ON PARTITIONS WITH k CORNERS NOT CONTAINING THE STAIRCASE WITH ONE MORE CORNER.

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Abstract. We give three proofs of the following result conjectured by Carriegos, De Castro-García and Muñoz Castañeda in their work on enumeration of control systems: when $\binom{k+1}{2} \leq n < \binom{k+2}{2}$, there are as many partitions of n with k corners as pairs of partitions (α, β) such that $\binom{k+1}{2} + |\alpha| + |\beta| = n$.

1. Introduction

The *partitions with k corners* are the integer partitions whose diagram has k corners. They are also the partitions with parts of k distinct sizes. For instance, $(7, 4, 4, 2, 2, 2, 1)$ is a partition of 22 with 4 corners, since it has parts of 4 different sizes (7,4,2, and 1).

The enumeration of the partitions with k corners has interesting connections with number theory. This is already seen for partitions of n with one corner: they are in bijection with the divisors of n. These connections have been explored in detail by MacMahon [6], and, with a focus on the asymptotics, by Andrews [1]. The function counting the partitions of n with k corners is given by sequence number A116608 in The On-Line Encyclopedia Of Integer Sequences [3].

Recently, the problem of counting partitions with k corners has arised in a different context, the enumeration of linear control systems with coefficients in a commutative rings, in a paper of Carriegos, De Castro-García and Muñoz Castañeda [2]. The present paper is devoted to proving the following result that they conjectured.

Theorem 1 ([2, Conjecture 30])**.** When $\binom{k+1}{2} \leq n < \binom{k+2}{2}$, there are as many partitions of n with k corners as pairs of partitions (α, β) such that $|\alpha| + |\beta| = n - \binom{k+1}{2}$.

Note that $\binom{k+1}{2}$ is the size of ρ_k, the staircase partition with k corners, which is $(k, k-1, k-2, \dots, 1)$.

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We actually give three proofs of Theorem 1. The first one (section 3) is based on generating series. The second one (section 4) is based on a result due to Fine on some statistics on partitions. The last one (section 5) is a bijective proof of the following more general result.

Theorem 2. *For any $k \geq 0$ and $m \geq 0$, there are as many pairs of partitions (α, β) with $|\alpha| + |\beta| = m$ whose lengths fulfill $\ell(\alpha) + \ell(\beta) \leq k$, as partitions of $m + \binom{k+1}{2}$ with k corners whose diagram does not contain the staircase with $k+1$ corners.*

Theorem 1 is then obtained as a corollary of Theorem 2 (see section 5.1). Figure 1 shows the bijection used to prove Theorem 2, in action, for Theorem 1 with $k = 3$.

2. Basic facts and notations

2.1. **Partitions and their diagrams.** In this section, we recall classical operations and notations for integer partitions. See [5, I.1] for further details.

Given a partition λ, we denote with λ' its conjugate and $\ell(\lambda)$ its length. By $\lambda \vdash n$ we means that λ is a partition of n. We call n the *weight* of λ, and denote it often with $|\lambda|$. The *diagram* of $\lambda = (\lambda_1, \lambda_2, \ldots)$ is the set of integer points (i, j) such that $1 \leq j \leq \ell(\lambda)$ and $1 \leq i \leq \lambda_j$.

Given a partition λ, denote for each i with m_i the multiplicity of i as a part of λ. Then the *exponential notation* for λ is $(1^{m_1}2^{m_2}3^{m_3}\cdots)$. For instance, the exponential notation for $(7, 4, 4, 2, 2, 2, 2, 1)$ is $(1^1 2^4 4^2 7^1)$.

Given two partitions $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $\beta = (\beta_1, \beta_2, \ldots)$ with exponential notations $(1^{m_1}2^{m_2}3^{m_3}\cdots)$ and $(1^{n_1}2^{n_2}3^{n_3}\cdots)$ respectively, their *sum* is the partition $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots)$ and their *union*, $\alpha \cup \beta$, is the partition with exponential notation $(1^{m_1+n_1}2^{m_2+n_2}3^{m_3+n_3}\cdots)$. For instance, if $\alpha = (7, 4, 4, 2, 2, 2, 2, 1)$ and $\beta = (4, 2, 1)$ then $\alpha + \beta = (7 + 4, 4 + 2, 4 + 1, 2 + 0, 2 + 0, 2 + 0, 2 + 0, 1 + 0) = (11, 6, 5, 2, 2, 2, 2, 1)$ and $\alpha \cup \beta = (7, 4, 4, 4, 2, 2, 2, 2, 2, 1, 1)$. The two operations are related by the identity $(\alpha + \beta)' = \alpha' \cup \beta'$.

2.2. **Corners.** A corner of (the diagram of) λ is a point (i, j) in the diagram of λ, such that neither $(i + 1, j)$ nor $(i, j + 1)$ is in the diagram of λ. The partitions with k corners are exactly the partitions with parts of k sizes. This shows in particular that if λ is a partition with parts of k sizes, so is its conjugate λ'.

We will denote with $\nu(n; k)$ the number of partitions of n with k corners.

Remember that we denote with ρ_k the "staircase partition of length k", which is $(k, k - 1, \ldots, 1)$. This is the smallest partition with k corners, in a sense made precise by the following lemma that we will use implicitly in the sequel.

Lemma 1. *For any $k \geq 0$, the diagram of any partition with k corners contains the diagram of ρ_k.*

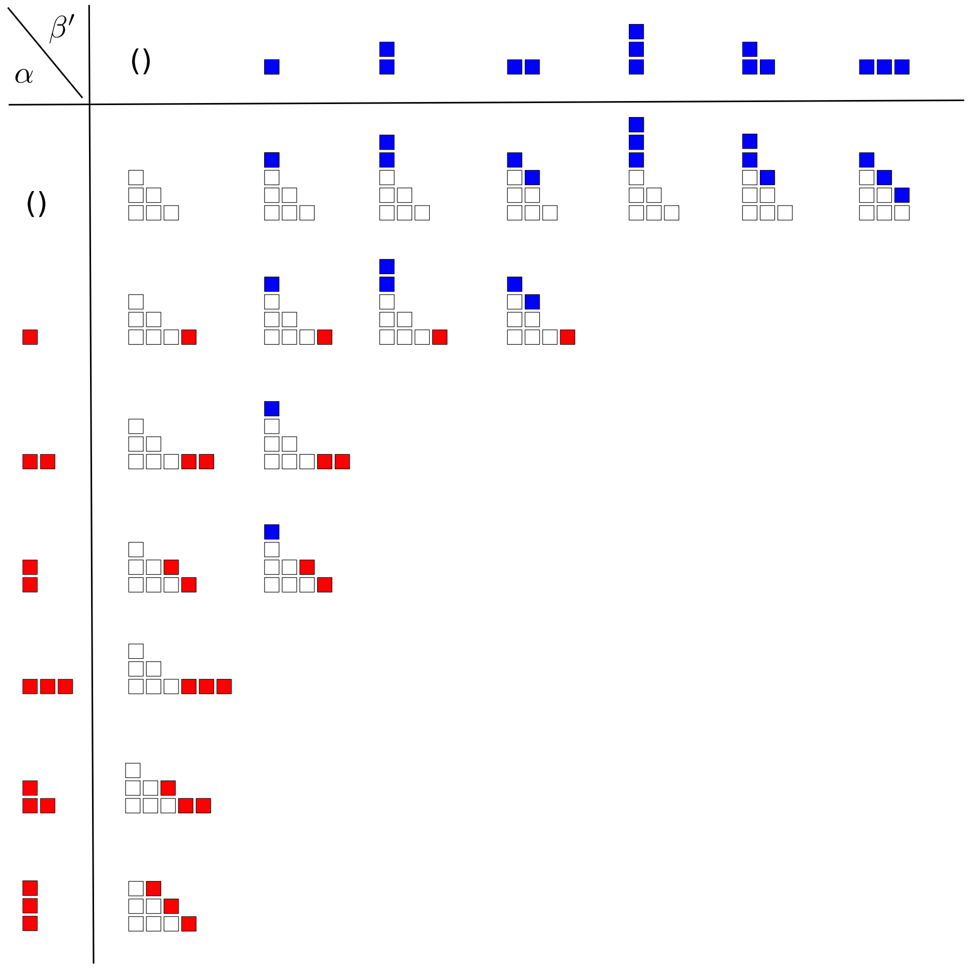

FIGURE 1. The bijection $(\alpha, \beta) \mapsto (\rho_3 \cup \beta') + \alpha$ described in Lemma 4, putting in correspondence the pairs of partitions (α, β) with $|\alpha| + |\beta| \le 3$, with the partitions λ with 3 corners and weight $< \binom{5}{2}$. This illustrates Theorem 1 for $k = 3$.

Proof. Let $k \ge 0$ and let λ be a partition with k corners. Then λ is of the form $(p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k})$ with $p_1 > p_2 > \cdots > p_k > 0$ and all $m_i > 0$.

For each i, set $\mu_i = p_i - (k - i + 1)$. Then $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_k \ge 0$, so $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ is a partition. We have:

$$\lambda = (\rho_k + \mu) \cup (p_1^{m_1-1} p_2^{m_2-1} \cdots p_k^{m_k-1}).$$

This shows that the diagram of λ contains the diagram of ρ_k. $\square$

We will also make repeated use of the following converse of Lemma 1.

Lemma 2. *Let $k \geq 0$, and let λ be a partition. If the diagram of λ does not contain the diagram of ρ_{k+1}, then λ has at most k corners. This is the case in particular for any partition λ of weight less than $\binom{k+2}{2}$.*

3. Proof with generating series.

In this section, we prove Theorem 1 by means of generating series.

The generating series of the numbers $\nu(n;k)$, of partitions of n with k corners, is

$$F(x,q) = \sum_{k,n} \nu(n;k) x^k q^n.$$

It easily seen to be given by:

$$F(x,q) = \prod_{i=1}^{\infty} \left(1 + x\frac{q^i}{1-q^i}\right).$$

Thus

$$\begin{aligned} F(x,q) &= \prod_{i=1}^{\infty} \frac{1+(x-1)q^i}{1-q^i} \\ &= \prod_{i=1}^{\infty} (1+(x-1)q^i) \cdot \prod_{i=1}^{\infty} \frac{1}{1-q^i} \end{aligned}$$

But after [7, (1.86)],

$$\prod_{i=1}^{\infty}(1+xq^i) = \sum_{j=0}^{\infty} \frac{x^j q^{\binom{j+1}{2}}}{(1-q)(1-q^2)\cdots(1-q^j)}. \tag{1}$$

With $x-1$ for x (a substitution that is allowed since the above identity is for series in q whose coefficients are polynomials in x), we get

$$\prod_{i=1}^{\infty}(1+(x-1)q^i) = \sum_{j=0}^{\infty} \frac{(x-1)^j q^{\binom{j+1}{2}}}{(1-q)(1-q^2)\cdots(1-q^j)}.$$

Therefore,

$$F(x,q) = \sum_{j=0}^{\infty} \frac{(x-1)^j q^{\binom{j+1}{2}}}{(1-q)(1-q^2)\cdots(1-q^j)} \cdot \prod_{i=1}^{\infty} \frac{1}{1-q^i}.$$

We now consider n and k as in Theorem 1. In the above sum, the summands with indices $j < k$ have degree in x at most $k-1$, and thus don't contribute to the coefficient of $x^k q^n$. The expansions of the summands with indices $j > k$ involve only monomials $x^m q^d$ with $d \geq \binom{k+2}{2} > n$, and thus these summands don't contribute to the coefficient of $x^k q^n$.

Therefore, $\nu(n;k)$ is the coefficient of $x^k q^n$ in the summand with index $j=k$, which is

$$\frac{(x-1)^k q^{\binom{k+1}{2}}}{(1-q)(1-q^2)\cdots(1-q^k)}\cdot\prod_{i=1}^{\infty}\frac{1}{1-q^i}$$

After expanding $(x-1)^k$, we get that this is simply the coefficient of q^n in

$$\frac{q^{\binom{k+1}{2}}}{(1-q)(1-q^2)\cdots(1-q^k)}\cdot\prod_{i=1}^{\infty}\frac{1}{1-q^i}$$

which is the coefficient of $q^{n-\binom{k+1}{2}}$ in

$$\frac{1}{(1-q)(1-q^2)\cdots(1-q^k)}\cdot\prod_{i=1}^{\infty}\frac{1}{1-q^i}$$

The coefficient of q^h in this last series is the number of pairs of partitions (λ,μ) such that $|\lambda|+|\mu|=h$, and λ has length at most k. For $h\leq k$, the condition on the length can be dropped. This is the case in particular for $h=n-\binom{k+1}{2}$, since $n<\binom{k+2}{2}=\binom{k+1}{2}+k+1$.

4. Proof from statistics on partitions

We now give another proof of Theorem 1, based on the following result due to Fine.

Theorem 3 ([4, Theorem 4 in Chapter 2])**.** *For any $r\geq 0$,*

$$\sum_{\lambda\vdash n}\binom{Q(\lambda)}{r}=\sum_{\lambda\vdash n}m_1(\lambda)m_2(\lambda)\cdots m_r(\lambda) \tag{2}$$

where $Q(\lambda)$ is the number of corners of λ, and $m_i(\lambda)$ stands for the multiplicity of i as a part of λ.

To prove Theorem 1 from Theorem 3, consider n and k such that

$$\binom{k+1}{2}\leq n<\binom{k+2}{2}.$$

After Lemma 2, the diagram of any partition of n has at most k corners. Apply Theorem 3 with $r=k$. For any $\lambda\vdash n$, either λ has less than k corners, and then $\binom{Q(\lambda)}{k}=0$, or λ has exactly k corners, and then $\binom{Q(\lambda)}{k}=1$. The right-hand side in (2) is thus $\nu(n;k)$.

For the right-hand side, observe that for any partition $\lambda\vdash n$, either λ has a part bigger than k, but then it can't have parts of all sizes $1,2,\ldots,k$ (else its weight would be at least $1+2+\cdots+k+(k+1)=\binom{k+2}{2}$), and thus $m_1(\lambda)m_2(\lambda)\cdots m_k(\lambda)=0$; or λ has all its parts smaller than or equal to k. We get thus:

$$\nu(n;k)=\sum m_1(\lambda)m_2(\lambda)\cdots m_k(\lambda)$$

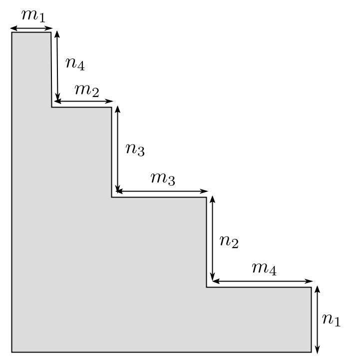

FIGURE 2. The border coordinates $(m_1, m_2, m_3, m_4; n_1, n_2, n_3, n_4)$ of a partition with 4 corners.

where the sum is over all partitions $\lambda \vdash n$ with all their parts in $\{1, 2, \ldots, k\}$. Each summand in the right-hand side counts the ways to decompose $(m_1 - 1, m_2 - 1, \ldots, m_k - 1)$ as a sum of two vectors of nonnegative integers:

$$(m_1 - 1, m_2 - 1, \ldots, m_k - 1) = (a_1, a_2, \ldots, a_k) + (b_1, b_2, \ldots, b_k)$$

This is also the ways of decomposing $\gamma = (1^{m_1-1} 2^{m_2-1} \cdots k^{m_k-1})$ as $\alpha \cup \beta$ (with $\alpha = (1^{a_1} 2^{a_2} \cdots k^{a_k})$ and $\beta = (1^{b_1} 2^{b_2} \cdots k^{b_k})$.) So the right-hand side counts the pairs of partitions (α, β) with $|\alpha| + |\beta| = n - \binom{k+1}{2}$ and all parts $\leq k$. The condition on the maximal size of the parts can be dropped, because the weights are already bounded by k.

5. Bijective proof

We will prove here Theorem 2. Before, we explain how Theorem 1 is derived from Theorem 2.

5.1. Derivation of Theorem 1 from Theorem 2. To deduce Theorem 1 from Theorem 2, consider $n < \binom{k+2}{2}$ and set $m = n - \binom{k+1}{2}$. Then $m \leq k$. Observe now that all pairs of partitions (α, β) with $|\alpha| + |\beta| = m$ fulfill $\ell(\alpha) + \ell(\beta) \leq |\alpha| + |\beta| = m \leq k$. Also, all partitions λ of n fulfill that their diagram does not contain the diagram of ρ_{k+1}, after Lemma 2.

5.2. Border coordinates of a partition. In order to prove Theorem 2, we will use the *border coordinates* for partitions, that we introduce now.

Let λ be a partition with parts of k sizes. Let $q_1, q_2, \ldots, q_k$ be the distinct parts of λ, with $q_1 > q_2 > \cdots > q_k$, and $p_1 > p_2 > \cdots > p_k$ be the distinct parts of the conjugate partition λ'. For each i, let m_i (resp. n_i) be the multiplicity of p_i (resp. q_i) in λ' (resp. λ). We call the pair of sequences $(m_1, m_2, \ldots, m_k; n_1, \ldots, n_k)$ the *border coordinates* of λ, since they are the lengths of the horizontal and vertical segments in the border of the diagram of λ.

Lemma 3. *Let λ be a partition with k corners and border coordinates $(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_k)$.*

(1) *Let α be a partition whose parts are all parts of λ. For each i, let a_i be the multiplicity of q_i in α. Then the border coordinates of $\lambda \cup \alpha$ are $(m_1, m_2, \ldots, m_k; n_1 + a_1, n_2 + a_2, \ldots, n_k + a_k)$.*

(2) *Let α be a partition such that all parts of α' are among the parts of λ'. For each i, let b_i be the multiplicity of p_i in α'. Then the border coordinates of $\lambda + \alpha$ are $(m_1 + b_1, m_2 + b_2, \ldots, m_k + b_k; n_1, n_2, \ldots, n_k)$.*

Proof. Let $p_1 > p_2 > \ldots > p_k$ be the parts of the conjugate λ' of λ. Set also $p_{k+1} = 0$. Then part 1 of Lemma 3 follows from the fact that, for each i, one has $m_i = p_i - p_{i+1}$, a fact that is not affected by making the union with a partition α whose parts are already parts of λ. For part 2, one can use that $\lambda + \alpha = (\lambda' \cup \alpha')'$. □

5.3. **Proof of Theorem 2.** Theorem 2 follows from the more precise lemma below.

Lemma 4. *The map $(\alpha, \beta) \mapsto (\rho_k \cup \beta') + \alpha$ establishes a bijection between the pairs of partitions (α, β) such that $\ell(\alpha) + \ell(\beta) \leq k$, and the partitions λ with k corners whose diagram does not contain the diagram of ρ_{k+1}.*

Proof. Let α and β be two partitions whose lengths have sum at most k. There exist $p \geq \ell(\alpha)$ and $q \geq \ell(\beta)$ with $p + q = k$. Set b_i (resp. a_i) for the multiplicity of i in β' (resp. α'). After Lemma 3, the border coordinates of $\rho_k \cup \beta'$ are $(1^k; 1 + b_1, 1 + b_2, \ldots, 1 + b_p, 1^q)$ where a^b stands for "the sequence of b occurrences of a". Next, again after Lemma 3, the border coordinates of $(\rho_k + \beta') + \alpha$ are $(1 + a_1, 1 + a_2, \ldots, 1 + a_q, 1^p; 1 + b_1, 1 + b_2, \ldots, 1 + b_p, 1^q)$. This shows that the map $(\alpha, \beta) \mapsto (\rho_k + \beta') + \alpha$ injects the pairs of partitions whose lengths have sum at most k in the set of partitions with k corners. Observe also that the operation of union with β' does not add any box in column $p + 1$, and the operation of sum with α does not add any box in row $q + 1$. Thus the box $(p + 1, q + 1)$, which lies in the diagram of ρ_{k+1}, is not in the diagram of $(\rho_k + \beta') + \alpha$. Thus the injection is with values in the set of partitions with k corners whose diagram does not contain the diagram of ρ_{k+1}.

Reciprocally, λ be a partition with k corners, whose diagram does not contain the diagram of ρ_{k+1}. Since λ has k corners, its diagram contains the diagram of ρ_k. Therefore, there exists (i_0, j_0) not in the diagram of λ, lying in the set difference of the diagram of ρ_{k+1} and the diagram of ρ_k. Thus $i_0 + j_0 = k + 2$. Let p be the number of corners (i, j) of the diagram of λ with $i < i_0$, and let q be the number of corners (i, j) with $j < j_0$. We have $p \leq i_0 - 1$ since there is at most one corner in each column. Similarly, $q \leq j_0 - 1$, because there is at most one corner in each row. Also, any point (i, j) in the diagram of λ fulfills $i \leq p$ or $j \leq q$. This is the case in particular for the k corners. Thus $k \leq p + q$. Altogether we get $k \leq p + q \leq (i_0 - 1) + (j_0 - 1) = k$. As a consequence, $k = p + q$ and

$p = i_0 - 1$, $q = j_0 - 1$. There are p corners in the first p columns and q corners, different from the previous ones, in the first q rows. We conclude that λ has border coordinates of the form $(f_1, \ldots, f_q, 1^p; g_1, \ldots, g_p, 1^q)$ for some positive numbers f_i and g_i. Thus $\lambda = (\rho_k \cup \beta') + \alpha$ for $\alpha = (1^{f_1-1} 2^{f_2-1} \cdots q^{f_q-1})$ and $\beta = (1^{g_1-1} 2^{g_2-1} \cdots p^{g_p-1})$. $\square$

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