On a matrix function interpolating between determinant and permanent

Emmanuel Briand

Departamento Matemáticas, Facultad de Ciencias, Estadística y Computación, Universidad de Cantabria, Avenidad de Los Castros s/n, 39005 Santander, Spain

Abstract

A matrix function, depending on a parameter t, and interpolating between the determinant and the permanent, is introduced. It is shown this function admits a simple expansion in terms of determinants and permanents of sub-matrices. This expansion is used to explain some formulas occurring in the resolution of some systems of algebraic equations.

AMS classification: 05E05; 13P10; 15A15

Keywords: Matrix functions; Multisymmetric functions; Zero-dimensional algebraic systems

1. A function interpolating between permanent and determinant

Given a permutation σ of $\{1, \ldots, r\}$, denote with $\mathcal{O}(\sigma)$ the set of its orbits, and define the following class function with parameter:

$$\gamma_t(\sigma) = \prod_{\omega \in \mathcal{O}(\sigma)} (1 - t^{|\omega|}).$$

Here $|\omega|$ is the cardinality of ω. One can now consider the following function, defined upon the square matrices of order r:

$$\Gamma_t(M) = \sum_{\sigma \in \mathfrak{S}_r} \gamma_t(\sigma) \prod_i m_{i,\sigma(i)}. \tag{1}$$

The object of this note is the following identity:

Theorem 1

$$\Gamma_t(M) = \sum (-t)^{|I|} \det M[I] \operatorname{per} M[I^*], \tag{2}$$

where the sum is carried over the subsets I of $\{1, \ldots, r\}$.

The notation $M[I]$ refers to the submatrix of M obtained by keeping only the lines with index in I, and as well the columns with index in I; and I^* is the complement of I.

Proof. Expanding the product

$$\gamma_t(\sigma) = \prod_{\omega \in \mathcal{O}(\sigma)} (1 - t^{|\omega|}),$$

one gets

$$\sum_{\mathcal{U} \subset \mathcal{O}(\sigma)} (-t)^{\sum_{\omega \in \mathcal{U}} |\omega|}.$$

Associate to each subset $\mathcal{U}$ of $\mathcal{O}(\sigma)$ the union I of the orbits that are elements of $\mathcal{U}$. This way each subset stabilized by σ is obtained exactly once. The permutation σ induces a permutation σ_I of I and a permutation σ_{I^*} of its complement; the signature of σ_I is $(-1)^{|I|-|\mathcal{U}|}$. Thus the product is also equal to

$$\sum_{I | \sigma(I) = I} (-t)^{|I|} \varepsilon(\sigma_I).$$

Using this in the formula (1) where $\Gamma_t(M)$ is defined, it comes:

$$\sum_{\sigma \in \mathfrak{S}_r, I | \sigma(I) = I} (-t)^{|I|} \varepsilon(\sigma_I) \prod_i m_{i,\sigma(i)},$$

which decomposes into

$$\sum_{\sigma \in \mathfrak{S}_r, I \,|\, \sigma(I) = I} (-t)^{|I|} \varepsilon(\sigma_I) \prod_{i \in I} m_{i,\sigma_I(i)} \prod_{i \in I^*} m_{i,\sigma_{I^*}(i)}$$
$$= \sum_I (-t)^{|I|} \det M[I] \operatorname{per} M[I^*].$$

And this is the announced result. □

The function Γ_t interpolates between the permanent and the determinant because:

- when $t = 0$, that is: when considering the constant term, one obtains the permanent.

- when $t \to -\infty$, that is: when considering the leading coefficient, that of t^n, one obtains the determinant.

Another interesting specialization is obtained with $t = 1$; it gives an identity due to Muir [1]:

$$0 = \sum_I (-1)^{|I|} \det M[I] \operatorname{per} M[I^*].$$

Yet another specialization, $t = 1/2$, appears in the study of certain systems of algebraic equations, where the identity (2) happens to be useful. This is what we expose in the next part.

2. An application to some systems of polynomial equations

Let a system of r quadratic equations $F_1 = \cdots = F_r = 0$ in r unknowns $X_1, \ldots, X_r$, with the following particular shape:

$$F_i = X_i^2 + \sum_{j=1}^{r} u_{i,j} X_j + c_i.$$

Such a system has always finitely many roots, namely 2^r if each root is counted as many times as indicated by its multiplicity. We will denote with U the matrix of the u_{ij}. One wants to calculate the multisymmetric power sums of the multiset of the roots of this system [1], denote them with p_α for $\alpha \in \mathbb{N}^r$. This means that p_α is the sum of the evaluations of the monomial $X_1^{\alpha_1} \cdots X_r^{\alpha_r}$ at the roots (repeated when there are multiplicities). The power sums which may be difficult to obtain are the first ones: the p_α with α with $0 - 1$ coordinates [2]). These first power sums may be computed using the *identities of Aizenberg and Kytmanov* [3] [3]:

$$\sum_{\alpha \in \mathbb{N}^r} \frac{p_\alpha}{X^\alpha} = \Omega \mathscr{S} \frac{X_1 \cdots X_r \mathrm{Jac}}{F_1 \cdots F_r}. \tag{3}$$

In the right-hand side of the formula, Jac is the Jacobian of $F_1, \ldots, F_r$; the $\mathscr{S}$ symbol means that a series expansion, following the decreasing total degree, is applied to

---

[1] Such analogues of the symmetric functions where introduced by several authors at the end of the nineteenth century. See [2] for more references and specially connections to systems of polynomial equations with finitely many solutions.

[2] The other power sums are easily deduced from the first ones. For instance, the first equation of the system yields the inductive formula

$$p_{(\alpha_1+2),\alpha_2,\ldots,\alpha_r} + \sum_j u_{1,j} p_{\alpha_1,\ldots,(1+\alpha_i),\ldots,\alpha_r} + c_1 p_\alpha = 0.$$

[3] Such identities were first written by Jacobi, in [4].

the rational fraction (this is possible because the denominator has exactly one term of maximal total degree, that is: $X_1^2 \cdots X_r^2$), and next that all the Laurent monomials *with numerator* are removed from this expansion (Ω symbol).

Applying this to the first examples of the considered systems, one finds that

- for $r = 1$,
  $$p_1 = -u_{11},$$
- for $r = 2$,
  $$p_{11} = u_{11}u_{22} + 3u_{12}u_{21},$$
- for $r = 3$,
  $$p_{111} = -u_{11}u_{22}u_{33} - 3u_{11}u_{23}u_{32} - 3u_{13}u_{22}u_{33} - 3u_{12}u_{21}u_{33} \\ -7u_{12}u_{23}u_{31} - 7u_{13}u_{21}u_{32}.$$

These observations, and a few next ones, lead to conjecture that

$$p_{11\cdots 1} = (-1)^r \sum_{\sigma \in \mathfrak{S}_r} \prod_{\omega \in \mathcal{O}(\sigma)} (2^{|\omega|} - 1) \prod_i u_{i,\sigma(i)}. \tag{4}$$

Making use of the identity (2), we will prove the above formula, and the more general one:

Theorem 2

$$p_A = (-1)^{|A|} 2^{n-|A|} \sum_{\sigma \in \mathfrak{S}_A} \prod_{\omega \in \mathcal{O}(\sigma)} (2^{|\omega|} - 1) \prod_{i \in A} u_{i,\sigma(i)}. \tag{5}$$

Here was made the following abuse of notation: given a subset A of $\{1, \ldots, n\}$, it was also denoted with A the vector whose ith coordinate is 1 if $i \in A$, 0 if $i \notin A$.

Proof. It is convenient to re-write the quotient under the form:

$$\frac{\mathrm{Jac}/(X_1 \cdots X_r)}{(F_1/X_1^2) \cdots (F_r/X_r^2)},$$

so that each of its term lies in the sub-algebra of Laurent monomials generated by the $1/X_i$ and the X_i/X_j^2; that is on the one hand

$$\frac{\mathrm{Jac}}{X_1 \cdots X_r} = \sum_I 2^{n-|I|} \frac{\det U[I]}{X^I},$$

and in the other

$$\frac{F_i}{X_i^2} = 1 + \sum_j u_{i,j} \frac{X_j}{X_i^2} + \frac{c_i}{X_i^2}.$$

Now we may work *modulo* the ideal generated by the $1/X_i^2$ and the $(X_j/X_i^2)^2$, since we are only interested in the monomials with shape $1/X^A$, which all lie outside this ideal.

One has then

$$\frac{F_i}{X_i^2} \equiv 1 + \sum_j u_{i,j} \frac{X_j}{X_i^2} \equiv \prod_j \left(1 + u_{i,j} \frac{X_j}{X_i^2}\right)$$

which has as inverse

$$\prod_j \left(1 - u_{i,j} \frac{X_j}{X_i^2}\right).$$

Thus it remains to compute the terms in $1/X^A$ in the expression:

$$\sum_I 2^{n-|I|} \frac{\det U[I]}{X^I} \prod_{i,j} \left(1 - u_{i,j} \frac{X_j}{X_i^2}\right),$$

and in this aim, it is now sufficient to work *modulo* the module over the polynomials in $1/X_1, \ldots, 1/X_r$ generated by the X_j/X_i^2 for $i \neq j$. One has then:

$$\prod_{i,j} \left(1 - u_{i,j} \frac{X_j}{X_i^2}\right) \equiv \sum_I (-1)^{|I|} \frac{\operatorname{per} U[I]}{X^I},$$

because the $X^J/(X^I)^2$ that appear in the expansion of the product are in the submodule unless $I = J$. Finally the coefficient of $1/X^A$ is given by

$$\sum_{I \subset A} 2^{n-|I|} \det U[I] (-1)^{|A|-|I|} \operatorname{per} U[A \backslash I],$$

and one recognizes in it

$$(-1)^{|A|} 2^n \Gamma_{1/2}(U[A]),$$

which re-writes easily as formula (5). □

References

[1] T. Muir, A relation between permanents and determinants, Proc. Roy. Soc. Edinburgh 22 (1897) 134–136.

[2] E. Briand, Polynômes multisymétriques, Thèse de doctorat, Université de Rennes 1 et Universidad de Cantabria, 2002.

[3] L.A. Aĭzenberg, A.M. Kytmanov, Multidimensional analogues of Newton's formulas for systems of nonlinear algebraic equations and some of their applications, Sibirsk. Mat. Zh. 22 (2) (1981) 19–30; English transl. in Sib. Math. J. 22 (1981) 180–189.

[4] C. Jacobi, Theoremata nova algebrica systema duarum aequationum inter duas cariabiles propositarum, J. Reine Angew. Math. 14 (1835) 281–288 [in Latin].