

THE 144 SYMMETRIES OF THE LITTLEWOOD-RICHARDSON COEFFICIENTS OF SL_3

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ABSTRACT. We compute with *SageMath* the group of all linear symmetries for the Littlewood-Richardson associated to the representations of SL_3 . We find that there are 144 symmetries, more than the 12 symmetries known for the Littlewood-Richardson coefficients in general.

1. INTRODUCTION

The *Littlewood–Richardson coefficients* $c'_{\lambda,\mu}$ are among the most studied constants in geometry and representation theory. In geometry, they are the structure constants for the multiplication in the cohomology ring of the Grassmannian, in the basis of the Schubert cycles [1]. They can also be interpreted as cardinalities of the intersection of some triples of Schubert varieties [2]. In the representation theory of the general linear group, they are the multiplicities in the tensor product of irreducible representations, and also the dimensions of the subspace of invariants in the triple tensor products of irreducible representations [3]. In the representation theory of the symmetric groups, they are the multiplicities in the restrictions of the irreducible representations of a symmetric group S_{m+n} to its Young subgroup $S_m \times S_n$. In the theory of symmetric functions, they are the structure constants for the ordinary multiplication in the basis of Schur functions. They receive also a number a combinatorial interpretations: they are known to count Littlewood-Richardson tableaux, hives, BZ triangles, . . .

Together, these different interpretations of the Littlewood-Richardson coefficients make clear some symmetries they afford. For instance, the descriptions as structural constants for commutative products (of Schur functions, in the ring of symmetric functions; of Schubert cycles, in the cohomology ring of the Grassmannian) make clear the invariance $c'_{\lambda,\mu} = c'_{\mu,\lambda}$, under exchanging the two partitions λ and μ . The descriptions as cardinalities of intersection of triples of Schubert varieties [4], or as dimensions of subspaces of invariants in triple tensor products [3] reveals the existence of a S_3 -symmetry of the Littlewood–Richardson

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coefficients (that comprises the previous symmetry):

$$(1) \quad c_{\lambda, \mu}^{\nu} = c_{\mu, \lambda}^{\nu} = c_{\mu, \nu^{\square}}^{\lambda^{\square}} = c_{\nu^{\square}, \mu}^{\lambda^{\square}} = c_{\nu^{\square}, \lambda}^{\mu^{\square}} = c_{\lambda, \nu^{\square}}^{\mu^{\square}}$$

where λ^{\square} , μ^{\square} and ν^{\square} are obtained from the Young diagrams of λ , μ and ν by “taking complements” in suitable rectangles (see Figure 1).

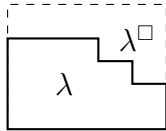


FIGURE 1. New Young diagram obtained as complement of a Young diagram in a rectangle. When considering the Littlewood–Richardson coefficients associated to SL_N with λ and μ of length at most $N - 1$ and ν of length at most N , one takes complements in rectangles of height N and: for (1), of length ν_1 ; for (2), of length respectively λ_1 , μ_1 , $\lambda_1 + \mu_1$ for λ^{\square} , μ^{\square} and ν^{\square} .

Duality provides an additional symmetry (see [5]):

$$(2) \quad c_{\lambda, \mu}^{\nu} = c_{\lambda^{\square}, \mu^{\square}}^{\nu^{\square}}.$$

Together, these “known symmetries” generate a group of 12 symmetries, isomorphic to $S_2 \times S_3$. Note that it is not straightforward to understand these symmetries all together from the combinatorial descriptions of the Littlewood–Richardson coefficients (see [6, 4, 7]).

Is it possible that the Littlewood–Richardson coefficients afford additional, unknown symmetries?

To settle this question, we adopt an experimental approach. We consider the family of Littlewood–Richardson coefficients indexed by partitions λ , μ and ν with restricted length. The combinatorial object they count can be interpreted as lattice points in a (rational convex polyhedral) cone. As a consequence (see [8]), $c_{\lambda, \mu}^{\nu}$ is a piecewise quasipolynomial function (actually polynomial, by an additional argument given again in [8]) of the parts of the partitions λ , μ and ν . The domains of polynomiality are the maximal cones (“chamber”) of a fan (a complex of rational polyhedral convex cones, called the “Chamber Complex”). Explicit knowledge of such a description provides data making possible an exhaustive search of all linear symmetries.

The Chamber Complex and the polynomial formulas for the Littlewood–Richardson coefficients $c_{\lambda, \mu}^{\nu}$ related to the representations of $GL_3(\mathbb{C})$ (i.e. where all three partitions have length at most 3) were explicitly given in [8]. We exploit and analyze these data with *SageMath* [9], and obtain:

Theorem 1. *The group of linear symmetries of the Littlewood–Richardson coefficients $c'_{\lambda,\mu}$ associated to SL_3 , has order 144. It is isomorphic to $S_2 \times (S_3 \wr S_2)$, and acts transitively on the chambers of the Chamber Complex. It is generated by the well-known group of 12 symmetries, and the additional symmetry:*

$$c_{(\lambda_1,\lambda_2),(\mu_1,\mu_2)}^{(\nu_1,\nu_2,\nu_3)} = c_{(\lambda_1+\mu_1-\nu_2,\lambda_2+\mu_1-\nu_2),(\nu_2,\mu_2)}^{(\nu_1,\mu_1,\nu_3)}$$

The calculations are presented in detail in a *SageMath* Notebook available online [10] and as an ancillary file.

2. THE LITTLEWOOD–RICHARDSON COEFFICIENTS ASSOCIATED TO SL_3 AND THEIR KNOWN SYMMETRIES

2.1. The Littlewood–Richardson coefficients associated to SL_3 .

In this work we restrict our study to the family of Littlewood–Richardson coefficients related to SL_3 . These are the Littlewood–Richardson coefficients $c'_{\lambda,\mu}$ such that λ and μ have length at most 2 and ν has length at most 3. Since $c'_{\lambda,\mu} = 0$ when $|\lambda| + |\mu| \neq |\nu|$, we have for any non-zero coefficient $c_{(\lambda_1,\lambda_2),(\mu_1,\mu_2)}^{(\nu_1,\nu_2,\nu_3)}$ that

$$(3) \quad \nu_3 = \lambda_1 + \lambda_2 + \mu_1 + \mu_2 - \nu_1 - \nu_2.$$

We thus consider the function C defined on \mathbb{Z}^6 by

$$C(\lambda_1, \lambda_2, \mu_1, \mu_2, \nu_1, \nu_2) = c_{(\lambda_1,\lambda_2),(\mu_1,\mu_2)}^{(\nu_1,\nu_2,\nu_3)}$$

when $\lambda_1 \geq \lambda_2 \geq 0$, $\mu_1 \geq \mu_2 \geq 0$, $\nu_1 \geq \nu_2 \geq \nu_3 \geq 0$ (with ν_3 defined by (3)), and $C(\lambda_1, \lambda_2, \mu_1, \mu_2, \nu_1, \nu_2) = 0$ otherwise.

Note that considering instead the Littlewood–Richardson coefficients associated to GL_3 (the $c_{(\lambda_1,\lambda_2,\lambda_3),(\mu_1,\mu_2,\mu_3)}^{(\nu_1,\nu_2,\nu_3)}$) gives equivalent results (but hides symmetries). This is because of the invariance properties:

$$c_{(\lambda_1,\lambda_2,\lambda_3),(\mu_1,\mu_2,\mu_3)}^{(\nu_1,\nu_2,\nu_3)} = c_{(1+\lambda_1,1+\lambda_2,1+\lambda_3),(\mu_1,\mu_2,\mu_3)}^{(1+\nu_1,1+\nu_2,1+\nu_3)}$$

and

$$c_{(\lambda_1,\lambda_2,\lambda_3),(\mu_1,\mu_2,\mu_3)}^{(\nu_1,\nu_2,\nu_3)} = c_{(\lambda_1,\lambda_2,\lambda_3),(1+\mu_1,1+\mu_2,1+\mu_3)}^{(1+\nu_1,1+\nu_2,1+\nu_3)}$$

which imply

$$c_{(\lambda_1,\lambda_2,\lambda_3),(\mu_1,\mu_2,\mu_3)}^{(\nu_1,\nu_2,\nu_3)} = c_{(\lambda_1,\lambda_2),(\mu_1,\mu_2)}^{(\nu_1-\mu_3-\lambda_3,\nu_2-\mu_3-\lambda_3,\nu_3-\mu_3-\lambda_3)}.$$

2.2. Known symmetries. We now look for all linear symmetries of C , i.e. all invertible linear maps $F : \mathbb{Z}^6 \rightarrow \mathbb{Z}^6$ such that $C \circ F = C$. The group S_3 of symmetries (1) is generated by

$$\lambda, \mu, \nu \mapsto \mu, \lambda, \nu, \quad \text{and} \quad \lambda, \mu, \nu \mapsto \nu^\square, \mu, \lambda^\square,$$

that correspond in this context to:

$$S : (\lambda_1, \lambda_2 \mid \mu_1, \mu_2 \mid \nu_1, \nu_2) \mapsto (\mu_1, \mu_2 \mid \lambda_1, \lambda_2 \mid \nu_1, \nu_2),$$

$$U : (\lambda_1, \lambda_2 \mid \mu_1, \mu_2 \mid \nu_1, \nu_2) \mapsto (\nu_1 - \nu_3, \nu_1 - \nu_2 \mid \mu_1, \mu_2 \mid \nu_1, \nu_1 - \lambda_2).$$

Chamber	Generators	Formula for C
κ_1	b, c, d_1, e_2, d_2, e_1	$1 - \lambda_2 - \mu_2 + \nu_1$
κ_2	b, c, d_1, g_1, d_2, g_2	$1 + \nu_2 - \nu_3$
κ_3	b, c, e_2, g_1, e_1, g_2	$1 + \lambda_1 + \mu_1 - \nu_1$
κ_4	b, f, d_1, e_2, d_2, e_1	$1 + \nu_1 - \nu_2$
κ_5	b, f, d_1, g_1, d_2, g_2	$1 + \lambda_2 + \mu_2 - \nu_3$
κ_6	b, f, e_2, g_1, e_1, g_2	$1 - \lambda_3 - \mu_3 + \nu_3$
κ_7	b, c, d_1, g_1, d_2, e_1	$1 + \lambda_3 + \mu_1 - \nu_3$
κ_8	b, c, d_1, e_2, d_2, g_2	$1 + \lambda_1 + \mu_3 - \nu_3$
κ_9	b, c, d_1, e_2, e_1, g_2	$1 + \lambda_1 - \lambda_2$
κ_{10}	b, c, e_2, g_1, d_2, e_1	$1 + \mu_1 - \mu_2$
κ_{11}	b, c, d_1, g_1, e_1, g_2	$1 - \lambda_2 - \mu_3 + \nu_2$
κ_{12}	b, c, e_2, g_1, d_2, g_2	$1 - \lambda_3 - \mu_2 + \nu_2$
κ_{13}	b, f, d_1, g_1, d_2, e_1	$1 - \lambda_1 - \mu_3 + \nu_1$
κ_{14}	b, f, d_1, e_2, d_2, g_2	$1 - \lambda_3 - \mu_1 + \nu_1$
κ_{15}	b, f, d_1, g_1, e_1, g_2	$1 + \mu_2 - \mu_3$
κ_{16}	b, f, e_2, g_1, d_2, g_2	$1 + \lambda_2 - \lambda_3$
κ_{17}	b, f, d_1, e_2, e_1, g_2	$1 + \lambda_1 + \mu_2 - \nu_2$
κ_{18}	b, f, e_2, g_1, d_2, e_1	$1 + \lambda_2 + \mu_1 - \nu_2$

TABLE 1. The Chamber Complex for C , from [8, Table 1]. Note that there is a typo in [8, Table 1]: there one should read ν_1 instead of ν_3 in the quasipolynomial formulas for chambers κ_{13} and κ_{14} .

The additional involution (2) corresponds to

$$T : (\lambda_1, \lambda_2 \mid \mu_1, \mu_2 \mid \nu_1, \nu_2) \\ \longmapsto (\lambda_1, \lambda_1 - \lambda_2 \mid \mu_1, \mu_1 - \mu_2 \mid \lambda_1 + \mu_1 - \nu_3, \lambda_1 + \mu_1 - \nu_2)$$

where, again, ν_3 is given by (3).

Rassart's Chamber Complex and polynomial formulas for the GL_3 -Littlewood-Richardson coefficients restrict to a Chamber Complex and polynomial formulas for the function C (set $\lambda_3 = \mu_3 = 0$). The rays of this Chamber Complex are given in Table 2, and the chambers and polynomial formulas are shown in Table 1.

3. COMPUTATION OF THE SYMMETRIES

In this section, we prove Theorem 1.

3.1. Reduction to the symmetries of the Chamber Complex.

In order to find all symmetries of the function C , we look for symmetries of simpler objects: first we check that any symmetry of C must be a

$$\begin{aligned} b &= (2, 1, 2, 1, 3, 2) & c &= (1, 1, 1, 1, 2, 1) & f &= (1, 0, 1, 0, 1, 1) \\ d_1 &= (1, 1, 1, 0, 1, 1) & e_1 &= (1, 1, 0, 0, 1, 1) & g_1 &= (1, 0, 0, 0, 1, 0) \\ d_2 &= (1, 0, 1, 1, 1, 1) & e_2 &= (0, 0, 1, 1, 1, 1) & g_2 &= (0, 0, 1, 0, 1, 0) \end{aligned}$$

TABLE 2. The minimal generators for the rays of the Chamber Complex for C .

symmetry of the chamber complex. Afterwards, we will show that such a symmetry must be also a symmetry of an even simpler object: the set of the ray generators of the Chamber Complex.

Lemma 1. *Any linear symmetry of C is necessarily a linear symmetry of the chamber complex, i.e. an invertible linear map from \mathbb{Z}^6 to itself that permutes the cells of the chamber complex.*

Proof. We will call \mathcal{C} the support of the Chamber Complex, i.e. the cone that is the union of all chambers.

Let F be a symmetry of C . On any chamber κ of the chamber complex, the function C coincides with a polynomial function P_κ . Let κ be a chamber of the chamber complex. Since $C \circ F = C$, we have that $C \circ F$ also coincides with P_κ on κ . Therefore C coincides with $P_\kappa \circ F^{-1}$ on $F(\kappa)$. The cone $F(\kappa)$ can't meet the exterior of the cone \mathcal{C} . Indeed, if $F(\kappa)$ would meet the exterior of \mathcal{C} , the intersection would contain a full-dimensional cone of \mathbb{R}^6 . But two polynomial functions that coincide on the integer points of a full-dimensional cone must be equal. We would have $P_\kappa \circ F^{-1} = 0$, and thus $P_\kappa = 0$, which is false. Therefore $F(\kappa) \subset \mathcal{C}$. Again, it is impossible that $F(\kappa)$ meet two chambers τ_1 and τ_2 full-dimensionally, because this would imply $P_{\tau_1} = P_{\tau_2}$, and there is no such coincidence in the list of formulas for C . As a consequence, there exists one cell τ such that $F(\kappa) \subset \tau$. Applying the same reasoning to F^{-1} , we see that we must have also $F^{-1}(\tau) \subset \kappa$. As a conclusion, $F(\kappa) = \tau$. Therefore, F permutes the chambers of the chamber complex. It follows that F is a symmetry of the chamber complex. \square

3.2. Reduction to the Symmetries of the rays. After Lemma 1, any linear symmetry of C is also a linear symmetry of the chamber complex. Obviously, any linear symmetry of the chamber complex induces also a linear symmetry of the set of its ray generators $\mathcal{R} = \{b, c, f, d_1, d_2, e_1, e_2, g_1, g_2\}$ (i.e. a permutation of \mathcal{R} induced by a linear invertible map from \mathbb{Z}^6 to itself). We will denote with $S(\mathcal{R})$ the group of linear symmetries of \mathcal{R} .

We compute with *SageMath* the group $S(\mathcal{R})$, thanks to the following lemma. Firstly, note that the group of symmetries of a set of vectors $R = \{v_1, v_2, \dots, v_n\}$ embeds in the symmetric group \mathfrak{S}_n .

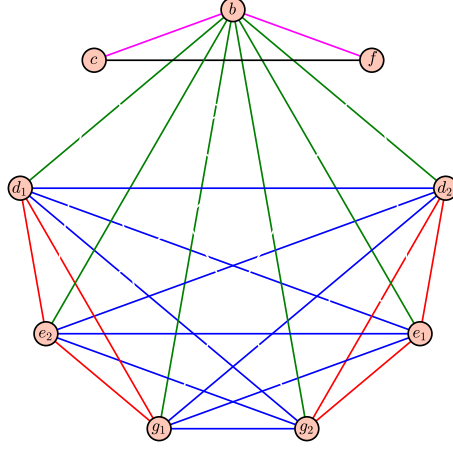


FIGURE 2. The edge-colored graph $H(\mathcal{R})$. Its automorphism group is the group of linear symmetries of the ray generators of the chamber complex.

Lemma 2 ([11, Proposition 3]). *Let $R = \{v_1, v_2, \dots, v_n\}$ be a set of vectors of \mathbb{Z}^m , spanning \mathbb{Z}^m . Then the linear symmetries of R are the automorphisms of the edge-colored complete graph $H(R)$ whose vertices are the v_i , with the edge $v_i - v_j$ colored by the entry (i, j) of the matrix $Q = V^t(VV^t)^{-1}V$, where V is the matrix of order $m \times n$ whose columns are the v_i .*

In the case under consideration, the edge-colored graph $H(\mathcal{R})$ is shown in Figure 2, and its automorphism group is readily obtained: $S(\mathcal{R})$ is the direct product of the group S_2 of permutations of $\{c, f\}$ and of the wreath product $S_3 \wr S_2$ of the permutations of $\{d_1, e_2, g_1, d_2, e_1, g_2\}$ that stabilize or swap the subsets $\{d_1, e_2, g_1\}$ and $\{d_2, e_1, g_2\}$. In particular, $S(\mathcal{R})$ is generated by $v = (c, f)$, $x = (e_1, g_2)$ and $y = (e_1, d_2)$ that permute $\{e_1, g_2, d_2\}$, and $s = (d_1, d_2)(e_1, e_2)(g_1, g_2)$ that swaps $\{d_1, e_2, g_1\}$ and $\{d_1, e_1, g_2\}$.

Note that s is precisely the automorphism of \mathcal{R} induced by the known symmetry S of the Littlewood–Richardson coefficients. Let us consider the automorphisms t, u of the rays induced by the other two known symmetries T, U . It is easily calculated that

$$t = (d_1, d_2)(e_1, g_1)(e_2, g_2)(c, f), \quad u = (d_1, g_2)(d_2, e_2)(e_1, g_1).$$

Again with *SageMath*, we check that s, t, u and x already generate $S(\mathcal{R})$. More precisely, we get that $v = txsx$ and $y = usxsu$, which is easily checked by hand.

3.3. The symmetries of the rays are also symmetries of the Littlewood–Richardson coefficients. Note that any automorphism of the rays lifts uniquely to a linear automorphism of \mathbb{R}^6 . Let X be the lifting of x . Let G be the group of all liftings of the elements of $S(\mathcal{R})$.

After Lemmas 1 and 2, the group G contains all symmetries of C . We will check by explicit calculation that G is actually equal to the group of symmetries of C . Since G is generated by S, T, U and X , and that S, T and U are known to be symmetries of C , it is enough to check that X is a symmetry of C .

To check that X is a symmetry of C we proceed as follows: for each chamber κ , with generators $w_1, w_2, w_3, w_4, w_5, w_6$, we calculate $x(w_1), x(w_2), \dots$ (This is immediate since x just swaps e_1 and g_2 and leaves all other ray generators fixed). We check that they are the generators of a chamber $g(\kappa)$ of the Chamber Complex. Then we check by inspection that g is indeed a permutation of the chambers. Last, we check that for each κ , the polynomials $P_{g(\kappa)}$ and $P_\kappa \circ X^{-1}$ are equal. The latter calculation is performed within *SageMath*, and gives the expected result: X is indeed a symmetry of C . Finally, one calculates that X is given by

$$(\lambda_1, \lambda_2 \mid \mu_1, \mu_2 \mid \nu_1, \nu_2) \longmapsto (\lambda_1 + \mu_1 - \nu_2, \lambda_2 + \mu_1 - \nu_2 \mid \nu_2, \mu_2 \mid \nu_1, \mu_1).$$

3.4. Transitivity of the action on the chambers. By direct inspection of Table 1, one can check that the rays of each chamber are: b , one of $\{c, f\}$, two of $\{d_1, e_2, g_1\}$ and two of $\{d_2, e_1, g_2\}$. This proves that G permutes transitively the chambers, since G is exactly the stabilizer of $(\{c, f\}, \{\{d_1, e_2, g_1\}, \{d_2, e_1, g_2\}\})$.

4. FINAL REMARKS

4.1. Littlewood–Richardson coefficients associated to GL_3 . If we consider the Littlewood–Richardson coefficients associated to GL_3 , instead of SL_3 , we get one more generator for the group of symmetries, yielding in total $144 \times 2 = 288$ symmetries. The additional generator sends $(\lambda_1, \lambda_2, \lambda_3 \mid \mu_1, \mu_2, \mu_3 \mid \nu_1, \nu_2, \nu_3)$ to

$$(\lambda_1 - m, \lambda_2 - m, \lambda_3 - m \mid \mu_1 + m, \mu_2 + m, \mu_3 + m \mid \nu_1, \nu_2, \nu_3),$$

where $m = \lambda_3 - \mu_3$. In Rassart’s description [8] of the chamber complex for the Littlewood–Richardson coefficients associated to GL_3 , this generator swaps the additional rays a_1 and a_2 while fixing all other rays.

4.2. The case of SL_N for $N \geq 4$. One finds that the linear symmetries of the Littlewood–Richardson coefficients associated to SL_N , for N in $\{4, 5, 6, 7\}$, are only the 12 known symmetries. Indeed, the group of symmetries of these Littlewood–Richardson coefficients embeds in the group of linear symmetries of the ray generators of their support. But one computes that this group has only 12 elements[12].

5. PERSPECTIVES

There are two contributions in this work: the results (144 linear symmetries for the Littlewood-Richardson coefficients associated to the representations of SL_3 , a much bigger number than expected); and the method (embedding the group of the linear symmetries in the group of symmetries of a chamber complex, and the group of symmetries of its rays). This method may be applied to the study of the symmetries of other families of Littlewood-Richardson coefficients, and even more, to families of other representation-theoretic structural coefficients (such as Kostka coefficients, Kronecker coefficients, plethysm coefficients) as long as we have an explicit description of their piecewise quasipolynomial formulas. Explicit descriptions of piecewise quasipolynomial for such structural coefficients appear in the literature. For example, for the plethysm coefficients in the Schur expansion of $s_\mu[s_k]$ with μ any fixed partition of 3, 4 or 5 [13], for the Kronecker coefficients $g_{\lambda,\mu,\nu}$ with λ and μ of length ≤ 2 [14] or with λ, μ, ν of length ≤ 3 [15].

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