Brill's equations of the subvariety of the products of linear forms

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Abstract

The products of linear forms in N variables are a subvariety of the space of the degree forms of degree n in N variables. At the end of the nineteenth century, Brill and Gordan used invariant theory to design a method to derive a system of equations defining this subvariety (*Brill's equations*). We show how to compute efficiently Brill's equations, and compare them with the ideal of the subvariety of products of linear forms.

Introduction

In the affine space of the forms of degree n in N variables with complex coefficients, the set of the products of n linear forms is a closed algebraic subvariety. It is proper as soon as $N > 2$. We denote this subvariety with $V(N;n)$ and its ideal with $I(N;n)$. The affine subvariety $V(N,n)$ is the affine cone over a projective variety that we denote with $C(N;n)$.

At the end of the nineteenth century, Alexander von Brill looked for several ways of deriving systems of equations of $V(N;n)$.

His first solution uses invariant theory ([3]): a set of covariants is produced, whose simultaneous vanishing is a necessary and sufficient condition for the complete factorizability of a form. Gordan ([6]) distinguished in Brill's set of covariants a particular one (*Brill's covariant*) that gives already a necessary and sufficient condition for complete factorisability. He also gave geometric insight on the meaning of this covariant. More recently, the modern representation-theoretic meaning of Brill's covariant was exposed in [5].

Brill's second idea ([2]), mainly developped by his student Junker ([8]) , consists in using the diagonal invariants of the symmetric groups $\mathfrak{S}_n$ (often called *multisymmetric polynomials* now). They are some kind of analogues of the symmetric polynomials. There exist *elementary multisymmetric polynomials* that, contrary to the elementary symmetric polynomials, are connected by algebraic relations. The latter are very closely related to the equations of $V(N;n)$. In [1], an algorithm to compute these relations was presented, and it was explained how to deduce from it a generating set of $I(N;n)$ through Gröbner-basis methods.

Brill's covariant depends of, besides the coefficients of the basis form f, three sets of N variables: $x = (x_1, \ldots, x_N), y = (y_1, \ldots, y_N), z = (z_1, \ldots, z_N)$, so it has has the following shape:

$$B(f,x,y,z) = \sum b_{\alpha,\beta,\gamma}(f)x^\alpha y^\beta z^\gamma,$$

where the α, β, γ are multi-exponents of length N. The necessary and sufficient condition for complete factorisability of f is given by the *identical* vanishing of $B(f,x,y,z)$, that is the simultaneous vanishing of all the coefficients $b_{\alpha,\beta,\gamma}(f)$, that are all forms of degree $n+1$ in the coefficients of f. These conditions are Brill's equations.

In this article, after shortly recalling how is built Brill's covariant, we explain how to compute Brill's equations efficiently. The problem is the following: Brill's covariant is a big polynomial, and we are only interested in knowing its coefficients separetely. In a naive computation, the derivation of the whole covariant is a bottleneck. We show that classical invariant theory gives the tool to bypass the bottleneck. Last, we exploit the results of the achieved computations to compare Brill's equations to $I(N;n)$.

1 Brill's covariant

We recall how Brill's covariant is defined.

Before, we need to introduce a definition. Let g a from of degree k in the vector variable $x = (x_1, \ldots, x_N)$. Introduce a twin vector variable $y = (y_1, \ldots, y_N)$ and decompose

$$g(x+y) = \sum_{i=0}^{k} g^{(i)}(x,y),$$

where $g^{(i)}(x,y)$ is the degree i homogeneous part in the x-variables (equivalently: the degree $(k-i)$ homogeneous part on the y variables). The $g^{(i)}$ are the *polarizations* of g.

Then:

- consider the polynomial in a variable u, monic, of degree n, whose term of degree i has as coefficient:
$$C_i(f;x;z) = (-1)^i f^{(i)}(x;z) f(z)^{i-1}.$$
- Compute $P_{f,z}(x)$, its power sum of degree n (for instance using the *Newton identities*).
- Brill's covariant is the *apolar covariant* of $f(x)$ and $p_{f,z}(x)$, that is the object defined from the polarizations of f and $P_{f,z}$ as follows:

$$B(f;x;y;z) = \frac{1}{n+1}\sum_{i=0}^{n}(-1)^i i!(n-i)! f^{(i)}(x;y) P_{f,z}^{(i)}(y;x).$$

Brill's covariant is a polynomial, homogeneous of degree $n+1$ in the coefficients of the basis form f, homogeneous of degree n in the x-variables, homogeneous of degree n in the y-variables and homogeneous of degree $n(n-1)$ in the z-variables.

2 The differential equation

Because B is a covariant, it fulfills some linear partial differential equations that can be used to compute its coefficients by induction. This kind of properties has been well-known for a long time in invariant theory (see [7], and [9] for a modern interpretation).

We shall here write the basis form f as

$$f = \sum_{\omega} a_\omega x^\omega,$$

where the ω are multi-exponents of length N with coordinate sum n.

The fact that B is a covariant means precisely that for any linear automorphism θ with determinant 1 of $\mathbb{C}^N$, one has:

$$B(f \circ \theta, x, y, z) = B(f, \theta(x), \theta(y), \theta(z)). \tag{1}$$

Specially, this holds when θ is the one-parameter automorphism:

$$\theta_t : (v_1, \ldots, v_{j-1}, v_j, v_{j+1}, \ldots) \longmapsto (v_1, \ldots, v_{j-1}, v_j + tv_1, v_{j+1}, \ldots)$$

for some j.

If Equality (1), with $\theta = \theta_t$, is derivated with respect to t, and if next it is set $t = 0$, then the following linar partial differential equations come:

$$\Delta_j(B) = x_1 \frac{\partial B}{dx_j} + y_1 \frac{\partial B}{dy_j} + z_1 \frac{\partial B}{dz_j}$$

where

$$\Delta_j = \sum_{\omega} (1 + \omega_j)\, a_{\omega + \xi_j - \xi_1} \frac{\partial}{da_\omega},$$

with ξ_j the j-th vector of the canonical basis of $\mathbb{Z}^n$. Extract from the last equation the coefficient of $x^\alpha y^\beta z^\gamma$, it is obtained:

$$\Delta_j(b_{\alpha,\beta,\gamma}) = (1 + \alpha_j) b_{\alpha+\xi_j-\xi_1,\beta,\gamma} + (1 + \beta_j) b_{\alpha,\beta+\xi_j-\xi_1,\gamma} + (1 + \gamma_j) b_{\alpha,\beta,\gamma+\xi_j-\xi_1}.$$

Now one can write it as follows:

$$(1 + \gamma_j) b_{\alpha,\beta,\gamma+\xi_j-\xi_1} = \Delta_j b_{\alpha,\beta,\gamma} - (1 + \alpha_j) b_{\alpha+\xi_j-\xi_1,\beta,\gamma} - (1 + \beta_j) b_{\alpha,\beta+\xi_j-\xi_1,\gamma}.$$

It gives a way to compute all the $b_{\alpha,\beta,\gamma}$, successively, from the $b_{\alpha,\beta,(n(n-1),0,\ldots,0))}$'s. These quantities are precisely the x, y-coefficients of $B(f, x, y, \xi_1)$; by analogy with the classical case of the covariants of binary forms ([7]), we call it the *source* of Brill's covariant.

This way the size of the biggest object in the computation has been considerably reduced: the z-variables (in which there was a dependance of degree $n(n-1)$) have been eliminated.

3 Experimental study of Brill's equations

Some questions about Brill's equations are:

- do they generate the ideal $I(N,n)$?
- if not, do they at least define the projective subvariety $C(N;n)$ locally ? That is: do, in any affine chart, the restrictions of Brill's equations generate the ideal of the trace of $C(N,n)$?

One can use the achieved computations to investigate these properties.

For the first question, remark that all of Brill's equations have degree exactly $n+1$. One can show ([4]) that the dimension of the degree $n+1$ component of $I(N;n)$ is at least $K_1 - K_2$ with

$$K_1 = \binom{k_1+n}{n+1}, \qquad K_2 = \binom{k_2+n-1}{n},$$

where

$$k_1 = \binom{n+N-1}{n}, \qquad k_2 = \binom{n+N}{n+1}.$$

When $(N,n) = (3,4)$, Brill's equations span a space of dimension 396, while $K_1 - K_2 = 1002$. When $(N,n) = (4,3)$, they span a space of dimension 875, while $K_1 - K_2 = 1085$. So in both cases, Brill's equations don't span the degree $n+1$ component of $I(N;n)$, and, *a fortiori*, don't generate the ideal.

Using Brill's second idea, it is possible to compute a Gröbner base for the ideal of the trace of $C(N;n)$ in an affine chart. This is detailed in [1]. By comparing this Gröbner base with the restrictions of Brill's equations to the chart, one observes that when $(N,n) = (3,4)$, Brill's equations cut out $C(N,n)$ locally, but when $(N,n) = (4,3)$ they don't (see [1] for the computations, and [4] for an alternative way of obtaining this).

When $(N,n) = (3,3)$, the span of Brill's equations has dimension $K_1 - K_2 = 35$. From a Gröbner base for the ideal of the trace of $C(N;n)$ one can always (in principle) compute another Gröbner base (through a *Gröbner walk*) from which a Gröbner base for $I(N;n)$ is then straightforwardly deduced. This is easy when $(N,n) = (3,3)$ and shows that Brill's equations generate $I(3;3)$.

This experiments can be (and will be) continued for bigger values of (N,n). Actually the biggest step in the computation of Brill's equations in the cases $(N,n) = (4,4), (5,3)$ has already been performed.

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