The width of a convex set on the sphere

F.J. Cobos J.C. Dana C.I. Grima A. Márquez
Departamento de Matemática Aplicada I
Universidad de Sevilla (Spain)
e-mail:{cobos|dana|grima|almar}@cica.es
Fax: +34-5-4557878

Abstract

We study the relationship between some alternative definitions of the concept of the width of a convex set on the sphere. Those relations allow to characterize whether a convex set on the sphere can pass through a spherical interval by rigid motions. Finally, we give an optimal algorithm to compute the width on the sphere.
Key words: Width, sphere, rigid motions, convex sets.

1 Introduction

In the plane, the width of a finite set of points is the minimum distance between parallel lines of support of the set [8]. This concept and the computation of the width of a finite set of points have applications in several fields such as in robotic (more specifically in collision–avoidance problems [18]), in approximating polygonal curves (see [9], [10] and [11]), etc. Moreover, the width of a set is familiar in Operations Research as a minimax location problem, in which we seek a line (the bisector to the lines of support given the width) whose greatest distance to any point of the set is a minimum.

The definition of the width of a finite set in the plane can be extended to Euclidean spaces of dimension greater than two. So, if we consider a finite set of points P in the space $\mathbf{R}^d$, the width of P is the minimum distance between parallel hyperplanes of support of P [8].

As the width of a finite set in the plane is the width of its convex hull [8], many authors have studied the width of convex polygons, because convex polygons are simple sets and they have many applications in pattern recognition [1], image processing [14] and stock cutting and allocation (see [5], [16] and [6]). By using the rotating caliper technique [15] or geometric transforms [3] it is possible to find the width of a convex polygons in linear time and space.

We will see that it is possible to adapt the rotating caliper technique to design an algorithm for computing the width of a set in the sphere. The study of the width in non-planar surfaces is motived by motion plannig [13], and more concretely a subfield of motion planning of considerable practical interest as it is planning the motion of an articulated robot arm, since, as it is well known, in most cases the points accessible by them are not, in general, in the plane but in a non–planar surface.

G. Strang [17] proved that the width of a convex set in the plane is equivalent to the concept of *door* of the set. The door of a set is the minimum closed interval such that the set can pass through it by a continuous family of rigid motions (translations combined with rotations). Nevertheless, in dimension three this is not true and H. Stark has constructed convex sets which can pass through a door, either square or circular, although no projection of the set will fit in the doorway (see [17]). We will see that, with regard to this problem, the behavior of the sphere is exactly the same as in the plane.

In addition to the problem of the door of a set, there exist another problems that can be solved knowing the width of a set. For instance, it is easy to see that the *line center* (that line minimizing the maximum distance to each point of a set) is the median (the equidistance parallel to the a pair of parallels) of the pair of support giving the width.

The goal of this paper is to try to generalize these concepts to convex sets on the sphere and, in addition, to seek necessary and sufficient conditions that those convex sets may verify to pass through a spherical interval by rigid motions on this surface, and the relationships between the concepts described above. These conditions and relationships allow us to design an algorithm which solves the problem of the width of a finite set of points on the sphere.

Our generalizations will use the concept of convex set on the sphere. We can define as in [12] that a set C in the sphere is convex if given two points of C the minimum geodesic joining these points is contained in C.

The angular length of a geodesic arc joining the points P and Q on the sphere is the angle between the two radii joining the center of the sphere with the points P and Q respectively. Observe that no convex set on the sphere contains a geodesic arc with angular length greater than $\dfrac{\pi}{2}$.

2 Width on the sphere

Before trying to generalize the concept of width of a finite set in the plane to the sphere, if we want to give a similar treatment of this idea, we will examine several alternative definitions that are considered in the plane, keeping in mind that we will try to preserve those properties when trying the extension to the sphere. So, firstly, we would replace the idea of lines of support of a set by geodesics of support of a set. In this way, if given a convex set C on the sphere, we will call *meridians of support of* C to the meridians which intersect C and leave the set on one hemisphere. We will call *lune of support of* C to the region delimited by two meridians of support of C that contains C. As, in the sphere, two different meridians have two points in common and they define only one great circle called equator, thus a lune of support defines one equatorial arc.

According to these definitions, we can say that, given a convex set C on the sphere, the *time width* $\mathcal{H}(C)$ of C is the minimum length between the equatorial arcs defined associated to lunes of support of C.

The main difference between this definition and the definition of width in the plane is that, in the plane, two parallel lines of support have empty intersection, whereas on the sphere two meridians of support have two points in common. If we want to preserve the property that the arcs of support of a convex set have empty intersection, similarly as in the plane, we could give another possible definition. Given a convex set C on the sphere, we will call *parallel of support of* C to a parallel which intersects C and leaves the set on one cap, where a cap is a part of the sphere divided by this parallel. If we use the idea of pair of parallels of support, we will conserve the concept of parallelism that we had in the plane (in the sense that they have empty intersection), but note that parallels in the sphere are not geodesics.

According to the definition above, we can say that given a convex set C on the sphere, the *tropical width* $\mathcal{T}(C)$ of C is the minimum distance between all possible pairs of parallels of support of C. Observe, that with this definition the tropical width of a set in the sphere can be used, as in the plane, in Operations Research as a minimax location problem, in which we seek a great circle (the equator of the parallels of support given the tropical width) whose greatest distance to any point of the set is a minimum.

On the other hand, and following the paper of Strang [17] who proved that the width of a convex set in the plane is the minimum length of an closed interval for the set can pass through it by a continuous family of rigid motions, we can give other definitions of width in the sphere as follows, given a convex set C on the sphere, the *door* $\mathcal{P}(C)$ of C is the minimum length between all possible closed arcs of meridians for the set C can pass through them by continuous family of rigid motions (translations combined with rotations) on the sphere.

G. Strang proved that the width of a convex set coincides with its door in the plane. But, as it was pointed out in the introduction, in dimension three this is not true and H. Stark has constructed convex sets which can pass through a door, either square or circular, although no projection of the set will fit in the doorway (see [17]. Thus it is interesting to ask if the behavior of the sphere is, in this point, similar to the plane or to the three dimensional space.

In the sphere, we have the following properties

Lemma 1 *Let C be a convex set on the sphere. Then, $\mathcal{P}(C) \leq \mathcal{T}(C)$.*

Proof: It suffices to consider the arc of meridians orthogonal and contained between the parallels which define $\mathcal{T}(C)$. The length of this arc is greater or equal than $\mathcal{P}(C)$ and, obviously, less or equal than $\mathcal{T}(C)$. □

Lemma 2 *Let C be a convex set on the sphere. Then, $\mathcal{T}(C) \leq \mathcal{H}(C)$.*

Proof: Let $\mathcal{H}$ be the lune that defines $\mathcal{H}(C)$. This lune is defined by meridian arcs which intersect C in two points P and Q. We consider the parallels tangent to C in the points P and Q. The distance $\mathcal{T}^*$ between these parallels is equal to $\mathcal{H}(C)$, so $\mathcal{T}(C) \leq \mathcal{T}^* = \mathcal{H}(C)$. □

Therefore, $\mathcal{P}(C) \leq \mathcal{T}(C) \leq \mathcal{H}(C)$. Next theorem says, that, as it happens in the plane, these three numbers agree in the sphere.

Theorem 3 *A convex set C on the sphere can pass through a meridian arc of length $\mathcal{P}(C)$ if and only if $\mathcal{H}(C) \leq \mathcal{P}(C)$.*

Proof: If $\mathcal{H}(C) \leq \mathcal{P}(C)$ and as C is contained in the lune which its equatorial arc has length $\mathcal{H}(C)$, obviously C can pass through this equatorial arc by rigid motions. To prove the converse, assume first that the boundary ∂C of C is smooth, through every boundary point there is a unique tangent line on the sphere, and it varies continuously along ∂C. Let I be an arc of meridian of length $\mathcal{P}(C)$ and denote by S the spherical surface. As C can pass through I, it is possible to define a continuous composition of motions $M : [0,1] \to S$ where $M(0)$ is the situation of C before going into I and $M(1)$ the situation after passing through I. For all $t \in [0,1]$, we can define two applications $f_1 : [0,1] \to [0,\pi]$ and $f_2 : [0,1] \to [0,\pi]$ as follows: $f_1(t)$ and $f_2(t)$ are the angular lengths between the points P_1 and P and the points P_2 and P respectively, where P_1 and P_2 are the intersection of ∂C with the arc I and P is the intersection between the tangents to C in the points P_1 and P_2 (see Figure 1).

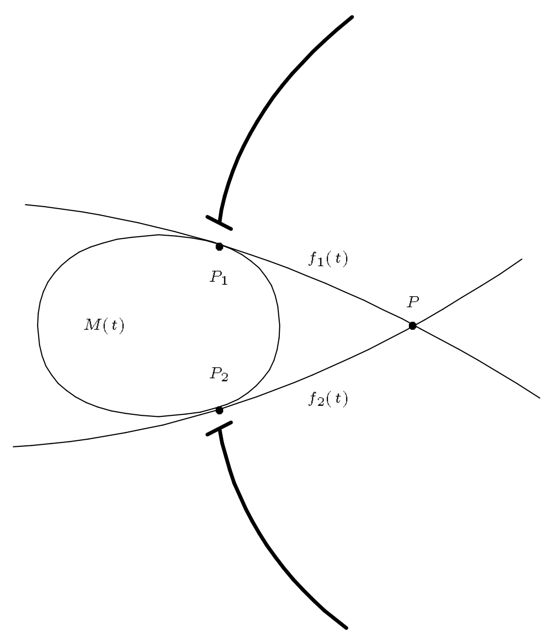

Figure 1

The application $f_1 + f_2 : [0,1] \to [0, 2\pi]$ is continuous and $f_1(0) + f_2(0) = 0$ and $f_1(1) + f_2(1) = 2\pi$, so there exists $t^* \in [0,1]$ such that $f_1(t^*) + f_2(t^*) = \pi$.

If $f_1(t^*) = f_2(t^*) = \dfrac{\pi}{2}$, then the meridian arc which defines $\mathcal{H}(C)$ is contained in I, so $\mathcal{H}(C) \leq \mathcal{P}(C)$. Else, $f_1(t^*) - \dfrac{\pi}{2} = \dfrac{\pi}{2} - f_2(t^*)$. Suppose that $f_1(t^*) > \dfrac{\pi}{2}$ and so $f_2(t^*) < \dfrac{\pi}{2}$. Then, the situation is as in Figure 2.

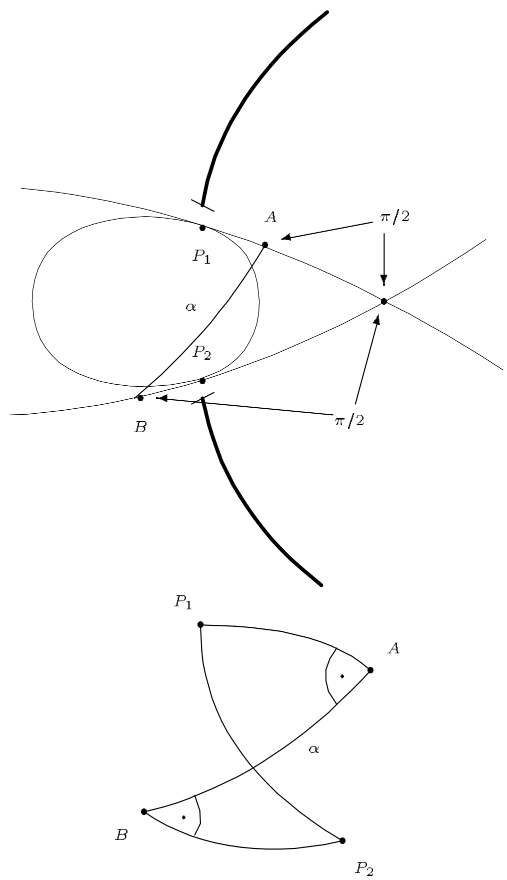

Figure 2

As the angles in the points A and B are of ninety degree, the length of the meridian arc joining P and Q is greater or equal than the length of the meridian arc joining A and B. So, $\mathcal{H}(C) \leq \mathcal{P}(C)$

The conclusion remains true for a convex set C even if ∂C is not smooth. We will proceed introducing a sequence of smooth convex subsets C_n converging to C. As C passes through I so do the C_n and their time widths must satisfy $\mathcal{H}(C_n) < \mathcal{P}(C)$. Therefore, $\mathcal{H}(C) \leq \mathcal{P}(C)$ and the theorem is proved. □

Then, the three definitions of width we have considered agree and we can talk about the width of a set.

As an immediate consequence of Theorem 3 we get

Corollary 4 *The minimum equatorial arc of a convex set C is included in C.*

3 Algorithm of the width on the sphere

Recall, that in the plane the width of a convex polygon is the minimum distance between parallel lines of support passing through an antipodal vertex-edge pair (to each antipodal vertex-edge pair, we associated the lune define by the meridian containing the edge and that containing the vertex such that the equator arc joins the vertex with the edge). In the sphere this is not true and it can be achieved in an antipodal edge-edge pair as Figure 3 shows, but we have

Lemma 5 *The width of a convex polygon is the minimum distance between meridians of support passing through either an antipodal vertex-edge pair or an edge-edge pair.*

Proof: It is an immediate consequence of Corollary 4. □

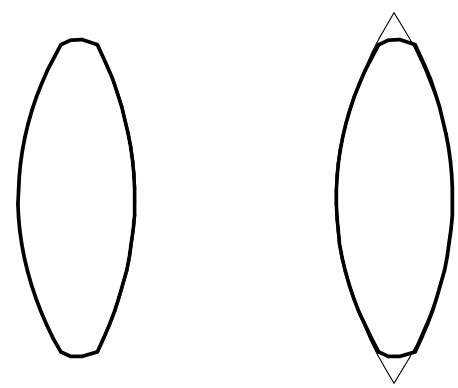

Figure 3

In any case, Lemma 5 says us that it is possible to adapt the rotating caliper algorithm to find the width of a convex polygon C (the number of events is linear). Thus we can give the following algorithm

WIDTH(C)

1.- Find an initial antipodal vertex-edge pair.

2.- If the associated lune to the vertex-edge pair contains C, compute its equator arc, otherwise compute the equator arc of the pair edge-edge associated to the original vertex-edge pair (this edge-edge pair is defined from the vertex-edge

pair by considering the edge incident with the vertex that is not contained in the lune).
3.- Use rotating caliper to generate all pairs as in (1)-(2).
4.- Compute the minimum obtained in previous steps.

It is straightforward to check the following result

Theorem 6 *Algorithm* WIDTH(C) *computes the width of a convex polygon C in optimal linear time.*

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