A complete set of covariants of the four qubit system

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Abstract. We obtain a complete and minimal set of 170 generators for the algebra of $SL(2, \mathbb{C})^{\times 4}$-covariants of a binary quadrilinear form. Interpreted in terms of a four qubit system, this describes in particular the algebraic varieties formed by the orbits of local filtering operations in its projective Hilbert space. Also, this sheds some light on the local unitary invariants, and provides all the possible building blocks for the construction of entanglement measures for such a system.

PACS numbers: O2.20.Qs, 03.65.Ud, 02.10.Xm

1. Introduction

The invariant theory of fourth rank hypermatrices (or quadrilinear forms) over a two-dimensional complex vector space, already considered at the end of the nineteenth century, has experienced recently a regain of interest, mainly due to its potential applications to the understanding of entanglement in four qubit systems [17, 11, 21, 9]. We shall give here the first complete solution to the problem of describing the polynomial covariants, an investigation started by Le Paige as early as 1881 [7]. The theory was further advanced by C Segre in 1922 [16], using only geometric methods which led him close to a complete classification of the orbits. Such a complete classification was obtained only recently by Verstratete et al. [21], by exploiting the local isomorphism between SO_4 and $SL_2 \times SL_2$, which permits a reduction of the problem to the classification of complex symmetric matrices up to orthogonal transformations.

A complete picture would require a description of the orbit closures as algebraic varieties and an understanding of their ordering with respect to inclusion. In classical invariant theory, such descriptions were usually given in terms of invariants and covariants. The main result of [9] was a complete description of the algebra of polynomial functions $f(a_{ijkl})$ in the components of a four qubit state

$$|\Psi\rangle = \sum_{i,j,k,l=0}^{1} a_{ijkl}|i\rangle \otimes |j\rangle \otimes |k\rangle \otimes |l\rangle \tag{1}$$

which are invariant under the natural action of the so-called SLOCC‡ group $G = SL(2,\mathbb{C})^{\times 4}$ on the local Hilbert space $\mathcal{H} = V^{\otimes 4}$, where $V = \mathbb{C}^2$.

One motivation for this investigation was to test on the four qubit case Klyachko's proposed definitions of entanglement and complete entanglement [5]. These consist in identifying entangled states as being precisely those for which at least one SLOCC polynomial invariant is not zero, and completely entangled states as the vectors of minimal norm in closed SLOCC orbits, which are unique up to local unitary transformations.

Klyachko's definition of complete entanglement seems to be supported by the recent numerical experiments of Verstraete et al. [22]. Indeed, these authors propose a numerical algorithm converging to a normal form, which, in the case of a stable state, is a state of minimal norm in its SLOCC orbit $\mathcal{O}$, and otherwise in the unique closed orbit contained in the closure $\bar{\mathcal{O}}$. Thus, in both cases, the normal form is a completely entangled state in the sense of [5].

We take the opportunity to point out that, as conjectured in [22], the normal form is indeed unique up to local unitary transformations §.

In [9], the polynomial invariants were constructed by means of the classical notion of a covariant. If we interpret our state $|\Psi\rangle$ as a quadrilinear form

$$f(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t}) = \sum_{i,j,k,l=0}^{1} a_{ijkl} x_i y_j z_k t_l \tag{2}$$

on $V \times V \times V \times V$, a covariant of A is a multi- homogeneous G-invariant polynomial in the form coefficients a_{ijkl} and in the original variables x_i, y_j, z_k, t_l.

Since the spaces $S^\mu(V)$ ($\mu \in \mathbb{N}^4$) of homogeneous polynomials of multidegree μ in $\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t}$ exhaust all finite dimensional representations of G, a covariant of degree d in the a_{ijkl} and μ in the variables can be regarded as a G-equivariant map $S^d(\mathcal{H}) \to S^\mu(V)$ from the space $S^d(\mathcal{H})$ of homogeneous polynomials of degree d in A to the irreducible representation $S^\mu(V)$. Such a map is determined by the image of a highest weight vector, so that covariants are in one to one correspondence with highest weight vectors in $S^d(\mathcal{H})$, these being known as semi-invariants in the classical language (cf. [12]).

The covariants form an algebra, which is naturally graded with respect to d and μ. We denote by $\mathcal{C}_{d;\mu}$ the corresponding graded pieces. The knowledge of their dimensions $c_{d;\mu}$ is equivalent to the decomposition of the character of $S^d(\mathcal{H})$ into irreducible characters of G, and the knowledge of a basis of $\mathcal{C}_{d;\mu}$ allows one to write down a Clebsch-Gordan series with respect to G for any polynomial in the a_{ijkl}. Also, it is known that the equations of any G-invariant closed subvariety of the projective space $\mathbb{P}(\mathcal{H})$ are given by the simultaneous vanishing of the coefficients of some covariants.

‡ for *Stochastic Local Operations assisted by Classical Communication*, see [1].

§ The results implying this conjecture, as well as the necessary background in Geometric Invariant Theory, are collected in [23]: the Kempf-Ness criterion (theorem 6.18) proves the result in the case of a closed orbit, and by the corollary of theorem 4.7, there is a unique closed orbit in the closure of an arbitrary orbit.

Finally, let us point out a connection with the approach of [3] and local unitary invariants. The spaces $S^\mu(V)$, and hence the $\mathcal{C}_{d;\mu}$ are also Hilbert spaces in a natural way. If $\Psi^\alpha_{d;\mu}$ is a linear basis of $\mathcal{C}_{d;\mu}$, the scalar products $\langle \Psi^\alpha_{d;\mu} | \Psi^\beta_{d;\mu} \rangle$ (taken with respect to the variables, the coefficients a_{ijkl} being treated as scalars) form a basis of the space of $U(2)^{\times 4}$ invariants of degree $2d$ (that is, d in A and d in $\bar{A}$), and the $\langle \Psi^\alpha_{d;\mu} | \Psi^\beta_{e;\mu} \rangle$ form a basis of the space of $SU(2)^{\times 4}$ invariants of bidegree (d, e) in $(A, \bar{A})$. Such expressions are used for example in [1], in the case of three qubits.

2. Summary of method and results

A minimal generating set consisting of 170 covariants is found by means a computer search through *iterated transvectants* (see section 3), guided by the knowledge of the Hilbert series (see section 4), and simplified by taking into account some special properties of multilinear forms. The following table gives the number of covariants of degree d in A and multidegree λ in the variables, where λ is in nondecreasing order. There are similar covariants for each of the n_λ permutations μ of the degrees. For example, in degree 5, we have 12 covariants which are cubic in one variable and linear in the other three, and one quadrilinear covariant.

$\lambda\backslash d$	n_λ	1	2	3	4	5	6	7	8	9	10	11	12
0000	1		1		2		1						
1111	1	1		2		1							
2200	6		1		1		1						
2220	4				2		2		2				
3111	4			1		3		3		1			
3311	6							1		2		1	
4000	4				1				1				
4200	12						1		1		1		
5111	4							1		2		1	
6000	4												1

(3)

3. Multiple transvectants

Transvectants, or Cayley's Omega-process, are the basic tools for constructing complete systems of covariants, and play a key rôle in Gordan's and Hilbert's proofs that the ring of covariants is finitely generated. The notion of a transvectant extends with little modifications to forms in several series of variables, and appears to have been first exploited by Le Paige [6] in the case of binary trilinear forms, and by Peano [13], who computed the complete systems for forms of bidegrees $(1,1)$, $(2,1)$ and $(2,2)$ in two independent binary variables. Complete systems for bidegrees $(3,1)$ and $(4,1)$ have been given by Todd [18, 19], and, to the best of our knowledge, the only forms in more that two binary variables for which the complete system is known are the $(1,1,1)$

[6, 14, 15] and the $(2,1,1)$, due to Gilham [2]. The geometry of the quadrilinear form is discussed by Segre [16] but no attempt is made to describe the covariants.

If f and g are forms in the binary variable $\mathbf{x} = (x_1, x_2)$, we identify their tensor product $f \otimes g$ with the polynomial $f(\mathbf{x}')g(\mathbf{x}'')$ in two independent binary variables $\mathbf{x}'$, $\mathbf{x}''$. Following [12], the multiplication map $f \otimes g \mapsto fg$ is denoted by tr. So, $\mathrm{tr}\,(f(\mathbf{x}')g(\mathbf{x}'')) = f(\mathbf{x})g(\mathbf{x})$.

The Cayley operator $\Omega_{\mathbf{x}}$ acts on such a tensor product by the differential operator

$$\Omega_{\mathbf{x}} = \begin{vmatrix} \frac{\partial}{\partial x_1'} & \frac{\partial}{\partial x_1''} \\ \frac{\partial}{\partial x_2'} & \frac{\partial}{\partial x_2''} \end{vmatrix} \tag{4}$$

If f and g are two p-tuple forms in p independent binary variables $\mathbf{x}_i$, one defines for any $(i_1, i_2, \ldots, i_p) \in \mathbb{N}^p$ a multiple transvectant of f and g by

$$(f,g)^{i_1 i_2 \ldots i_p} = \mathrm{tr}\, \Omega_1^{i_1} \Omega_2^{i_2} \cdots \Omega_p^{i_p} f(\mathbf{x}_1', \ldots, \mathbf{x}_p') g(\mathbf{x}_1'', \ldots, \mathbf{x}_p''), \tag{5}$$

where $\Omega_i = \Omega_{\mathbf{x}_i}$, and tr acts on all variables by $\mathbf{x}_i', \mathbf{x}_i'' \mapsto \mathbf{x}_i$.

It can be proved that the complete system of covariants of any number of forms can be reached in a finite number of steps by building iterated transvectants, starting with the ground forms.

4. The Hilbert series

The (multivariate) Hilbert series for the algebra of covariants is defined by

$$h(t, u_1, u_2, u_3, u_4) = \sum_{d,\mu} c_{d;\mu} t^d \mathbf{u}^\mu, \tag{6}$$

where $c_{d;\mu}$ is the dimnension of the space of homogeneous covariants which are of degree d in the a_{ijkl} and of multidegree μ in the variables. Let

$$S = \prod_{i=1}^{4} (1 - u_i^{-2}) \prod_{\alpha \in \{-1,1\}^4} (1 - \mathbf{u}^\alpha t)^{-1}. \tag{7}$$

Here S has to be considered as the formal power series obtained by expansion with respect to the variable t. Let $\mathcal{L}$ be the linear operator acting on a formal series in $t, \mathbf{u}$ by leaving unchanged every monomial $t^d \mathbf{u}^\mu$ with $\mu \in \mathbb{N}^4$, and annihilating those with u-exponent having some negative coordinate. It follows from standard considerations about the characters of G that $h = \mathcal{L}S$.

By successive decompositions into partial fractions (with respect to u_1, next u_2, u_3, u_4) we have computed this series, which guided us in the search for the covariants. The numerator is too large to be printed, but if one substitutes $u_1 = u_2 = u_3 = u_4 = u$, one finds after simplification $h = P/Q$, where the numerator P is

$$\begin{aligned} & 1 - u^2 t + \left(3\,u^4 - 2\,u^2\right) t^2 + \left(u^6 + 4\,u^4\right) t^3 \\ & + \left(10\,u^4 - u^2\right) t^4 + \left(-4\,u^8 - 2\,u^6 + 2\,u^4\right) t^5 \\ & + \left(2\,u^{10} + 6\,u^8 - 2\,u^6 + 8\,u^4\right) t^6 + \left(2\,u^{10} + 6\,u^8\right) t^7 \\ & + \left(-8\,u^{12} + u^{10} + 13\,u^8 - 2\,u^6 + 4\,u^4\right) t^8 \end{aligned}$$

$$+\left(-8\,u^{12}-u^{10}+12\,u^{8}-u^{6}\right)t^{9}$$
$$+\left(2\,u^{14}-13\,u^{12}+13\,u^{8}-2\,u^{6}\right)t^{10}$$
$$+\left(u^{14}-12\,u^{12}+u^{10}+8\,u^{8}\right)t^{11}$$
$$+\left(-4\,u^{16}+2\,u^{14}-13\,u^{12}-u^{10}+8\,u^{8}\right)t^{12}$$
$$+\left(-6\,u^{12}-2\,u^{10}\right)t^{13}+\left(-8\,u^{16}+2\,u^{14}-6\,u^{12}-2\,u^{10}\right)t^{14}$$
$$+\left(-2\,u^{16}+2\,u^{14}+4\,u^{12}\right)t^{15}+\left(u^{18}-10\,u^{16}\right)t^{16}$$
$$+\left(-4\,u^{16}-u^{14}\right)t^{17}+\left(2\,u^{18}-3\,u^{16}\right)t^{18}+u^{18}t^{19}-u^{20}t^{20}$$

and the denominator Q is

$$(1-tu^2)(1-tu^4)(1-t^2)(1-t^2u^2)^2(1-t^2u^4)^3$$
$$\times(1-t^4)(1-t^4u^2)(1-t^4u^4)(1-t^6)\,.$$

The algebra of covariants is Cohen-Macaulay (see, e.g., [23]). This means that it is a free module of finite rank over a subalgebra generated by a finite familly of homogeneous, algebraically independent elements $f_1,\ldots,f_k$. Then k is the Krull dimension of the algebra of covariants (the maximum number of algebraically independent elements) and $h(t,t,t,t,t)$ has a pole at $t=1$, of order k. We found in this way that $k=12$.

When substituting $u_i=0$, the Hilbert series of the invariants is recovered.

5. A fundamental set of covariants

The fundamental covariants are tabulated in Appendix A. They are denoted by symbols X^m_{pqrs}, in which the letter X indicates the degree in the form coefficients a_{ijkl} (1 for A, 2 for B, etc.), the subscripts $ijkl$ indicate the degrees in the variables $\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t}$, and the optional exponent m serves to distinguish between covariants having the same degrees.

The only covariant of degree 1 is the ground form $f=A_{1111}$. In degree two, we have one invariant, the basic hyperdeterminant $H=\frac{1}{2}B_{0000}$, which is defined in general for all multilinear form with an even number of variables (see [8] for some applications), and six biquadratic forms such as B_{2200}. In degree 3, we find two quadrilinear forms and four cubico-trilinear covariants. There are three other invariants, which occur in degree 4 (D^1_{0000}, D^2_{0000}) and 6 (F_{0000}). The maximal degree is 12, where we find four binary sextics.

We see that some of the covariants are of a type for which the invariant theory is well understood (binary forms of degree 4 and 6, biquadratic forms), or are again quadrilinear forms. Other are of essentially unexplored types (3111, 2220, 4200, 3311, 5111).

Naturally, the covariants of a covariant are again covariants (although not necessarily irreducible). Some of these are discussed in Appendix C.

Let us now sketch the application to the description of the orbit closures. The normal forms obtained by Verstraete et al. [21] are

$$G_{abcd} \quad = \frac{a+d}{2}(|0000\rangle+|1111\rangle)+\frac{a-d}{2}(|0011\rangle+|1100\rangle)$$

$$
\begin{aligned}
& + \frac{b+c}{2}(|0101\rangle + |1010\rangle) + \frac{b-c}{2}(|0110\rangle + |1001\rangle) \\
L_{abc_2} &= \frac{a+b}{2}(|0000\rangle + |1111\rangle) + \frac{a-b}{2}(|0011\rangle + |1100\rangle) \\
& \quad + c(|0101\rangle + |1010\rangle) + |0110\rangle \\
L_{a_2b_2} &= a(|0000\rangle + |1111\rangle) + b(|0101\rangle + |1010\rangle) \\
& \quad + |0110\rangle + |0011\rangle \\
L_{ab_3} &= a(|0000\rangle + |1111\rangle) + \frac{a+b}{2}(|0101\rangle + |1010\rangle) \\
& \quad + \frac{a-b}{2}(|0110\rangle + |1001\rangle) \\
& \quad + \frac{i}{\sqrt{2}}(|0001\rangle + |0010\rangle + |0111\rangle + |1011\rangle) \\
L_{a_4} &= a(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle) \\
& \quad + (i|0001\rangle + |0110\rangle - i|1011\rangle) \\
L_{a_2 0_{3\oplus\bar{1}}} &= a(|0000\rangle + |1111\rangle) + (|0011\rangle + |0101\rangle + |0110\rangle) \\
L_{0_{5\oplus\bar{3}}} &= |0000\rangle + |0101\rangle + |1000\rangle + |1110\rangle \\
L_{0_{7\oplus\bar{1}}} &= |0000\rangle + |1011\rangle + |1101\rangle + |1110\rangle \\
L_{0_{3\oplus\bar{1}} 0_{3\oplus\bar{1}}} &= |0000\rangle + |0111\rangle
\end{aligned}
$$

where in our notation a ket $|ijkl\rangle$ is to be identified with the monomial $x_i y_j z_k t_l$. The invariants B_{0000}, D^1_{0000}, D^2_{0000} and F_{0000} are shown in [9] to separate the normal forms G_{abcd}, L_{abc_2}, L_{ab_3}, $L_{a_2b_2}$, L_{a_4} and $L_{a_2 0_{3\oplus\bar{1}}}$. But they vanish for $L_{0_{5\oplus\bar{3}}}$, $L_{0_{7\oplus\bar{1}}}$ and $L_{0_{3\oplus\bar{1}} 0_{3\oplus\bar{1}}}$. The knowledge of the fundamental set of covariants is more than sufficient to separate the last three forms. Indeed,

$$C_{3111}(L_{0_{5\oplus\bar{3}}}) = 2(x_2y_2z_1t_1 - x_1y_2z_1t_1) \tag{8}$$

$$D_{2200}(L_{0_{5\oplus\bar{3}}}) = 0 \tag{9}$$

$$C_{3111}(L_{0_{7\oplus\bar{1}}}) = 2x_2(y_1z_1t_2 + y_1z_2t_1 - 2y_2z_1t_1) \tag{10}$$

$$D_{2200}(L_{0_{7\oplus\bar{1}}}) = -16x_2^2 z_1 z_2 \tag{11}$$

$$C_{3111}(L_{0_{3\oplus\bar{1}} 0_{3\oplus\bar{1}}}) = 0 \tag{12}$$

$$D_{2200}(L_{0_{3\oplus\bar{1}} 0_{3\oplus\bar{1}}}) = 0 \tag{13}$$

From this, it is not difficult (although somewhat tedious) to obtain the equations of the closures of the orbits, and to study the inclusions between them. This will be left for a future paper on the subjet, in which we expect to be able to produce first a better choice of the fundamental covariants (with greater geometrical or physical significance).

6. Rational covariants

The algebra of rational covariants is simpler than the algebra of polynomial covariants. It is a field of rational functions over 12 homogeneous independent generators. A way to compute a fundamental set of rational semi-invariants, consists in using the so-called associated forms [10]. Let F be the polynomial obtained from the ground form by applying the following series of substitutions

$$x_1 \to a_{0000}x_1 - a_{1000}x_2,\; x_2 \to a_{0000}x_2,$$
$$y_1 \to a_{0000}y_1 - a_{0100}y_2,\; y_2 \to a_{0000}y_2,$$
$$z_1 \to a_{0000}z_1 - a_{0010}z_2,\; z_2 \to a_{0000}z_2,$$
$$t_1 \to a_{0000}t_1 - a_{0001}t_2,\; t_2 \to a_{0000}t_2.$$

The semi-invariants which are the sources of the associated forms are the coefficients of the monomials $x_iy_jz_kt_l$ in F, divided by $a_{0000}^{9-i-j-k-l}$ We obtain in this way a list of semi-invariants which are the sources of some polynomial covariants given below. Here, $H = \frac{1}{2}(f,f)^{1111}$, $b_{xy} = \frac{1}{2}(f,f)^{0011}$, etc. are as in [9].

Source c_α	Covariant $\mathcal{C}_\alpha$
c_{0000}	1
c_{1000}	0
c_{0100}	0
c_{0010}	0
c_{0001}	0
c_{0011}	b_{xy}
c_{0101}	b_{xz}
c_{0110}	b_{xt}
c_{1001}	b_{yz}
c_{1010}	b_{yt}
c_{1100}	b_{zt}
c_{0111}	$-C_{3111}$
c_{1011}	$-C_{1311}$
c_{1101}	$-C_{1131}$
c_{1110}	$-C_{1113}$
c_{1111}	$Hf^2 - b_{xy}b_{zt} - b_{xz}b_{yt} - b_{xt}b_{yz}$

(14)

The 12 homogeneous independent generators of the field of rational covariants are the 11 non-trivial associated forms above, together with the ground form f. Actually, the last one ($\mathcal{C}_{1111}$) can be advantageously replaced by H.

Now, each covariant can be written as a rational function in these 12 generators. It suffices to make the substitutions

$$a_{ijkl} \to \mathcal{C}_{ijkl}f^{1-(i+j+k+l)} \tag{15}$$

in the source of the covariant, where $\mathcal{C}_\alpha$ is the covariant with source c_α.

For example, the source of D_{4000} is

$$2a_{0111}a_{0100}a_{0000}a_{0011} - 4a_{0111}a_{0010}a_{0001}a_{0010}$$
$$-a_{0000}^2a_{0111}^2 + 2a_{0111}a_{0000}a_{0001}a_{0110}$$
$$+2a_{0111}a_{0000}a_{0010}a_{0101} + 2a_{0110}a_{0001}a_{0011}a_{0100}$$
$$-a_{0001}^2a_{0110}^2 + 2a_{0110}a_{0001}a_{0010}a_{0101}$$

$$-4a_{0110}a_{0011}a_{0000}a_{0101}+2a_{0101}a_{0010}a_{0011}a_{0100}$$
$$-a_{0010}^2a_{0101}^2-a_{0011}^2a_{0100}^2$$

and the above substitutions give

$$D_{4000}=-\frac{1}{\mathcal{C}_{0000}^2}(\mathcal{C}_{0111}^2+4\mathcal{C}_{0110}\mathcal{C}_{0011}\mathcal{C}_{0101})$$
$$=-\frac{1}{f^2}(C_{3111}^2+4b_{xt}b_{xy}b_{xz}). \qquad (16)$$

This yields a syzygy

$$f^2D_{4000}+C_{3111}^2+4b_{xy}b_{xz}b_{xt}=0\,. \qquad (17)$$

7. Conclusion

It is remarkable that the investigation of the fine structure of the four qubit system has led to the first complete solution of a mathematical problem which had already been considered as early as 1881 [7]. This problem was among the very few ones which were out of reach of the computational skills of the classical invariant theorists, though accessible to a computer treatment. The number of fundamental covariants, here 170, is not, however, the highest ever found ‖, and we expect to be able to produce in the near future a human readable proof, together with a better choice of the generators, i.e., to find, at least for the lowest degrees, generators with a transparent geometrical interpretation.

A complete description of the ring of covariants should in principle include a generating set of the syzygies. However, we can see from the Hilbert series that this is a hopeless task, as it is already for the previously known specializations. We have computed all the syzygies up to degree 7, and formula (15) allows one to find at least one syzygy for each covariant which is not one of the $\mathcal{C}_\alpha$.

Turning back to the issue of entanglement, we see that we have now at our disposal all the possible building blocks for the construction of entanglement measures for systems with no more than four qubits. It is to be expected that further investigations will allow one to select among them the most relevant ones, and that the analysis of their geometric significance will give a clue for the general case.

‖ For example, Turnbull obtained in 1910 a system of 784 forms for the case of three ternary quadratics, and in 1947, Todd proved that 603 of them formed a complete minimal system.

Appendix A. Fundamental covariants

Degree 2

Symbol	Transvectant
B_{0000}	$(f,f)^{1111}$
B_{2200}	$(f,f)^{0011}$
B_{2020}	$(f,f)^{0101}$
B_{2002}	$(f,f)^{0110}$
B_{0220}	$(f,f)^{1001}$
B_{0202}	$(f,f)^{1010}$
B_{0022}	$(f,f)^{1100}$

Degree 3

Symbol	Transvectant
C^1_{1111}	$(f,B_{2200})^{1100}$
C^2_{1111}	$(f,B_{2020})^{1010}$
C_{3111}	$(f,B_{2200})^{0100}$
C_{1311}	$(f,B_{2200})^{1000}$
C_{1131}	$(f,B_{2020})^{1000}$
C_{1113}	$(f,B_{2002})^{1000}$

Degree 4

Symbol	Transvectant
D^1_{0000}	$(f,C^1_{1111})^{1111}$
D^2_{0000}	$(f,C^2_{1111})^{1111}$
D_{2200}	$(f,C_{3111})^{1011}$
D_{2020}	$(f,C^1_{1111})^{0101}$
D_{2002}	$(f,C_{3111})^{1110}$
D_{0220}	$(f,C_{1311})^{1101}$
D_{0202}	$(f,C_{1311})^{1110}$
D_{0022}	$(f,C_{1131})^{1110}$

Symbol	Transvectant
D_{4000}	$(f,C_{3111})^{0111}$
D_{0400}	$(f,C_{1311})^{1011}$
D_{0040}	$(f,C_{1131})^{1101}$
D_{0004}	$(f,C_{1113})^{1110}$
D^1_{2220}	$(f,C_{1311})^{0101}$
D^2_{2220}	$(f,C^1_{1111})^{0001}$
D^1_{2202}	$(f,C_{1113})^{0011}$
D^2_{2202}	$(f,C_{1311})^{0110}$
D^1_{2022}	$(f,C_{1113})^{0101}$
D^2_{2022}	$(f,C^1_{1111})^{0100}$
D^1_{0222}	$(f,C_{1113})^{1001}$
D^2_{0222}	$(f,C_{1311})^{1100}$

Degree 5

Symbol	Transvectant
E_{1111}	$(f,D_{2200})^{1100}$
E^1_{3111}	$(f,D_{2200})^{0100}$
E^2_{3111}	$(f,D^1_{2202})^{0101}$
E^3_{3111}	$(f,D^2_{2022})^{0011}$
E^1_{1311}	$(f,D_{2200})^{1000}$
E^2_{1311}	$(f,D_{0202})^{0001}$
E^3_{1311}	$(f,D_{0220})^{0010}$
E^1_{1131}	$(f,D^1_{0222})^{0101}$
E^2_{1131}	$(f,D^2_{2022})^{1001}$
E^3_{1131}	$(f,D_{2020})^{1000}$
E^1_{1113}	$(f,D^1_{2022})^{1010}$
E^2_{1113}	$(f,D^2_{2022})^{1010}$
E^3_{1113}	$(f,D_{0004})^{0001}$

Degree 6

Symbol	Transvectant
F_{0000}	$(f, E_{1111})^{1111}$
F_{2200}	$(f, E^1_{3111})^{1011}$
F_{2020}	$(f, E_{1111})^{0101}$
F_{2002}	$(f, E^1_{1113})^{0111}$
F_{0220}	$(f, E^1_{1311})^{1101}$
F_{0202}	$(f, E^3_{1113})^{1011}$
F_{0022}	$(f, E^1_{1113})^{1101}$
F^1_{2220}	$(f, E^1_{1311})^{0101}$
F^2_{2220}	$(f, E^2_{1311})^{0101}$
F^1_{2202}	$(f, E^2_{3111})^{1010}$
F^2_{2202}	$(f, E^3_{3111})^{1010}$
F^1_{2022}	$(f, E^1_{1113})^{0101}$
F^2_{2022}	$(f, E^2_{1113})^{0101}$
F^1_{0222}	$(f, E^1_{1131})^{1010}$
F^2_{0222}	$(f, E^2_{1131})^{1010}$

Symbol	Transvectant
F_{4200}	$(f, E^1_{3111})^{0011}$
F_{4020}	$(f, E^2_{3111})^{0101}$
F_{4002}	$(f, E^2_{3111})^{0110}$
F_{0420}	$(f, E^3_{1311})^{1001}$
F_{0402}	$(f, E^2_{1311})^{1010}$
F_{0042}	$(f, E^1_{1131})^{1100}$
F_{2400}	$(f, E^1_{1311})^{0011}$
F_{2040}	$(f, E^1_{1131})^{0101}$
F_{2004}	$(f, E^1_{1113})^{0110}$
F_{0240}	$(f, E^1_{1131})^{1001}$
F_{0204}	$(f, E^1_{1113})^{1010}$
F_{0024}	$(f, E^1_{1113})^{1100}$

Degree 7

Symbol	Transvectant
G^1_{3111}	$(f, F_{2200})^{0100}$
G^2_{3111}	$(f, F_{4002})^{1001}$
G^3_{3111}	$(f, F^1_{2202})^{0101}$
G^1_{1311}	$(f, F_{0402})^{0101}$
G^2_{1311}	$(f, F_{2200})^{1000}$
G^3_{1311}	$(f, F_{0202})^{0001}$
G^1_{1131}	$(f, F^1_{0222})^{0101}$
G^2_{1131}	$(f, F^2_{0222})^{0101}$
G^3_{1131}	$(f, F_{2040})^{1010}$
G^1_{1113}	$(f, F^1_{2022})^{1010}$
G^2_{1113}	$(f, F^2_{2022})^{1010}$
G^3_{1113}	$(f, F_{0202})^{0100}$

Symbol	Transvectant
G_{5111}	$(f, F_{4002})^{0001}$
G_{1511}	$(f, F_{0402})^{0001}$
G_{1151}	$(f, F_{2040})^{1000}$
G_{1115}	$(f, F_{0024})^{0010}$
G_{3311}	$(f, F_{2400})^{0100}$
G_{3131}	$(f, F^2_{2022})^{0001}$
G_{3113}	$(f, F_{4002})^{1000}$
G_{1331}	$(f, F_{0240})^{0010}$
G_{1313}	$(f, F_{0402})^{0100}$
G_{1133}	$(f, F^2_{2022})^{1000}$

Degree 8

Symbol	Transvectant
H_{4000}	$(f, G_{5111})^{1111}$
H_{0400}	$(f, G^1_{1311})^{1011}$
H_{0040}	$(f, G_{1151})^{1111}$
H_{0004}	$(f, G^3_{1113})^{1110}$
H^1_{2220}	$(f, G^1_{1311})^{0101}$
H^2_{2220}	$(f, G^2_{1311})^{0101}$
H^1_{2202}	$(f, G^3_{3111})^{1010}$
H^2_{2202}	$(f, G^2_{1113})^{0011}$
H^1_{2022}	$(f, G^1_{1113})^{0101}$
H^2_{2022}	$(f, G^2_{1113})^{0101}$
H^1_{0222}	$(f, G^1_{1131})^{1010}$
H^2_{0222}	$(f, G^2_{1131})^{1010}$

Symbol	Transvectant
H_{4200}	$(f, G_{5111})^{1011}$
H_{4020}	$(f, G_{5111})^{1101}$
H_{4002}	$(f, G_{5111})^{1110}$
H_{0420}	$(f, G^1_{1311})^{1001}$
H_{0402}	$(f, G_{1313})^{1011}$
H_{0042}	$(f, G_{1151})^{1110}$
H_{2400}	$(f, G^1_{1311})^{0011}$
H_{2040}	$(f, G_{1151})^{0111}$
H_{2004}	$(f, G^1_{1113})^{0110}$
H_{0240}	$(f, G_{1151})^{1011}$
H_{0204}	$(f, G^1_{1113})^{1010}$
H_{0024}	$(f, G^1_{1113})^{1100}$

Degree 9

Symbol	Transvectant
I_{3111}	$(f, H_{4020})^{1010}$
I_{1311}	$(f, H^1_{2220})^{1010}$
I_{1131}	$(f, H_{0240})^{0110}$
I_{1113}	$(f, H_{2004})^{1001}$
I^1_{5111}	$(f, H_{4020})^{0010}$
I^2_{5111}	$(f, H_{4002})^{0001}$
I^1_{1511}	$(f, H_{0402})^{0001}$
I^2_{1511}	$(f, H_{2400})^{1000}$
I^1_{1151}	$(f, H_{0240})^{0100}$
I^2_{1151}	$(f, H_{0042})^{0001}$
I^1_{1115}	$(f, H_{2004})^{1000}$
I^2_{1115}	$(f, H_{0024})^{0010}$

Symbol	Transvectant
I^1_{3311}	$(f, H^1_{2220})^{0010}$
I^2_{3311}	$(f, H^2_{2220})^{0010}$
I^1_{3131}	$(f, H_{4020})^{1000}$
I^2_{3131}	$(f, H^1_{2220})^{0100}$
I^1_{3113}	$(f, H_{2004})^{0001}$
I^2_{3113}	$(f, H^1_{2022})^{0010}$
I^1_{1331}	$(f, H_{0240})^{0010}$
I^2_{1331}	$(f, H^1_{2220})^{1000}$
I^1_{1313}	$(f, H_{0204})^{0001}$
I^2_{1313}	$(f, H^1_{0222})^{0010}$
I^1_{1133}	$(f, H_{0024})^{0001}$
I^2_{1133}	$(f, H^1_{0222})^{0100}$

Degree 10

Symbol	Transvectant
J_{4200}	$(f, I^1_{5111})^{1011}$
J_{4020}	$(f, I^1_{5111})^{1101}$
J_{4002}	$(f, I^1_{3113})^{0111}$
J_{0420}	$(f, I^1_{1331})^{1011}$
J_{0402}	$(f, I^1_{1511})^{1110}$
J_{0042}	$(f, I^1_{1133})^{1101}$
J_{2400}	$(f, I^1_{1511})^{0111}$
J_{2040}	$(f, I^1_{3131})^{1101}$
J_{2004}	$(f, I^1_{3113})^{1110}$
J_{0240}	$(f, I^1_{1331})^{1101}$
J_{0204}	$(f, I^1_{1115})^{1011}$
J_{0024}	$(f, I^1_{1115})^{1101}$

Degree 11

Symbol	Transvectant
K_{3311}	$(f, J_{4200})^{1000}$
K_{3131}	$(f, J_{4020})^{1000}$
K_{3113}	$(f, J_{4002})^{1000}$
K_{1331}	$(f, J_{0420})^{0100}$
K_{1313}	$(f, J_{0402})^{0100}$
K_{1133}	$(f, J_{0042})^{0010}$
K_{5111}	$(f, J_{4200})^{0100}$
K_{1511}	$(f, J_{2400})^{1000}$
K_{1151}	$(f, J_{2040})^{1000}$
K_{1115}	$(f, J_{2004})^{1000}$

Degree 12

Symbol	Transvectant
L_{6000}	$(f, K_{5111})^{0111}$
L_{0600}	$(f, K_{1511})^{1011}$
L_{0060}	$(f, K_{1151})^{1101}$
L_{0006}	$(f, K_{1115})^{1110}$

Appendix B. Syzygies

The method of associated forms presented in Section 6 gives all the syzygies up to degree 5. In degree 6, one can check that, for example, the following two syzygies

$$\begin{aligned}
&D^2_{0000}f^2 + 2D^1_{0000}f^2 - \tfrac{3}{2}B_{2200}B_{0022}B_{0000} \\
&+\tfrac{3}{2}B_{2020}B_{0202}B_{0000} - \tfrac{9}{2}D_{2200}B_{0022} \\
&-4(C^2_{1111})^2 - \tfrac{9}{2}D_{0022}B_{2200} - 8C^1_{1111}C^2_{1111} \\
&+\tfrac{9}{2}D_{0220}B_{2002} + \tfrac{9}{2}D_{2002}B_{0220} \qquad = 0
\end{aligned}$$

$$\begin{aligned}
&(C^1_{1111})^2 + \tfrac{3}{4}D^1_{0000}f^2 - \tfrac{9}{8}B_{2200}B_{0022}B_{0000} \\
&+\tfrac{9}{8}B_{2020}B_{0202}B_{0000} - \tfrac{9}{4}D_{2200}B_{0022} \\
&+\tfrac{9}{8}D_{0202}B_{2020} - 2(C^2_{1111})^2 - \tfrac{9}{4}D_{0022}B_{2200} \\
&-2C_{1111}C_{1111} - \tfrac{3}{2}D_{2020}B_{0202} \\
&+\tfrac{9}{8}D_{0220}B_{2002} + \tfrac{9}{8}D_{2002}B_{0220} \qquad = 0
\end{aligned}$$

cannot be obtained by this method.

The second order syzygies arise in degree 7 and type (5333).

Appendix C. Invariants of the covariants

As already mentioned, the next step in the study of the four qubit system should be to find geometrical interpretations of the simplest covariants. Partial information is already available, as some of the fundamental covariants are of known types, for which the invariant theory is reasonably well understood. Among these, the most important ones are certainly the six biquadratic forms b_{uv}. A double binary form $g(\mathbf{u}, \mathbf{v})$ of bidegree (m, n) is usually interpreted as defining a space curve, lying of the projective quadric $XT - YZ = 0$, which may be parametrized by

$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix} = \begin{pmatrix} u_2v_2 & -u_2v_1 \\ -u_1v_2 & u_1v_1 \end{pmatrix}, \qquad \text{(C.1)}$$

so that $\mathbf{u}$ and $\mathbf{v}$ parametrize the rectilinear generatrices. Such a curve has generically m intersections with the generatrices of one system, and n with generatrices of the other one. The covariants b_{uv} can therefore be interpreted as six space quartics. It turns out that these have the same discriminant, which is proportional to the hyperdeterminant used in [11]. The nonvanishing of the hyperdeterminant is therefore the condition for these curves to be elliptic. On the normal forms of [21], it is easy to check that one has three isomorphisms $(b_{xy}) \simeq (b_{zt})$, $(b_{xz}) \simeq (b_{yt})$ and $(b_{xt}) \simeq (b_{yz})$. Geometric interpretations of the complete system of covariants of such forms of bidegree $(2, 2)$ (which is due to Peano [13]) are available in the works of Kasner [4] and Turnbull [20].

Other double binary forms of the same type occur as covariants in degrees 4 and 6.

There are also some simple binary forms among the covariants. Quartics, whose invariant theory is completely understood, occur in degree 4 and 8. We have already used the fact that the common discriminant of the quartics in degree 4 was the hyperdeterminant [9]. The fundamental covariants in degree 12, the highest degree, are four binary sextics, whose invariant of degree 2 is again the hyperdeterminant, as can be seen from the normal forms

$$L_{6000} = 144\,V(a^2,b^2,c^2,d^2)(x_1^4-x_2^4)x_1x_2 \tag{C.2}$$
$$L_{0600} = 96\,V(a^2,b^2,c^2,d^2)(y_1^4-y_2^4)y_1y_2 \tag{C.3}$$
$$L_{0060} = 288\,V(a^2,b^2,c^2,d^2)(z_1^4-z_2^4)z_1z_2 \tag{C.4}$$
$$L_{0006} = -\,96\,V(a^2,b^2,c^2,d^2)(t_1^4-t_2^4)t_1t_2 \tag{C.5}$$

where V denotes the Vandermonde determinant. One can see from this example that the normal forms of [21] are simple enough to allow the complete calculation of all the fundamental covariants. It is also interesting to observe that under the specialization G_{abcd} of [21], the covariants b_{uv} come directly in Kasner's normal form

$$k(u_1^2v_1^2+u_2^2v_2^2)+l(u_1^2v_2^2+u_2^2v_1^2)+4m\,u_1u_2v_1v_2 \tag{C.6}$$

and that the three quadrilinear covariants occuring in degrees 3 and 5 are also in normal form. Indeed,

$$\begin{aligned} C^1_{1111} = {} & \tfrac{a_1+d_1}{2}x_1y_1z_1t_1+\tfrac{a_1-d_1}{2}x_1y_1z_2t_2 \\ & +\tfrac{b_1+c_1}{2}x_1y_2z_1t_2+\tfrac{b_1-c_1}{2}x_1y_2z_2t_1 \\ & +\tfrac{b_1-c_1}{2}x_2y_1z_1t_2+\tfrac{b_1+c_1}{2}x_2y_1z_2t_1 \\ & +\tfrac{a_1-d_1}{2}x_2y_2z_1t_1+\tfrac{a_1+d_1}{2}x_2y_2z_2t_2\,, \end{aligned}$$

where

$$\begin{aligned} a_1 &= 3a^3-ad^2-ab^2-ac^2 \\ b_1 &= -\,bc^2-ba^2-bd^2+3b^3 \\ c_1 &= -\,b^2c+3c^3-ca^2-cd^2 \\ d_1 &= -\,a^2d+3d^3-db^2-dc^2\,, \end{aligned}$$

and

$$\begin{aligned} C^2_{1111} = {} & \tfrac{a_2+d_2}{2}x_1y_1z_1t_1+\tfrac{a_2-d_2}{2}x_1y_1z_2t_2 \\ & +\tfrac{b_2+c_2}{2}x_1y_2z_1t_2+\tfrac{b_2-c_2}{2}x_1y_2z_2t_1 \\ & +\tfrac{b_2-c_2}{2}x_2y_1z_1t_2+\tfrac{b_2+c_2}{2}x_2y_1z_2t_1 \\ & +\tfrac{a_2-d_2}{2}x_2y_2z_1t_1+\tfrac{a_2+d_2}{2}x_2y_2z_2t_2\,, \end{aligned}$$

where

$$\begin{aligned} a_2 &= 2ab^2+2ac^2+2ad^2+6dbc \\ b_2 &= 2ba^2+2bd^2+2bc^2+6cad \\ c_2 &= 2b^2c+6bad+2ca^2+2cd^2 \\ d_2 &= 2a^2d+6abc+2db^2+2dc^2 \end{aligned}$$

and finally

$$
\begin{aligned}
E_{1111} = & \tfrac{a_3+d_3}{2} x_1 y_1 z_1 t_1 + \tfrac{a_3-d_3}{2} x_1 y_1 z_2 t_2 \\
& + \tfrac{b_3+c_3}{2} x_1 y_2 z_1 t_2 + \tfrac{b_3-c_3}{2} x_1 y_2 z_2 t_1 \\
& + \tfrac{b_3-c_3}{2} x_2 y_1 z_1 t_2 + \tfrac{b_3+c_3}{2} x_2 y_1 z_2 t_1 \\
& + \tfrac{a_3-d_3}{2} x_2 y_2 z_1 t_1 + \tfrac{a_3+d_3}{2} x_2 y_2 z_2 t_2 \,,
\end{aligned}
$$

where

$$
\begin{aligned}
a_3 &= 8(-a^3d^2 - c^2a^3 - b^2a^3 + ac^2d^2 + ab^2c^2 + ab^2d^2) \\
b_3 &= 8(-b^3c^2 + bc^2d^2 + bc^2a^2 - b^3d^2 + ba^2d^2 - b^3a^2) \\
c_3 &= 8(ca^2d^2 + a^2b^2c + cb^2d^2 - b^2c^3 - c^3d^2 - c^3a^2) \\
d_3 &= 8(db^2c^2 + a^2b^2d + a^2c^2d - a^2d^3 - c^2d^3 - b^2d^3)
\end{aligned}
$$

On another hand, it seems that nothing is known about quadruple binary forms of multidegree $(3, 1, 1, 1)$, and the first thing to be done now is probably to set up a convenient geometric representation of such forms.

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