Diameter of a set on the cylinder

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Abstract

We present an algorithm that computes the diameter of a set of n points in the cylinder in optimal time $O(n \log n)$; this algorithm uses as a fundamental tool the farthest point Voronoi diagram.

1 Introduction

A well-known measure of the spread of a set is its *diameter* (i.e., the maximum distance between two points of the set). Intuitively, a cluster with small diameter has elements that are closely related, while the opposite is true when the diameter is large. This concept has led to several related problems producing a remarkable amount of literature (see, for instance, [1, 6, 7, 17, 18]). But most of the efforts have been concentrated in the plane or euclidean spaces, and, in many cases the set of points in which we are interested are not in an euclidean space but confined to some surface (or a more general space) and the usual techniques are not valid anymore.

It is known that the computation of the diameter of a set of n points in every Euclidean space requires $\Omega(n \log n)$ operations. The usual procedure to compute in optimal time the diameter in the plane uses the fact that the diameter of a set of points is equal to the diameter of its convex hull [9], then it is enough to compute all antipodal pairs and, in a convex polygon, this task can be completed in linear time, thus the total running time of the algorithm is $O(n \log n)$. Unfortunately, this method cannot be used in the space since the number of antipodal pairs in the space is $O(n^2)$. And, in fact it is not known a $O(n \log n)$ algorithm in dimension 3 (as far as we know, the best result for the running time of a deterministic algorithm for the three-dimensional diameter problem is an ($O(n \log^3 n)$ algorithm due to Amato, Goodrich and Ramos [2]). In this paper, we will show that although the procedure followed in the plane cannot be applied in the cylinder, it is possible to get an optimal algorithm in that surface by using the farthest point Voronoi diagram.

The main obstacle in the cylinder is that the convex hull of a set of points is, in general, too big [4], and therefore, it is not useful as a tool for other problems. In fact, it is not difficult to find examples of sets of points in the cylinder such that their diameters are not equal to the diameters of their convex hulls. Therefore, it is needed another technique to get an optimal algorithm, in this paper our goal is achieved by using the farthest point Voronoi diagram in the cylinder. The structure of this work is as follows, next section will be devoted to summarize some results of [4] about the convex hull of a set of point in the cylinder. In Section 3 we will develop the farthest-point Voronoi diagram in the cylinder, and in Section 4 we will present our algorithm. We will finish with some conclusions, related problems and open questions.

2 Convex hull in the cylinder

Several extensions of convexity to non-planar surfaces (or to non-euclidean spaces) have been considered in the literature. Most of them are based on metrical concepts, and more concretely, in the family of geodesics of the surface. In order to describe the family of geodesics, as usual, we identify the cylinder with the quotient space obtained from the plane by identifying those points with the same ordinate such that their abscissae differ in an integer number. With the metric obtained from this definition, the geodesics joining two points in the cylinder can be identified with the segments in the plane joining a fixed representative of one of the points and all of the representatives of the other point. We say that generatrices $\{(x,y) : x = x_0\}$ and $\{(x,y) : x = x_0 + 1/2\}$ are *opposite generatrices*.

Using this representation, we can define the *strip* of a set of points $P = \{(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)\}$ in the cylinder with $y_1 \le y_2 \le \ldots \le y_n$ as the open strip O delimited by the maximal circles in the extreme points of P with respect to the ordinates ($O = \{(x,y) : y_1 < y < y_n\}$). Equally, we define the *m-top* of P as the minimal arc containing all points of P with the ordinate y_n if that arc is shorter than a half of the circle or the whole maximal circle if that arc is greater than a half of the circle or the single point (x_n,y_n) otherwise (equivalently the *m-bottom*).

Then, as Hopf-Rinow's Theorem [10] proves that there exists always the shortest geodesic joining two points, we can define as in [14] that $C \subseteq S$ is *metrically convex* if given two points of C the minimum geodesic in S joining those points is contained in C. And, as usual, given a set P of points in S, the *metrically convex hull* of P is the smallest metrically convex set containing P.

It is possible to give the following characterization of the metrically convex hull

Theorem 1 [4]. *The metrically convex hull of a set of N points P in the cylinder is*

1. *The convex hull of P in the plane if P is contained between two opposite generatrices.*

2. *The open strip delimited by the points P union the m-top and the m-bottom of P otherwise.*

Moreover, this metrically convex hull can be computed in $O(N \log N)$ time in the first case and in linear time in the second case, and it can be decided in which one of the cases we are in linear time.

Thus, Theorem 1 says that in many cases convex hull is too big for many purposes. In fact Figure 1 shows a set of points in the cylinder such that the diameter of the set is not equal to the diameter of its convex hull.

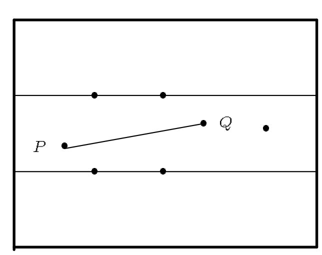

Figure 1

3 Voronoi diagrams on the cylinder

As it has been said in the introduction, the main tool in our algorithm to compute the diameter will be the farthest point Voronoi diagram. As in the euclidean spaces, given a set of points S in the cylinder, we denote by $V_f(i)$ the locus of points farther to $x_i \in S$ than to any other point of S. The set of all those loci is called the *farthest point Voronoi diagram* of S, $vor_f(S)$. Several methods to compute that structure are known in the plane and there exists a direct method, based on the divide and conquer scheme analogous to the algorithm for the closest-point diagram, which achieve the result in optimal $O(n \log n)$ time. On the other hand, Mazón presented in [13] an optimal algorithm to compute the closest-point Voronoi diagram of a set of points in the cylinder. Her method to considers three copies of the cylinder,and it constructs the diagram of the sets of $3n$ points, the diagram in the cylinder is the resulting diagram in the central copy. Therefore, it is not difficult to see that this method cannot be used to generate the farthest-point Voronoi diagram in the cylinder. Then we will try the divide and conquer approach.

Obviously, the first step will be to construct the bisector between two points. This can be done using Mazón's methods

Lemma 2 *The bisector of the points $P = (x_1, y_1)$, $Q = (x_2, y_2)$ in the cylinder with $x_1 < x_2$ with $y_1 < y_2$, is given by the bisectors in the plane of the points P and Q, Q and $P' = (x_1 + 1, y_1)$ and P and $Q' = (x_2 - 1, y_2)$ (see Figure 2).*

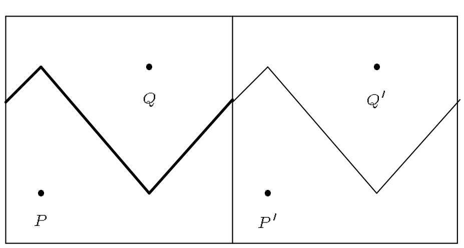

Figure 2

As far as the key step in the divide and conquer algorithm is to construct the dividing chain, we give some properties of that chain. In this order, we suppose that the original set S has been split in two parts S_1 and S_2 by a parallel c and if $x_i \in S_j$ $j = 1, 2$, we denote by $V_f^j(x_i)$ to its region in $vor_f(S_j)$. Then,

Lemma 3 *if $V_f^1(x_i) \cap V_f^2(x_j) \cap c \neq \emptyset$, then the section of the bisector between x_i and x_j contained in $V_f^1(x_i) \cap V_f^2(x_j)$ appears in the dividing chain of S_1 and S_2.*

Lemma 4 *The orthogonal projection of the dividing chain of S_1 y S_2 on c is a homeomorphism*

Lemma 3 is the key to construct an algorithm in the cylinder similar to the algorithm in the plane.

Algorithm DIVID-CHAIN(S_1, S_2, c):

(1) Find an initial point in the dividing chain by using Lemma 3.

(2) Construct the bisector between x and y.

(3) Determine the portion of bisector computed in (2) that is in $V_f^1(x) \cap V_f^2(y)$.

(4) Compute the extremes of the portion already computed of the dividing chain and update the points x and y.

Lemma 5
Algorithm DIVID-CHAIN(S_1, S_2, c) *computes the dividing chain between S_1 and S_2, subsets of S linearly separated by parallel c in linear time.*

Then we conclude

Theorem 6 *The farthest-point Voronoi diagram of n points in the cylinder can be constructed in optimal $O(n \log n)$ time.*

4 Diameter of a sets of points in the cylinder

Obviously, if the diameter of S is $d(u, v)$ for certain $u, v \in S$ then $u \in V_f(v)$. Therefore, the algorithm to compute the diameter will be

Algorithm DIAMETER(S)
(1) Construct the farthest-point Voronoi diagram of S.
(2) Localize in which region of the diagram is each point of S.
(3) Compute the distance between each point of S and the point defining the region obtained in (2).
(4) Report the maximum obtained in (3) as the diameter.

It is straightforward to check the validity of the algorithm DIAMETER(S) and then we have

Theorem 7 *It is possible to compute the diameter of a set of n points in the cylinder in optimal $O(n \log n)$.*

5 Open questions

Although an optimal algorithm to compute the diameter is presented, some open question arise related to the problem considered in this work.

First of all, it would be interested to find a structure that, as the convex hull in the plane, allows to find from it the diameter in the cylinder in linear time (observe that from the farthest-point Voronoi diagram we find the diameter in $O(n \log n)$ time). Moreover, it seems to be that our technique can be applied to other surfaces but building the farthest-point Voronoi diagram could be a difficult task.

Another interesting question that has been studied extensively in Euclidean spaces is, how many times can the maximum distance between n points occur? It is known that in the plane it can occur at most n times [5], and in the space $2n-2$ times [8]. The case of the cylinder is not the same as in the plane since it is possible to give a structure where the maximum can occur $4/3n$. This structure split the n points in three subsets of $n/3$ points each and each of those subsets are a regular polygon in a parallel in such a way that those polygons in the top and in the bottom parallels have their vertices on the same meridians and the other polygon has its vertices on the equidistant meridians to those considered before, see Figure 3.

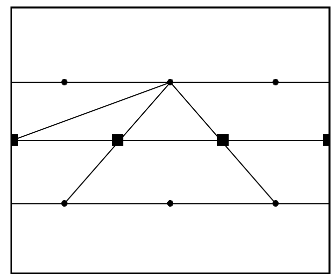

Figure 3

It remains to solve if this example is optimal or to find better bounds.

On the other hand, it is possible to answer completely a related question how many times can the maximum distance between n points occur?

Theorem 8 *The minimum distance between n points in the cylinder can occur at most $3n-6$ times.*

Proof: It is easy to see that the graph of the minimum distance between points in the cylinder is planar. Thus, by Euler's formula $3n-6$ is an upperbound and Figure 4 shows that this upperbound can be achieved. $\square$

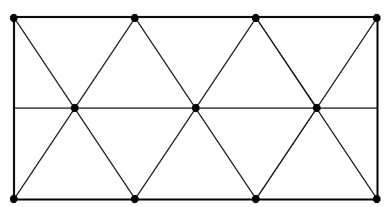

Figure 4

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