}^2$ of the parameters $(u:v:w)$ under the projective action of G_{25}. We have described this quotient as a 648-fold ramified covering $\mathbb{P}^2 \to \mathcal{M}$, and analyzed its ramification locus.

VIII. CONCLUSION

A problem of current interest in Quantum Information Theory has been connected to various important mathematical works, scattered on a period of more than one century from Ref. 2 in 1889 to Ref. 27 in 2000, in general independent of each other and apparently discussing different subjects. Relying on all these works, we have described the geometry of the normal forms of semistable orbits of three-qutrit states under the action of $\mathrm{SL}(3,\mathbb{C})^{\times 3}$, the group of local filtering (SLOCC) operations. From a physical point of view, our results can be expected to provide a good starting point for studying the richness of the entanglement of three qutrits and its differences with that of the simpler qubit systems. From a mathematical point of view, we have worked out an interesting example of a problem in invariant theory, using both classical algebraic and modern geometric methods, found a surprising connection with the geometry of complex polytopes, and applied Klein's vision of Galois theory to the explicit solution of an algebraic equation of degree 648.

Also, this example provides a good illustration of the ideas presented in Refs. 14 and 17.

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