On some partitioning problems for two-colored point sets

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Abstract

Let S be a two-colored set of n points in general position in the plane. We show that S admits at least $2\left\lfloor \frac{n}{17} \right\rfloor$ pairwise disjoint monochromatic triangles with vertices in S and empty of points of S. We further show that S can be partitioned into $3\left\lceil \frac{n}{11} \right\rceil$ subsets with pairwise disjoint convex hull such that within each subset all but at most one point have the same color. A lower bound on the number of subsets needed in any such partition is also given.

1 Introduction

An important class of problems in Computational Geometry deals with two-colored point sets [13] motivated, surely, by representing two different properties, facilities,... On the other hand, partitions of point sets have been the focus of an extensive research, see [8] for example. Regarding the union of both concepts, there can also be found results on partitioning a two-colored point set into monochromatic convex sets, that is, all points of a convex set belong to the same color class [10]. We study the following variants of this problem. In Section 2 we consider convex sets with a fixed number of vertices on its convex hull. In particular, we investigate how many monochromatic triangles can be found in a partition of S. Here, the monochromatic triangles do not contain points of S in the interior.

In Section 3 we consider partitions into "almost" monochromatic parts. For this case, the number of vertices on the convex hull of a part can be arbitrary, but in each convex set all but at most k of the points - the stains- have the same color, where k is a fixed value. This kind of study is potentially useful in noise reduction in digital signal processing [15]. Salt and pepper noise is a form of noise typically seen on images, where the image contains dark and white dots. Generally this type of noise will only affect a small number of image pixels. We derive bounds on the number of convex pieces that are needed for a partition of S into convex sets with at most one stain. A related problem on two-colored point sets (without considering convexity) is considered in [3], where the authors look for geometric monochromatic 2-factors, possibly adding some extra Steiner points.

2 Disjoint monochromatic triangles

Given a two-colored point set S of n points in general position in the plane, let $\kappa(S, m)$ be the maximum number of monochromatic convex m-gons that can be constructed with vertices in S, such that their convex hulls are pairwise disjoint and have interiors empty of points from S. Let

$$\kappa(n, m) = \min\{\kappa(S, m) \mid S \subset \mathbb{R}^2 \text{ is in general position}, |S| = n\}.$$

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For $m = 2$, $\kappa(S, 2)$ is the number of segments in the largest noncrossing matching in S such that every edge connects points of the same color. Dumitrescu and Steiger proved that there are sets of n points in general position such that any monochromatic matching consists of fewer than $(1 - \epsilon)n/2$ edges, for some constant ϵ and gave a lower bound for the constant [11], later improved by Dumitrescu and Kaye [9] who showed $3n/7 - O(1) \leq \kappa(n, 2) \leq 47n/95 + O(1)$.

For $m = 4$ it has been conjectured that $\kappa(S, 4) \geq 1$ provided that $|S|$ is large enough [7]; an example of 32 points with no monochromatic empty convex quadrilateral was given in [16]. Only recently it has been shown that each set of at least 5044 two-colored points contains an (not necessarily convex) empty monochromatic quadrilateral [2]. Arbitrarily large two-colored sets without monochromatic empty convex pentagons were described in [7], therefore $\kappa(n, m) = 0$ for all $m \geq 5$.

Here we focus on the number $\kappa(n, 3)$. As every set of 10 points contains an empty convex pentagon, and at least three of its vertices will have the same color, we immediately get

$$\left\lfloor \frac{n}{10} \right\rfloor \leq \kappa(n, 3).$$

A related result was proved in [7]: every two-colored set of n points in general position admits $\lceil n/4 \rceil - 2$ empty monochromatic *compatible* triangles, which is tight, where compatible means that any two of them have disjoint interiors and, for example all of them might share a vertex. For $\kappa(n, 3)$ we require a much stronger disjointness condition.

Let us recall a result that will be used repeatedly later:

Lemma 2.1. [12] *Let S be a set of n points in general position in the plane, and let G_x be the graph with vertex set S, two of them being adjacent when the line they define has exactly $\lceil \frac{n-2}{2} \rceil$ points on one side and $\lfloor \frac{n-2}{2} \rfloor$ on the other side. Then, for any n odd, the graph G_x is connected.*

We are now ready for proving the main result of this section:

Theorem 2.2. *Let $\kappa(n, 3)$ be the largest number such that any two-colored set of n points in general position in the plane admits $\kappa(n, 3)$ triangles with vertices in S, pairwise disjoint and empty of points of S. Then we have*

$$2\left\lfloor \frac{n}{17} \right\rfloor \leq \kappa(n, 3) \leq \begin{cases} \lfloor \frac{n-2}{6} \rfloor & \text{if } n \text{ is even} \\ \lfloor \frac{n-1}{6} \rfloor & \text{if } n \text{ is odd} \end{cases}$$

Proof. We first show that any set S_9 of 9 black and white points contains an empty monochromatic triangle. Let W denote the set of white points, B the set of black points, and assume $|W| > |B|$. W admits a triangulation into $2|W| - |conv(W)| - 2$ white triangles with pairwise disjoint interiors; $conv(W)$ denotes the convex hull of W. If $|W| \geq 6$ or $|conv(W)| = 3$ then the triangulation of W contains at least one white triangle also empty of black points. Assume that there is no triangulation of W that contains a triangle empty of points of B. This implies that $|W| = 5$ and $|conv(W)| \geq 4$. If $|conv(W)| = 5$ then at least 3 black points lie in the interior of $conv$(W) and form at least one empty triangle. Finally, if $|conv(W)| = 4$ then the 4 black points lie in the interior of $conv$(W). Triangulating B we obtain at least two black triangles, at most one of them contains a white point in its interior.

We now show that any set S_{17} of 17 black and white points contains two pairwise disjoint empty monochromatic triangles. This immediately implies the lower bound. Thereto, we use that S_{17} admits a segment joining one point of each color and leaving 8 points on one side and 7 on the other. Indeed, for each of the 17 points there exists another point of the set such that the line they define leaves 8 points on one side and 7 on the other. By Lemma 2.1, the graph defined by all these line segments is connected. Consequently not all of these line segments connect two points of the same color. Let x and y denote the two points of S_{17} of such a separating line segment, where x and y have different color. There is an empty monochromatic triangle in the set formed by the set of 7 points together with $\{x, y\}$. Since x and y have different color, this triangle uses at most one of x and y. Assume y is not used. We thus can find another empty monochromatic triangle in the set formed by the 8 points together with y. These two triangles are pairwise disjoint.

To prove the upper bound, consider the set in Figure 1. The set W of white points is in convex position and no two of them lie on a horizontal line. Near every white point, but the topmost and lowermost, a black point is placed such that it lies on the horizontal line through the white point and in the interior of $conv(\mathrm{W})$. Then this n-point set has no empty white triangle, because each white triangle can be stabbed by a horizontal line through its second white point, ordered by y-coordinates, thus containing a black point. Then, there are $\lfloor \frac{n-2}{6} \rfloor$ pairwise disjoint black triangles. For the case n is odd, remove the topmost white point in Figure 1. We remark that this example also appears in problems related to guarding triangles [6]. □

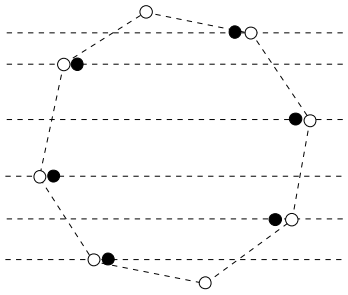


Figure 1: A point set S with $\kappa(S,3) = \lfloor \frac{|S|-2}{6} \rfloor$.

3 At most one stain

Given a two-colored point set S of n points in general position in the plane, let $\sigma(S,k)$ be the minimum number of subsets in a partition of S, such that their convex hulls are pairwise disjoint and all the points in each subset have the same color with the possible exception of at most k of them (the "stains"). Let

$$\sigma(n,k) = \max\{\sigma(S,k) \mid S \subset \mathbb{R}^2 \text{ is in general position}, |S| = n\}.$$

For $k = 0$, $\sigma(S,0)$ is the cardinality of the smallest partition of S into monochromatic subsets whose convex hulls are pairwise disjoint. Dumitrescu and Pach [10] proved that

$$\sigma(n,0) = \left\lceil \frac{n+1}{2} \right\rceil.$$

For generic k there is a trivial bound

$$\sigma(n,k) \leq \left\lceil \frac{n}{2k+1} \right\rceil,$$

because every subset of $2k+1$ points has at most k stains.

Here we focus on the number $\sigma(n,1)$ and prove the following result:

Theorem 3.1. *Let $\sigma(n,1)$ be the smallest number such that any two-colored set of n points in general position in the plane can be partitioned into $\sigma(n,1)$ subsets with pairwise disjoint convex hulls, each one having at most one stain. Then we have*

$$\left\lceil \frac{n+1}{4} \right\rceil \leq \sigma(n,1) \leq \begin{cases} 3\left\lfloor \frac{n}{11} \right\rfloor & \text{if } n' = 0 \\ 3\left\lfloor \frac{n}{11} \right\rfloor + \left\lceil \frac{n'+1}{4} \right\rceil & \text{if } n' \neq 0 \end{cases}$$

where n' is the residue of dividing n by 11.

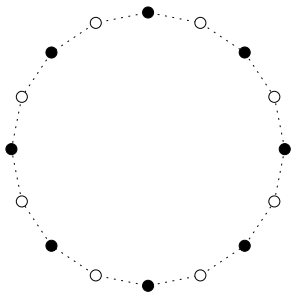


Figure 2: A point set S with $\sigma(S,1) = (|S|+2)/4$.

Proof. For the lower bound, we first consider the case n is even. Let S be a two-colored set of n points in convex position such that any two neighbored points on the convex hull have different colors, see Figure 2. We show that $\sigma(S,1) \geq \left\lceil \frac{n+1}{4} \right\rceil$. Let Π be a partition of S into the minimum number of subsets, such that their convex hulls are pairwise disjoint and such that each subset contains at most one stain. Define a directed tree T, whose nodes are the subsets of Π, as follows: Choose some subset B_s of Π as the source vertex of T. Draw directed arcs from B_s to all subsets of Π that can be "seen" from B_s, and take each of these subsets as the root of a subtree, defined iteratively. More formally, a subset B_i is a descendant of B_s if $conv(B_s \cup B_i)$ does not intersect any other subsets. Similarly, a subset B_j is a descendant of a subset $B_k \neq B_s$ if B_k is a descendant of some subset B_h, B_j is not a descendant of B_h and $conv(B_k \cup B_j)$ does not intersect any other subsets of Π. Figure 3 shows an example.

Assume that Π contains at least one subset B_s that contains at least four points, as the statement is obviously true otherwise. Choose B_s as the root of T.

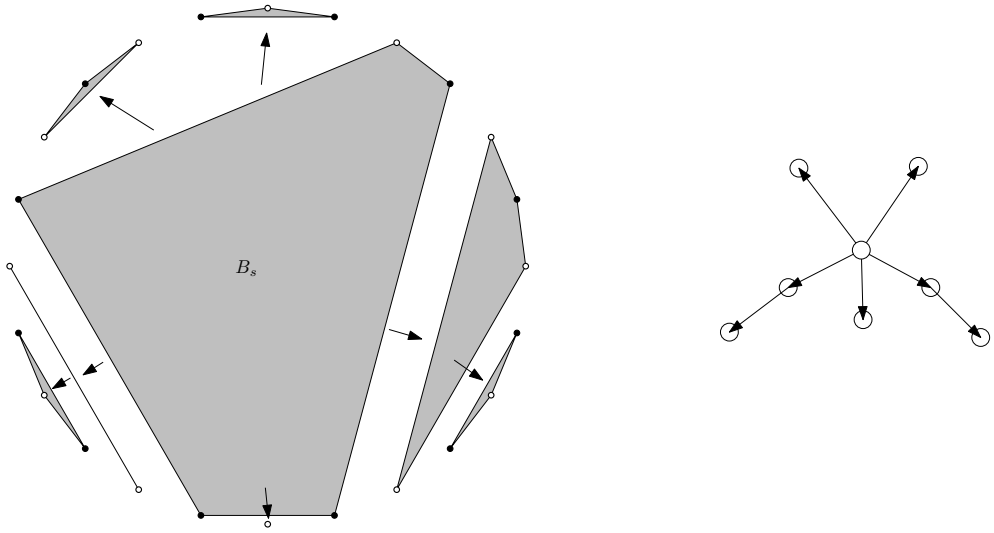


Figure 3: A partition of S and the associated directed tree.

Let n_k denote the number of interior nodes of T that correspond to subsets of Π containing exactly k points, and let h denote the number of leaves of T. Note that an interior node of T corresponds to a subset of Π of at least two points, and a leaf of T corresponds to a subset of Π of at most three points. In the following we give a lower bound on h in terms of the number of interior nodes of T. Thereto, we assign each leaf of T to a unique interior node of T, where a leaf ℓ can be assigned to an interior node v if there is a directed path from v to ℓ in T. We define this assignment recursively by first assigning leaves to the source vertex B_s and then considering the assignment for the children of B_s. A subset of Π contains at most two edges of the convex hull. We thus can assign at least $|B_s| - 2$ leaves to B_s. For $k \geq 5$, to each subset $B_p \neq B_s$ of Π containing k points assign at least $k-4$ leaves: B_p has at least

$k-3$ outgoing edges in T, where one of them is counted for a leaf assigned to an antecessor of B_p. We thus obtain

$$h \geq n_5 + 2n_6 + 3n_7 + \cdots + (t-4)n_t + 2, \tag{1}$$

where t is the number of points of the largest subset, and the last summand 2 comes from the two additional leaves assigned to B_s. On the other hand, the number of leaves of T is at least

$$h \geq \frac{n - 2n_2 - 3n_3 - 4n_4 - \cdots - tn_t}{3}. \tag{2}$$

Taking the sum of (1) and three times (2) we obtain

$$4h \geq n - 2n_2 - 3n_3 - 4n_4 - 4n_5 - 4n_6 - \cdots - 4n_t + 2.$$

Thus, we also have

$$4h \geq n - 4n_2 - 4n_3 - 4n_4 - 4n_5 - 4n_6 - \cdots - 4n_t + 2,$$

which implies that

$$\sigma(S,1) \geq h + n_2 + n_3 + n_4 + \cdots + n_t \geq \left\lceil \frac{n+2}{4} \right\rceil = \left\lceil \frac{n+1}{4} \right\rceil,$$

because n is even.

The case when n is odd can be treated the same way, just remove one point from S, apply the bound for the obtained even point set and add the removed point to a piece.

For the upper bound, we first show that $\sigma(S_7, 1) \leq 2$. Any set S_7 of 7 black and white points admits a segment joining one point of each color leaving 3 points on one side and 2 on the other. Indeed, for each of the 7 points there exists another point of the set such that the line they define leaves 3 points on one side and 2 on the other. By Lemma 2.1, the graph defined by all these line segments is connected. Consequently not all of these line segments connect two points of the same color. Then, the 3 points on one side of the segment together with one of the endpoints of the segment form a piece with at most one stain. The remaining three points form the other piece of the partition.

We are going to prove now that for any set S_{11} of 11 black and white points we have $\sigma(S_{11}, 1) \leq 3$; from this and $\sigma(S_7, 1) \leq 2$ the upper bound is immediately derived. Let W and B be the set of white and black points in S_{11}, respectively, and let us assume, without loss of generality, that $|W| > |B|$.

If $|B| \leq 3$, the claim is obvious; if $|B| = 4$ we can always find two black points b_1 and b_2 such that the other two black points lie in opposite halfplanes with respect to the line b_1b_2; then the two sets of white and black points lying in the same halfplane and $\{b_1, b_2\}$ give the claimed partition. Therefore, we are only left with the case $|B| = 5$.

For the case $|B| = 5$ observe that whenever we can separate with a line a set of four points with at most one stain, then the seven remaining points would require at most two pieces and we would be done.

Let us consider first the case in which at least one of the white points, w, is an extreme point, and let us denote by $x_1, \ldots, x_{10}$ the other points in counterclockwise radial order around w, in such a way that x_1 and x_{10} are neighbors of w on the convex hull $conv(S_{11})$. Assume that the set $\{x_1, x_2, x_3\}$ contains exactly two black points and one white point, as otherwise we would be done, because we would be able to isolate the subset $\{w, x_1, x_2, x_3\}$ with at most one stain. For the same reason we assume that x_4 is a white point, as otherwise we could separate the subset $\{x_1, x_2, x_3, x_4\}$. By symmetry reasons we also assume that $\{x_8, x_9, x_{10}\}$ contains exactly two black points and one white point and that x_7 is a white point. But now $\{x_5, x_6\}$ must consist of one white point and one black point, and we can take the partition $\{x_1, x_2, x_3\}, \{w, x_4, x_5, x_6, x_7\}, \{x_8, x_9, x_{10}\}$ (Figure 4).

Let us switch to the case in which none of the white points is an extreme point. If there are four or five black points that are extreme, we are done, because we can define two black monochromatic

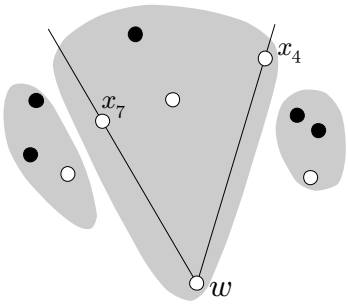
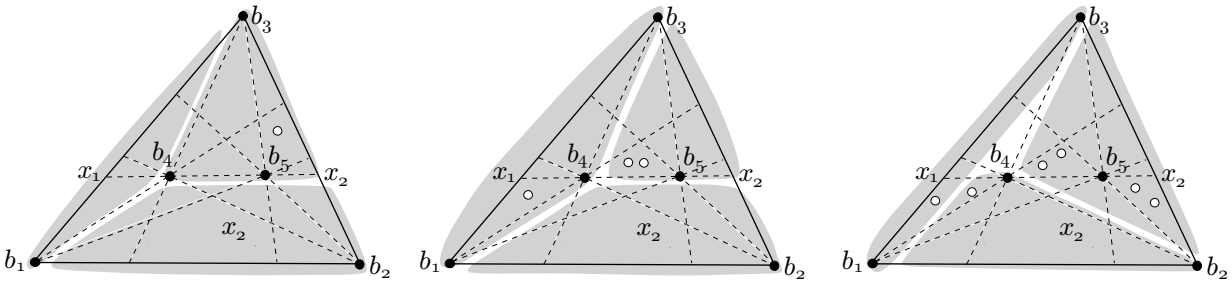


Figure 4: The case $|B| = 5$ with at least one white extreme point.

subsets by taking the endpoints of two disjoint convex hull edges, and the remaining black point would join as a stain the whole subset of white points.

Hence we are left with the situation in which the convex hull is a triangle with three black corners, which we denote by $\{b_1, b_2, b_3\}$. The line through the other two black points, b_4 and b_5, will cut two convex hull edges; for ease of description we assume that the line b_1b_2 is the horizontal base of the convex hull and that the line b_4b_5 is parallel to b_1b_2, which is no restriction of generality from the combinatorial viewpoint (refer to Figure 5).



(a) (b) (c)

Figure 5: The cases $|B| = 5$ with no white extreme point.

If the halfplane above the line b_4b_5 contains less than two white points, we can separate them together with b_4, b_5 and b_3, and get $\sigma(S_{11}, 1) = 3$; the same happens if the halfplane above the line b_4b_5 contains more than two white points, which we would separate together with b_3. Let us assume therefore that there are exactly two white points above b_4b_5. If the triangle $b_3b_4x_2$ contains at most one of them we can take it as part together with b_4, b_5 and b_3, a second part would consist of b_1 and any white point interior to $b_1b_4b_3$ and a third part would consist of b_2 and any white point interior to the quadrilateral $b_1b_2x_2b_4$ (Figure 5(a)), this gives $\sigma(S_{11}, 1) = 3$. The same argument applies to the triangle $b_5b_3x_1$, hence we assume hereafter that the two white points above b_4b_5 lie inside the triangle $b_3b_4b_5$.

Now, if the triangle $b_1b_4b_3$ contains at most one white point, we can take it together with b_1, b_4 and b_3 as a part, a second part would consist of b_5 and the two white points above b_4b_5, and a third part would consist of b_2 and all the white points inside the quadrilateral $b_1b_2x_2b_4$; again, we get $\sigma(S_{11}, 1) = 3$ (Figure 5(b)).

The same reasoning as in the preceding paragraph applies to the triangle $b_5b_2b_3$, hence we are finally left with the situation in which two white points are interior to $b_1b_4x_1$, two are interior to $b_5b_2x_2$ and two are interior to $b_3b_4b_5$ (Figure 5(c)). Now we can take b_1 and b_3 together with the white point closest to the line b_1b_3, the other white point inside $b_1b_4b_3$ joins b_2 and b_4 in a second subset, and the white points interior to $b_3b_4b_2$ together with b_5 is the third part. Again, we get $\sigma(S_{11}, 1) = 3$, which concludes the proof. □

4 Future work

We have presented some combinatorial studies on $\kappa(n, 3)$ (the number of monochromatic empty disjoint triangles in any two–colored n–set) and $\sigma(n, 1)$ (the smallest number of pieces with at most one stain in any two–colored n-set), so an algorithmic approach to both problems would be interesting. On the other hand, since in the first case we have that that any two–colored n–set admits a triangulation containing at least $2\left\lfloor \frac{n}{17} \right\rfloor$ monochromatic triangles, one can be interested, thinking in interpolation problems, in to decide, given a two–colored n–set of points, whether there exists a triangulation of it without monochromatic triangles. Our first steps in this direction lead us to suspect that this will be an NP–complete problem. A related question to determine $\kappa(n, 3)$ is to count the number of *all* (not necessarily disjoint) monochromatic empty triangles in a two-colored point set. This problem has been addressed in [1, 14]. While for uncolored point sets there always exists a quadratic number of empty triangles [4], determining the right asymptotic value for the two-colored case represents an intriguing open problem.

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