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The 16th Hilbert problem for discontinuous piecewise isochronous centers of degree one or two separated by a straight line

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M. Esteban,[1,a] J. Llibre,[2,b] and C. Valls[3,c]

AFFILIATIONS

[1] Dept. Matemática Aplicada II and Instituto de Matemáticas (IMUS), Escuela Técnica Superior de Ingeniería de la Universidad de Sevilla, Camino de los Descubrimientos s/n, 41092 Sevilla, Spain
[2] Dept. Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain
[3] Dept. Matemática, Instituto Superior Técnico, Universidade Técnica de Lisboa, Av. Rovisco Pais, 1049–001 Lisboa, Portugal

[a] **Author to whom correspondence should be addressed:** marinaep@us.es
[b] jllibre@mat.uab.cat
[c] cvalls@math.ist.utl.pt

ABSTRACT

In this paper, we deal with discontinuous piecewise differential systems formed by two differential systems separated by a straight line when these two differential systems are linear centers (which always are isochronous) or quadratic isochronous centers. It is known that there is a unique family of linear isochronous centers and four families of quadratic isochronous centers. Combining these five types of isochronous centers, we obtain 15 classes of discontinuous piecewise differential systems. We provide upper bounds for the maximum number of limit cycles that these fifteen classes of discontinuous piecewise differential systems can exhibit, so we have solved the 16th Hilbert problem for such differential systems. Moreover, in seven of the classes of these discontinuous piecewise differential systems, the obtained upper bound on the maximum number of limit cycles is reached.

To solve the 16th Hilbert problem, i.e., to find an upper bound for the maximum number of limit cycles that a given class of differential systems can exhibit, is in general an unsolved problem. For the classes of discontinuous piecewise differential systems here studied, we can obtain the solution using the first integrals of the linear and quadratic isochronous centers.

I. INTRODUCTION AND MAIN RESULTS

We consider planar differential systems of the form

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y),$$

where $P(x,y)$ and $Q(x,y)$ are polynomial functions, and the degree of the systems is the maximum degree of such polynomials. In particular, in this paper we consider discontinuous piecewise differential systems of the form

$$(\dot{x},\dot{y}) = \mathbf{F}(x,y) = \begin{cases} \mathbf{F}^-(x,y) = (f^-(x,y), g^-(x,y)) & \text{if } x < 0, \\ \mathbf{F}^+(x,y) = (f^+(x,y), g^+(x,y)) & \text{if } x > 0, \end{cases} \quad (1)$$

being bi-valued on the separation line $x = 0$. Following Ref. 9, a point $(0,y)$ is a *crossing point* if $f^-(0,y)f^+(0,y) > 0$. If there exists a periodic orbit of the discontinuous differential system (1) having exactly two crossing points, then we call it a *crossing periodic orbit*. A *crossing limit cycle* is an isolated periodic orbit in the set of all crossing periodic orbits of system (1). In what follows for simplicity, we shall say limit cycle instead of crossing limit cycle.

The analysis of planar continuous piecewise linear systems is well established when the number of linear zones is small, see Ref. 33 and the references therein. They frequently appear in many non-linear engineering devices, which are accurately modelled by piecewise linear vector fields, see Ref. 7. They appear also in mathematical biology, see Refs. 6 and 30–32. However, when the planar vector field is discontinuous, the adaptation of the 16th Hilbert's problem on the maximum number of existing limit cycles is an open problem. In the last years, many authors have worked in this problem, trying to determine how many limit cycles can appear in planar systems separated by a straight line, see, for instance, Refs. 1–4, 8, 10–14, 16–18, and 21–29. For details on the classical 16th Hilbert problem, see, for instance, Refs. 15, 19, and 20.

Let $p \in \mathbb{R}^2$ be a singularity of a differential system in the plane. The singularity p is a *center* if there exists an open neighbourhood U of p such that all the solutions in $U \setminus \{p\}$ are periodic. Denote by T_q the period of the periodic orbit through $q \in U \setminus \{p\}$. We say that p is an *isochronous center* if T_q is constant for all $q \in U \setminus \{p\}$.

In this paper, we work with the following five types of systems that cover the classes of a linear system having a center and of all quadratic polynomial differential systems having an isochronous center. For a proof of the linear system, see Lemma 1 of Ref. 27, and for a proof of the quadratic systems, see page 34 of Ref. 5.

(I) Any linear differential system having a center can be written as

$$\dot{x} = -Ax - (A^2 + \omega^2)y + B, \quad \dot{y} = x + Ay + C,$$

with $\omega > 0$, $A, B, C \in \mathbb{R}$ and $A \neq 0$. A first integral of this system is

$$H_1(x, y) = (x + Ay)^2 + 2(Cx - By) + y^2\omega^2.$$

Of course, every linear center is isochronous.

(II) The first family of quadratic isochronous differential systems can be obtained doing an affine transformation to the system

$$\dot{x} = -y + x^2 - y^2, \quad \dot{y} = x(1 + 2y),$$

with first integral

$$\tilde{H}_2(x, y) = \frac{x^2 + y^2}{1 + 2y}.$$

(III) The second family of quadratic isochronous differential systems can be obtained doing an affine transformation to the system

$$\dot{x} = -y + x^2, \quad \dot{y} = x(1 + y),$$

whose first integral is

$$\tilde{H}_3(x, y) = \frac{x^2 + y^2}{(1 + y)^2}.$$

(IV) The third family of quadratic isochronous differential systems can be obtained doing an affine transformation to the system

$$\dot{x} = -y + \frac{4}{3}x^2, \quad \dot{y} = x\Big(1 - \frac{16}{3}y\Big),$$

with first integral

$$\tilde{H}_4(x, y) = \frac{9(x^2 + y^2) - 24x^2y + 16x^4}{-3 + 16y}.$$

(V) The fourth family of quadratic isochronous differential systems can be obtained doing an affine transformation to the system

$$\dot{x} = -y + \frac{16}{3}x^2 - \frac{4}{3}y^2, \quad \dot{y} = x\Big(1 + \frac{8}{3}y\Big),$$

whose first integral is

$$\tilde{H}_5(x, y) = \frac{9(x^2 + y^2) + 24y^3 + 16y^4}{(3 + 8y)^4}.$$

Our objective is to solve the 16th Hilbert problem for the 15 classes of discontinuous piecewise differential systems separated by a straight line and formed by two arbitrary isochronous centers of degree 1 or 2, i.e., we shall provide for all these 15 classes an upper bound on the maximum number of limit cycles that each class can exhibit. Moreover, as we shall see in many cases, the upper bound that we shall provide is reached.

We must mention that, in general, it is very difficult (many times for the moment impossible) to provide an upper bound for the maximum number of limit cycles that a class of differential systems in the plane can exhibit, and of course, it is even more difficult to provide the exact upper bound, see, for instance, Refs. 15, 19, and 20.

It was proved in Theorem 3 of Ref. 27 or in Corollary 3 of Ref. 23 that discontinuous piecewise differential systems separated by a straight line and formed by two arbitrary linear centers have no limit cycles. So this case is not considered here.

Our first main result is to provide the maximum number of limit cycles that can exist for discontinuous piecewise differential systems of the form (1), where in $x < 0$ there is an arbitrary linear differential center (I), and for $x > 0$, there is one of the four quadratic isochronous differential systems (II), (III), (IV), or (V) after an arbitrary affine change of variables.

Theorem 1. *Consider discontinuous piecewise differential systems separated by the straight line $x = 0$ and formed by a linear differential center (I) after an affine change of variables in $x < 0$ and by a quadratic isochronous system of type either (II), or (III), or (IV), or (V) after an affine change of variables in $x > 0$. The maximum number of limit cycles of these discontinuous piecewise differential systems is*

(a) *at most one for systems of types (I) and (II), and there are systems of this type with exactly one limit cycle, see Fig. 1;*
(b) *at most one for systems of types (I)–(III), and there are systems of this type with exactly one limit cycle, see Fig. 2;*
(c) *at most two for systems of types (I)–(IV), and there are systems of this type with exactly one limit cycle, see Fig. 3; and*
(d) *at most two for systems of types (I)–(V), and there are systems of this type with exactly two limit cycles, see Fig. 4.*

Note that for all systems of type (I)-(k) with $k \in \{II, III, V\}$, the upper bound on the maximum number of limit cycles is reached.

The proof of Theorem 1 is given in Sec. III.

The second main result of the paper is to give the maximum number of limit cycles that can appear in discontinuous piecewise differential systems of the form (1) such that in a half-plane there is a general quadratic isochronous differential system of type (II), and in the other one, there is a general quadratic isochronous differential system of type (II), (III), (IV), or (V) after an arbitrary affine change of variables.

Theorem 2. *Consider discontinuous piecewise differential systems separated by the straight line $x = 0$ and formed by a quadratic isochronous center of type (II) after an affine change of variables in $x < 0$ and by a quadratic isochronous system of type either (II), or (III), or (IV), or (V) after an affine change of variables in $x > 0$. The maximum number of limit cycles of these discontinuous piecewise differential systems is*

(a) *at most one for systems of types (II) and (II), and there are systems of this type with exactly one limit cycle, see Fig. 5;*
(b) *at most one for systems of types (II) and (III), and there are systems of this type with exactly one limit cycle, see Fig. 6;*
(c) *at most three for systems of types (II)–(IV), and there are systems of this type with exactly two limit cycles, see Fig. 7; and*
(d) *at most three for systems of types (II)–(V), and there are systems of this type with exactly two limit cycles, see Fig. 8.*

Note that for systems of types (II) and (II) and (II) and (III) after an affine change of variables, the upper bound on the maximum number of limit cycles is reached.

The proof of Theorem 2 is given in Sec. IV.

The third main result of the paper is to give the maximum number of limit cycles that can appear in discontinuous piecewise differential systems of the form (1) such that in a half-plane there is a general quadratic isochronous differential system of type (III), and in the other one, a general quadratic isochronous differential system of type (III), (IV), or (V) after an arbitrary affine change of variables.

Theorem 3. *Consider discontinuous piecewise differential systems separated by the straight line $x = 0$ and formed by a quadratic isochronous center of type (III) after an affine change of variables in $x < 0$, and by a quadratic isochronous system of type either (III), or (IV), or (V) after an affine change of variables in $x > 0$. The maximum number of limit cycles of these discontinuous piecewise differential systems is*

(a) *at most one for systems of types (III) and (III), and there are systems of this type with exactly one limit cycle, see Fig. 9;*
(b) *at most three for systems of types (III) and (IV), and there are systems of this type with exactly two limit cycles, see Fig. 10; and*
(c) *at most three for systems of types (III)–(V), and there are systems of this type with exactly two limit cycles, see Fig. 9.*

Note that for systems of types (III) and (III) after an affine change of variables, the upper bound on the maximum number of limit cycles is reached.

The proof of Theorem 3 is given in Sec. V.

The following result gives the maximum number of limit cycles that can appear in discontinuous piecewise differential systems of the form (1) such that in a half-plane there is a general quadratic isochronous differential system of type (IV), and in the other one, a general quadratic isochronous differential system of type (IV) or (V) after an arbitrary affine change of variables.

Theorem 4. *Consider discontinuous piecewise differential systems separated by the straight line $x = 0$ and formed by a quadratic isochronous center of type (IV) after an affine change of variables in $x < 0$ and by a quadratic isochronous system of type either (IV), or (V) after an affine change of variables in $x > 0$. The maximum number of limit cycles of these discontinuous piecewise differential systems is*

(a) *at most three for systems of types (IV) and (IV), and there are systems of this type with exactly two limit cycles, see Fig. 12; and*
(b) *at most three for systems of types (IV) and (V), and there are systems of this type with exactly two limit cycles, see Fig. 13.*

The proof of Theorem 4 is given in Sec. VI.

The last main result gives the maximum number of limit cycles that can appear in discontinuous piecewise differential systems of the form (1) such that in both half-planes there is a quadratic isochronous differential system of type (V) after an affine change of variables.

Theorem 5. *The maximum number of limit cycles for discontinuous piecewise isochronous quadratic differential systems formed by two systems of type (V) separated by the straight line $x = 0$ after an affine change of variables is at most 12, and there are systems of this type with exactly two limit cycles, see Fig. 14.*

The proof of Theorem 5 is given in Sec. VII. See the remark at the end of the proof of Theorem 5 related with this theorem.

II. THE QUADRATIC ISOCHRONOUS DIFFERENTIAL SYSTEMS (II), (III), (IV), AND (V) AFTER AN AFFINE CHANGE OF VARIABLES

In this section, we show the expressions for the quadratic isochronous systems (II), (III), (IV), and (V) and their first integrals, after doing the general affine change of variables of the form

$$(x, y) \to (ax + by + c, \alpha x + \beta y + \gamma), \tag{2}$$

with $b\alpha - a\beta \neq 0$. Thus, the differential system (II) after this affine change of variables becomes

$$\begin{aligned}
\dot{x} = {} & \frac{1}{b\alpha - a\beta}\big(\beta\gamma^2 + 2b\gamma c + bc + \beta\gamma - \beta c^2 + (2ab\gamma + 2\alpha\beta\gamma \\
& + ab + \alpha\beta - 2a\beta c + 2\alpha bc)x + (2\gamma + 1)(b^2 + \beta^2)y + (-a^2\beta \\
& + \alpha^2\beta + 2\alpha ab)x^2 + 2\alpha(b^2 + \beta^2)xy + \beta(b^2 + \beta^2)y^2\big), \\
\dot{y} = {} & \frac{1}{b\alpha - a\beta}\big(-\alpha\gamma^2 - 2a\gamma c - ac - \alpha\gamma + \alpha c^2 - (2\gamma + 1) \\
& \times (a^2 + \alpha^2)x + (-2ab\gamma - 2\alpha\beta\gamma - ab - \alpha\beta - 2a\beta c + 2\alpha bc)y \\
& - \alpha(a^2 + \alpha^2)x^2 - 2\beta(a^2 + \alpha^2)xy - (\alpha\beta^2 + 2a\beta b - \alpha b^2)y^2\big),
\end{aligned} \tag{3}$$

whose first integral is

$$H_2(x, y) = \frac{(c + ax + by)^2 + (x\alpha + y\beta + \gamma)^2}{1 + 2(x\alpha + y\beta + \gamma)}.$$

The differential system (III) becomes

$$\dot{x} = \frac{1}{\alpha b - a\beta}\big(-b\gamma c - bc - \beta\gamma + \beta c^2 + (-ab\gamma - ab - \alpha\beta + 2a\beta c - \alpha bc)x - (b^2\gamma + b^2 + \beta^2 + \beta bc)y + a(a\beta - \alpha b)x^2 - b(\alpha b - a\beta)xy\big), \quad (4)$$

$$\dot{y} = \frac{1}{\alpha b - a\beta}\big(-a\gamma c - ac - \alpha\gamma + \alpha c^2 - (a^2\gamma + a^2 + \alpha^2 - \alpha ac)x + (-ab\gamma - ab - \alpha\beta - a\beta c + 2\alpha bc)y - a(a\beta - \alpha b)xy + b(\alpha b - a\beta)y^2\big),$$

whose first integral is

$$H_3(x, y) = \frac{(ax + by + c)^2 + (\gamma + \alpha x + \beta y)^2}{(\gamma + \alpha x + \beta y + 1)^2}.$$

The differential system (IV) becomes

$$\dot{x} = \frac{1}{3(\alpha b - a\beta)}\big(-16b\gamma c + 3bc + 3\beta\gamma + 4\beta c^2 + (-16ab\gamma + 3ab + 3\alpha\beta + 8a\beta c - 16\alpha bc)x + (-16b^2\gamma + 3b^2 + 3\beta^2 - 8\beta bc)y + 4a(a\beta - 4\alpha b)x^2 - 8b(a\beta + 2\alpha b)xy - 12b^2\beta y^2\big), \quad (5)$$

$$\dot{y} = \frac{1}{3(\alpha b - a\beta)}\big(16a\gamma c - 3ac - 3\alpha\gamma - 4\alpha c^2 + (16a^2\gamma - 3a^2 - 3\alpha^2 + 8\alpha ac)x + (16ab\gamma - 3ab - 3\alpha\beta + 16a\beta c - 8\alpha bc)y + 12a^2\alpha x^2 + 8a(2a\beta + \alpha b)xy - 4b(\alpha b - 4a\beta)y^2\big),$$

whose first integral is

$$H_4(x, y) = \frac{1}{16(\gamma + \alpha x + \beta y) - 3}\big(-24(ax + by + c)^2(\gamma + \alpha x + \beta y) + 9\left((ax + by + c)^2 + (\gamma + \alpha x + \beta y)^2\right) + 16(ax + by + c)^4\big).$$

Finally, the differential system (V) after the change of variables (2) becomes

$$\dot{x} = \frac{1}{3(\alpha b - a\beta)}\big(4\beta\gamma^2 + 8b\gamma c + 3bc + 3\beta\gamma - 16\beta c^2 + (8ab\gamma + 8\alpha\beta\gamma + 3ab + 3\alpha\beta - 32a\beta c + 8\alpha bc)x + (8b^2\gamma + 8\beta^2\gamma + 3b^2 + 3\beta^2 - 24\beta bc)y + 4(\alpha^2\beta - 4a^2\beta + 2\alpha ab)x^2 + 8(\alpha\beta^2 - 3a\beta b + \alpha b^2)xy - 4\beta y^2(2b^2 - \beta^2)y^2\big), \quad (6)$$

$$\dot{y} = \frac{1}{3(\alpha b - a\beta)}\big(16\alpha c^2 - 4\alpha\gamma^2 - 8a\gamma c - 3ac - 3\alpha\gamma - (8a^2\gamma + 8\alpha^2\gamma + 3a^2 + 3\alpha^2 - 24\alpha ac)x - (8ab\gamma + 8\alpha\beta\gamma + 3ab + 3\alpha\beta + 8a\beta c - 32\alpha bc)y + 4\alpha(2a^2 - \alpha^2)x^2 + 8(a^2(-\beta) - \alpha^2\beta + 3\alpha ab)xy - 4(\alpha\beta^2 + 2a\beta b - 4\alpha b^2)y^2\big),$$

whose first integral is

$$H_5(x, y) = \frac{1}{(8(\gamma + \alpha x + \beta y) + 3)^4}\big(9\left((ax + by + c)^2 + (\gamma + \alpha x + \beta y)^2\right) + 16(\gamma + \alpha x + \beta y)^4 + 24(\gamma + \alpha x + \beta y)^3\big).$$

III. PROOF OF THEOREM 1

A. Proof of Theorem 1 for systems (I)–(II)

We consider the planar linear differential system (I) with first integral $H_1(x, y)$ in the half-plane $x < 0$ and the quadratic polynomial differential system (3) with first integral $H_2(x, y)$ in the half-plane $x > 0$. If there exists a limit cycle of the discontinuous piecewise differential systems (I)-(3), it must intersect the discontinuity line $x = 0$ in two different points $(0, y)$ and $(0, Y)$. Clearly,, these two points must satisfy the system

$$\begin{aligned} H_1(0, y) - H_1(0, Y) &= (Y - y)\left(-4A^2y - 4A^2Y + 8B - y\omega^2 - \omega^2 Y\right) \\ &= (Y - y)P_1(y, Y) = 0, \\ H_2(0, y) - H_2(0, Y) &= \frac{(Y - y)Q_2(y, Y)}{[1 + 2(\beta y + \gamma)][1 + 2(\beta Y + \gamma)]} = 0, \end{aligned} \quad (7)$$

where P_1 and Q_2 are polynomials of degrees one and two, respectively. Since the points $(0, y)$ and $(0, Y)$ are different, from $P_1(y, Y) = 0$, we get Y as a function of y, that is, $Y = f(y)$. Substituting this expression in equation $Q_2(y, Y) = 0$, we obtain a quadratic equation in the variable y. Then, the maximum number of solutions of (7) is two, namely, (y_1, Y_1) and (y_2, Y_2), but in fact, these two solutions represent the same limit cycle because $Y_1 = y_2$ and $Y_2 = y_1$. So for the discontinuous piecewise differential system (I)–(3), there exists at most one limit cycle.

Now we give an example of a discontinuous piecewise differential system (I)–(3) having one limit cycle. On $x > 0$, we consider the linear differential system

$$\dot{x} = 1 - x - \frac{5}{4}y, \quad \dot{y} = x + y, \quad (8)$$

whose first integral is

$$H_1(x, y) = -8y + y^2 + 4(x + y)^2,$$

and on $x > 0$, we consider the quadratic isochronous differential system of type (3)

$$\dot{x} = -4 - 5x - 6y - x^2 - 4xy - 2y^2, \quad \dot{y} = 1 + 3x + y + x^2 + 2xy, \quad (9)$$

whose first integral is

$$H_2(x, y) = \frac{(x + y + 1)^2 + (y + 1)^2}{2(x + y + 1) + 1}.$$

We can take, without loss of generality, the solution of (7) satisfying $y < Y$, and so the pair $(y, Y) = \left(\frac{1}{5}(4 - 3\sqrt{14}), \frac{1}{5}(4 + 3\sqrt{14})\right)$ provides the limit cycle that exists for the discontinuous differential piecewise systems (8) and (9) shown in Fig. 1.

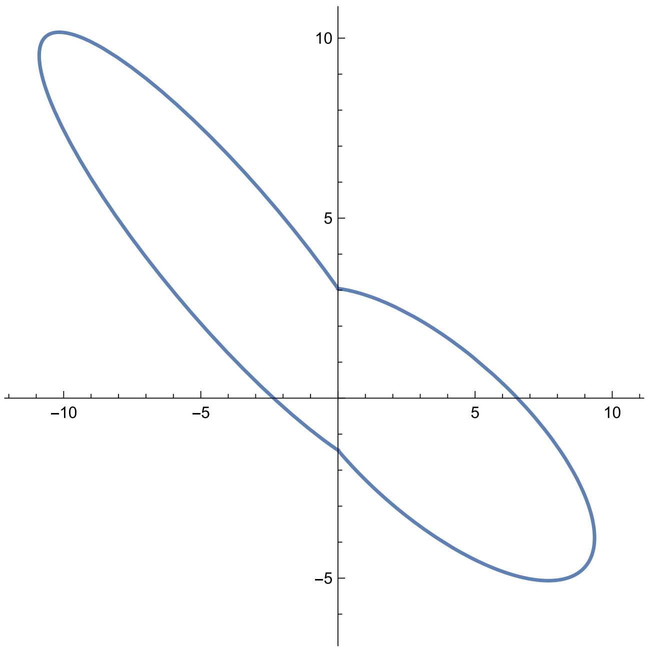

FIG. 1. The unique limit cycle that exists for systems (8) and (9) of classes (I) and (II). It is travelled in counter-clockwise sense.

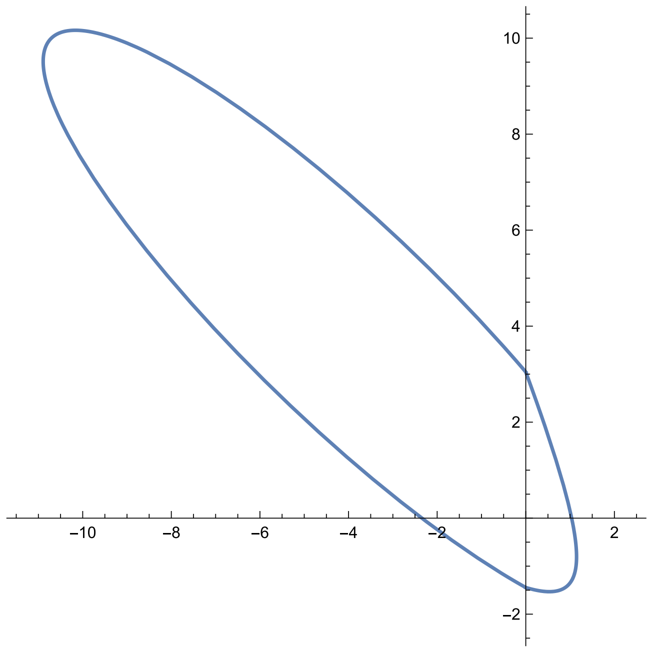

FIG. 2. The unique limit cycle that exists for systems (8)–(11) of classes (I)–(III). It is travelled in counter-clockwise sense.

B. Proof of Theorem 1 for systems (I)–(III)

We consider the linear differential system (I) with first integral $H_1(x, y)$ on the half-plane $x < 0$, and on the half-plane $x > 0$, we take the quadratic isochronous differential system (4) with its first integral $H_3(x, y)$. Then, if there exists some limit cycle for the discontinuous differential system (I)–(4), it must intersect the discontinuity line $x = 0$ at two different points $(0, y)$ and $(0, Y)$, satisfying the equations

$$\begin{aligned} H_1(0, y) - H_1(0, Y) &= (Y - y)P_1(y, Y) = 0, \\ H_3(0, y) - H_3(0, Y) &= \frac{(Y - y)Q_3(y, Y)}{(1 + \beta y + \gamma)^2(1 + \beta Y + \gamma)^2} = 0. \end{aligned} \tag{10}$$

In (10), P_1 and Q_3 are polynomials of degrees one and two, respectively. By following the same procedure as for the proof of systems (I) and (II), we solve the equation $P_1(y, Y) = 0$ obtaining the variable Y as a function of y, that is, $Y = f(y)$. By replacing Y in the equation $Q_3(y, Y) = 0$, we obtain again a quadratic polynomial equation in the variable y, so that the equation has at most two different solutions. As in the proof for systems (I) and (II), these two solutions represent, if they exist, the same limit cycle. Therefore, system (10) has only one solution with $y < Y$, and then the discontinuous piecewise differential system (I)–(4) has at most one limit cycle.

Next, we give a specific discontinuous piecewise differential system (I)–(4) having one limit cycle. On the half-plane $x < 0$, we consider the linear differential system (8), and on the half-plane $x > 0$, we consider the quadratic isochronous differential system of type (4)

$$\dot{x} = -2 - 2y + x^2 + xy, \quad \dot{y} = 2 + 2x + 3y + xy + y^2, \tag{11}$$

with first integral

$$H_3(x, y) = \frac{(x + y + 1)^2 + (y + 1)^2}{(y + 2)^2}.$$

In this case, the unique solution for system (10) with $y < Y$ is

$$(y, Y) = \left(\frac{1}{5}(4 - 3\sqrt{14}), \frac{1}{5}(4 + 3\sqrt{14})\right),$$

and the corresponding limit cycle of the discontinuous piecewise differential systems (8)–(11) associated to this solution is shown in Fig. 2.

C. Proof of Theorem 1 for systems (I)–(IV)

We consider again on the half-plane $x < 0$ the linear differential system (I) with its first integral $H_1(x, y)$, and on $x > 0$, we take the quadratic isochronous differential system (5) with its first integral $H_4(x, y)$. Then, if the discontinuous differential system (I)–(5) has a limit cycle, it must intersect the discontinuity line $x = 0$ at two different points $(0, y)$ and $(0, Y)$. These points must satisfy the equations

$$H_1(0,y) - H_1(0,Y) = (Y-y)P_1(y,Y) = 0,$$
$$H_4(0,y) - H_4(0,Y) = \frac{(Y-y)Q_4(y,Y)}{(-3+16y\beta+16\gamma)(-3+16Y\beta+16\gamma)} = 0, \quad (12)$$

where P_1 and Q_4 are polynomials of degrees one and four, respectively. We solve the equation $P_1(y,Y) = 0$ obtaining the variable Y as a function of y, that is, $Y = f(y)$. If we substitute $Y = f(y)$ in equation $Q_4(y,Y) = 0$, we obtain a polynomial equation of degree four in the variable y, and so system (12) has at most four real solutions. Taking into account the symmetry between these solutions, as in the previous statements, there can be only two different solutions (y, Y) of (12) satisfying $y < Y$.

Now, we show a concrete discontinuous piecewise differential system (I)-(5) having a unique limit cycle. On the half-plane $x < 0$, we consider the linear differential system

$$\dot{x} = 1 - x - 2y, \quad \dot{y} = x + y, \quad (13)$$

with first integral

$$H_1(x,y) = -2y + y^2 + (x+y)^2,$$

and on the half-plane $x > 0$, we consider the quadratic isochronous differential system of type (4) given by

$$\begin{aligned} \dot{x} &= -8.988\,89 - 17.538\,3x - 20.744\,7y - 2.216\,06x^2 \\ &\quad - 9.765\,45xy - 7.549\,39y^2, \\ \dot{y} &= 14.677\,9 + 21.205x + 25.538\,3y + 3.549\,39x^2 \\ &\quad + 12.432\,1xy + 8.882\,73y^2, \end{aligned} \quad (14)$$

with first integral

$$\begin{aligned} H_4(x,y) &= \frac{1}{16x + 0.031\,894\,3(1067y + 2091)} \big(16(1+x+y)^4 \\ &\quad - 24(1+x+y)^2(4.355\,69 + x + 2.126\,95y) \\ &\quad + 9((1+x+y)^2 + (4.355\,69 + x + 2.126\,95y)^2)\big). \end{aligned}$$

In this case, the solution to system (12) with $y < Y$ is

$$(y_1, Y_1) = (-1.898\,651\,543\,493\,539, 2.898\,651\,543\,493\,539),$$

and the corresponding limit cycle of the discontinuous piecewise differential systems (13) and (14) associated to these solutions is shown in Fig. 3.

Remark. For all these discontinuous piecewise differential systems, it is possible that the upper bound found for the maximum number of limit cycles cannot be reached. This is due to the fact that the solutions (y, Y) do not need to correspond necessarily to periodic solutions of the discontinuous piecewise differential systems.

D. Proof of Theorem 1 for systems (I)–(V)

We take again the linear differential system (I) with its first integral $H_1(x,y)$ on the half-plane $x < 0$, and on $x > 0$, we consider the quadratic isochronous differential system (6) with its first integral $H_5(x,y)$. Thus, if the discontinuous differential system (I)–(6) has a limit cycle, it must intersect the discontinuity line $x = 0$ at two different points $(0, y)$ and $(0, Y)$. These points must satisfy the equations

$$H_1(0,y) - H_1(0,Y) = (Y-y)P_1(y,Y) = 0,$$
$$H_5(0,y) - H_5(0,Y) = \frac{(Y-y)Q_5(y,Y)}{(3+8y\beta+8\gamma)^4(3+8Y\beta+8\gamma)^4} = 0, \quad (15)$$

where P_1 and Q_5 are polynomials of degrees one and five, respectively. We solve again the equation $P_1(y,Y) = 0$ obtaining the variable Y as a function of y, that is, $Y = f(y)$. If we substitute $Y = f(y)$ in equation $Q_5(y,Y) = 0$, we obtain a polynomial equation of degree 4 in the variable y, and so system (12) has at most four real solutions. Taking into account the symmetry between these solutions, as in the previous statements, there can be only two different solutions (y, Y) of (15) satisfying $y < Y$.

Finally, we show a discontinuous piecewise differential system (I)–(6) having two limit cycles. On the half-plane $x < 0$, we consider the linear differential system

$$\dot{x} = 1 + x - 2y, \quad \dot{y} = x - y, \quad (16)$$

with first integral

$$H_1(x,y) = -2y + x^2 - 2xy + 2y^2,$$

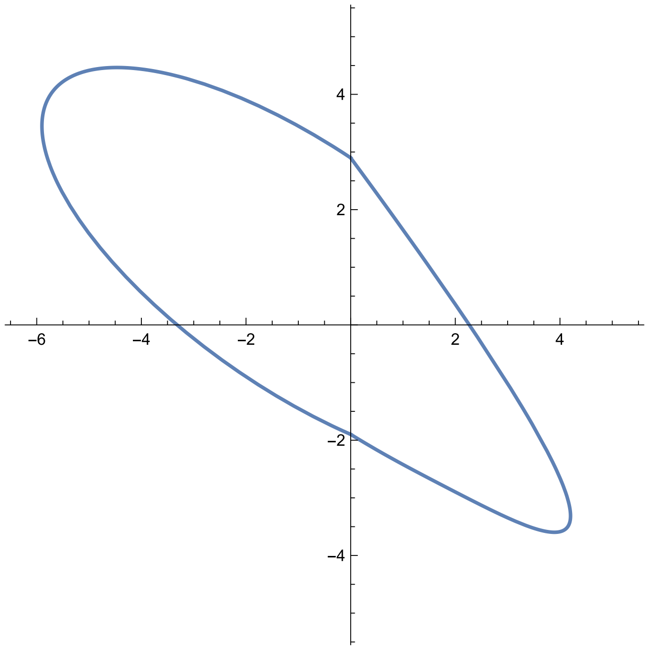

FIG. 3. The existing limit cycle for systems (13) and (14) of classes (I)–(IV). It is travelled in counter-clockwise sense.

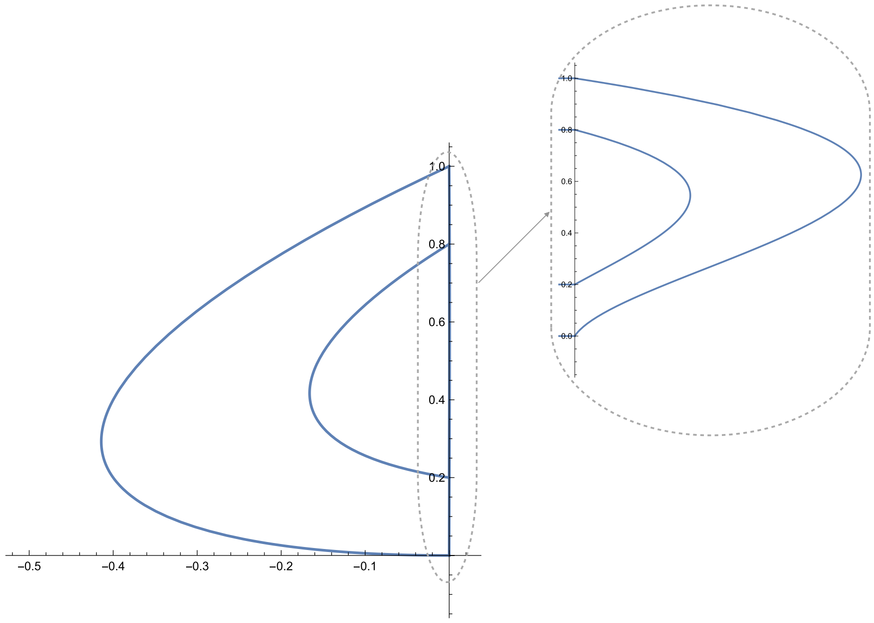

FIG. 4. The pair of limit cycles that exist for systems (16) and (17) of classes (I)–(V). They are travelled in counter-clockwise sense.

and on the half-plane $x > 0$, we consider the quadratic isochronous differential system of type (6)

$$\dot{x} = 0.001\,345\,83 + 7.888\,17x + 0.032\,69y - 5.333\,33x^2 + 5.810\,77xy - 0.073\,541\,8y^2, \tag{17}$$

$$\dot{y} = 3.825\,31 - 3.825\,31x + 5.445\,16y - 2.666\,67xy + 1.936\,92y^2,$$

with first integral

$$H_5(x,y) = \frac{1}{(8(-1y - 1.809\,49) + 3)^4}\Big(9\left((-1x + 0.726\,346y + 1)^2 + (-1y - 1.809\,49)^2\right) + 16(-1y - 1.809\,49)^4 + 24.(-1y - 1.809\,49)^3\Big).$$

For this case, the two solutions for system (15) with $y < Y$ are

$$(y_1, Y_1) = (0, 1), \quad (y_2, Y_2) = \left(\frac{1}{5}, \frac{4}{5}\right),$$

and the corresponding limit cycles of the discontinuous piecewise differential system (8)–(17) associated to this solution are shown in Fig. 4.

IV. PROOF OF THEOREM 2

A. Proof of statement (a) of Theorem 2

We consider the quadratic polynomial differential system (3) with first integral $H_2(x,y)$ in the half-plane $x < 0$. By changing the parameters $(a, \alpha, b, \beta, c, \gamma)$ to $(a_1, \alpha_1, b_1, \beta_1, c_1, \gamma_1)$ in system (3) and in its first integral, we obtain a second isochronous quadratic differential system of type (3) with the first integral $\widetilde{H}_2(x,y)$, namely,

$$\begin{aligned}\dot{x} = {} & \frac{1}{b_1\alpha_1 - a_1\beta_1}\big(\beta_1\gamma_1^2 + 2b_1\gamma_1c_1 + b_1c_1 + \beta_1\gamma_1 - \beta_1c_1^2 + (2a_1b_1\gamma_1 + 2\alpha_1\beta_1\gamma_1 \\ & + a_1b_1 + \alpha_1\beta_1 - 2a_1\beta_1c_1 + 2\alpha_1b_1c_1)x + (2\gamma_1 + 1)(b_1^2 + \beta_1^2)y + (-a_1^2\beta_1 + \alpha_1^2\beta_1 \\ & + 2\alpha_1a_1b_1)x^2 + 2\alpha_1(b_1^2 + \beta_1^2)xy + \beta_1(b_1^2 + \beta_1^2)y^2\big), \\ \dot{y} = {} & \frac{1}{b_1\alpha_1 - a_1\beta_1}\big(-\alpha_1\gamma_1^2 - 2a_1\gamma_1c_1 - a_1c_1 - \alpha_1\gamma_1 + \alpha_1c_1^2 - (2\gamma_1 + 1)(a_1^2 + \alpha_1^2)x \\ & + (-2a_1b_1\gamma_1 - 2\alpha_1\beta_1\gamma_1 - a_1b_1 - \alpha_1\beta_1 - 2a_1\beta_1c_1 + 2\alpha_1b_1c_1)y - \alpha_1(a_1^2 + \alpha_1^2)x^2 \\ & - 2\beta_1(a_1^2 + \alpha_1^2)xy - (\alpha_1\beta_1^2 + 2a_1\beta_1b_1 - \alpha_1b_1^2)y^2\big),\end{aligned} \tag{18}$$

whose first integral is

$$\widetilde{H}_2(x,y) = \frac{(c_1 + a_1x + b_1y)^2 + (x\alpha_1 + y\beta_1 + \gamma_1)^2}{1 + 2(x\alpha_1 + y\beta_1 + \gamma_1)}.$$

If a limit cycle of the discontinuous piecewise differential systems (3)–(18) has two different intersection points $(0, y)$ and $(0, Y)$ with the line $x = 0$, then they must satisfy the system

$$H_2(0,y) - H_2(0,Y) = \frac{(Y-y)P_2(y,Y)}{(2\gamma + 2\beta y + 1)(2\gamma + 2\beta Y + 1)} = 0, \tag{19}$$
$$\widetilde{H}_2(0,y) - \widetilde{H}_2(0,Y) = \frac{(Y-y)Q_2(y,Y)}{(2\gamma_1 + 2\beta_1 y + 1)(2\gamma_1 + 2\beta_1 Y + 1)} = 0,$$

where both polynomials P_2 and Q_2 are of degree two. From the equation $Q_2(y, Y) = 0$, we get Y as a function of y, that is, $Y = f(y)$, and substituting this expression in the equation $P_2(y, Y) = 0$, we obtain a quadratic polynomial equation in the variable y. Then, the maximum number of solutions of (19) is two, namely, (y_1, Y_1) and (y_2, Y_2), but in fact, these two solutions represent the same limit cycle because they are symmetric in the sense of the proof of Theorem 1. Thus system (3)–(3) has at most one limit cycle.

Now we give an example of a discontinuous piecewise differential system of type (3)–(3) having one limit cycle. On $x < 0$ we consider the quadratic isochronous differential system

$$\dot{x} = -4 - x - 6y + x^2 - 2y^2, \quad \dot{y} = 3 + 3x + 5y + 2xy + 2y^2, \tag{20}$$

with a first integral

$$H_2(x,y) = \frac{(1+y)^2 + (1+x+y)^2}{1 + 2(1+y)},$$

and on $x > 0$ we consider the quadratic isochronous differential system of type (3)

$$\dot{x} = -2 + x - 6y + x^2 - 4xy + 2y^2, \quad \dot{y} = 1 + 3x - 5y + x^2 - 2xy, \tag{21}$$

whose first integral is

$$\widetilde{H}_2(x,y) = \frac{(1+x-y)^2 + (1+y)^2}{1 + 2(1+x-y)}.$$

The solution of (19) satisfying $y < Y$ is $(y, Y) = \left(\frac{1}{2}(-1-\sqrt{3}), \frac{1}{2}(-1+\sqrt{3})\right)$, which provides the limit cycle for the discontinuous differential piecewise systems (20) and (21) shown in Fig. 5.

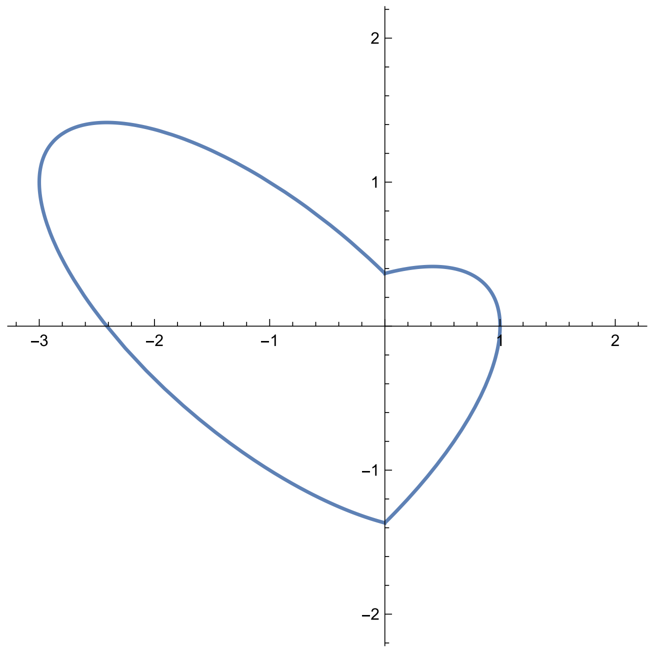

FIG. 5. The unique limit cycle that exists for systems (20) and (21) of types (II)–(II). It is travelled in counter-clockwise sense.

B. Proof of statement (b) of Theorem 2

We consider again the quadratic polynomial differential system (18) with first integral $\widetilde{H}_2(x, y)$ in the half-plane $x < 0$, and for $x > 0$, we take the isochronous differential system (4) whose first integral is $H_3(x, y)$.

If there exists a limit cycle of the discontinuous piecewise differential system (18)–(4), then it has two different intersection points $(0, y)$ and $(0, Y)$ with the line $x = 0$, which satisfy the system

$$\widetilde{H}_2(0,y) - \widetilde{H}_2(0,Y) = \frac{(Y-y)P_2(y,Y)}{(2\gamma_1 + 2\beta_1 y + 1)(2\gamma_1 + 2\beta_1 Y + 1)} = 0, \tag{22}$$
$$H_3(0,y) - H_3(0,Y) = \frac{(Y-y)Q_2(y,Y)}{(\gamma_2 + \beta_2 y + 1)^2(\gamma_2 + \beta_2 Y + 1)^2} = 0,$$

where both polynomials P_2 and Q_2 have degree two. From the equation $P_2(y, Y) = 0$, we get Y as a function of y, that is, $Y = f(y)$, and substituting this expression in the equation $Q_2(y, Y) = 0$, we obtain a polynomial equation of degree two in the variable y. Then, the maximum number of solutions of (22) is two. But due to the symmetry of the solutions, the systems (3) and (4) have at most one limit cycle.

Now we write an example of a discontinuous piecewise differential system having a unique limit cycle. On $x < 0$, we consider the quadratic isochronous differential system (20), and on $x > 0$, we consider the quadratic isochronous differential system of type (4),

$$\dot{x} = -2 - 4y - xy, \quad \dot{y} = x - 3y - y^2, \tag{23}$$

whose first integral is

$$H_3(x,y) = \frac{(x-y+1)^2 + (y+1)^2}{(x-y+2)^2}.$$

Then, the obtained solution of system (22) satisfying $y < Y$ is $(y, Y) = \left(-\frac{4}{3}, 0\right)$. This pair provides the limit cycle that exists for

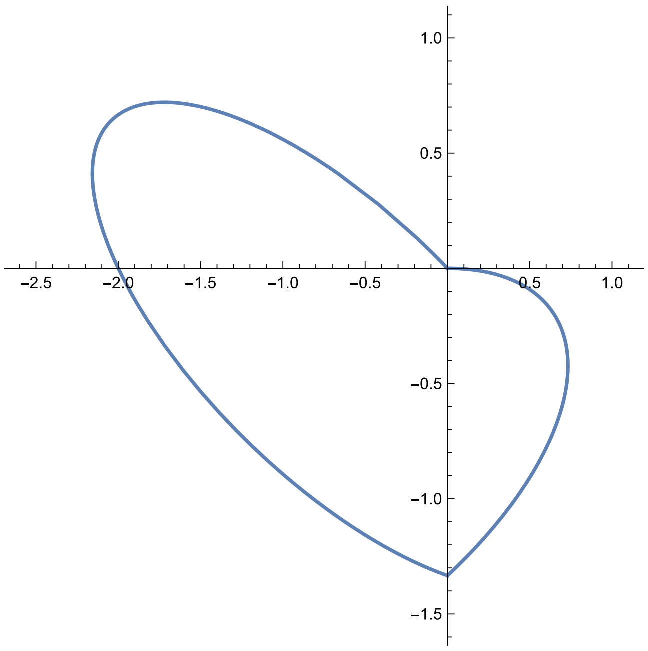

FIG. 6. The unique limit cycle that exists for systems (20)–(23) of types (II)–(III). It is travelled in counter-clockwise sense.

the discontinuous differential piecewise systems (20)–(23) shown in Fig. 6.

C. Proof of statement (c) of Theorem 2

We take the quadratic polynomial differential system (18) with the first integral $\widetilde{H}_2(x, y)$ in the half-plane $x < 0$, and for $x > 0$, we take the isochronous differential system (5) with the first integral $H_4(x, y)$.

If there exists a limit cycle of the discontinuous piecewise differential system (18)–(5), then it has two different intersection points $(0, y)$ and $(0, Y)$ with the separation line $x = 0$, satisfying the system

$$\widetilde{H}_2(0,y) - \widetilde{H}_2(0,Y) = \frac{(Y-y)P_2(y,Y)}{(2\gamma_1 + 2\beta_1 y + 1)(2\gamma_1 + 2\beta_1 Y + 1)} = 0,$$
$$H_4(0,y) - H_4(0,Y) = \frac{(Y-y)Q_4(y,Y)}{(-3+16y\beta+16\gamma)(-3+16Y\beta+16\gamma)} = 0, \tag{24}$$

where both polynomials P_2 and Q_4 are of degree two and four, respectively. From equation $P_2(y, Y) = 0$, we obtain Y as a function of y, that is, $Y = f(y)$, and if we put this expression in the second equation $Q_4(y, Y) = 0$, we obtain a polynomial equation of degree six in the variable y. Then, the maximum number of solutions of (22) is six, but due to the symmetry, there are at most three solutions of system (24) satisfying $y < Y$. Thus, systems (3)–(5) have at most three limit cycles.

Next, we give an example of discontinuous piecewise differential system of types (3)–(5) having two limit cycles. On $x < 0$, we consider the quadratic isochronous differential system

$$\begin{aligned}\dot{x} &= -0.427\,384 - 0.457\,523x - 4.273\,84y + 1.931\,7x^2 \\ &\quad + 6.838\,14xy - 0.427\,384y^2, \\ \dot{y} &= 0.116\,463 + 0.682\,96x + 1.871\,74y - 0.546\,368x^2 \\ &\quad + 0.136\,592xy + 3.652y^2,\end{aligned} \tag{25}$$

with the first integral

$$\widetilde{H}_2(x,y) = \frac{9 + 128x^2 + 9y(18+89y) - 16x(1 - 2\sqrt{2} + y - 20\sqrt{2}y)}{16(5 - 8x + y)},$$

and on $x > 0$, we consider the quadratic isochronous differential system of type (5)

$$\begin{aligned}\dot{x} &= -0.593\,985 - 6.893\,77x - 6.770\,32y + 3.034\,14x^2 + 0.667\,964xy - 1.899\,16y^2, \\ \dot{y} &= 4.772\,86 + 7.534\,17x + 9.754\,81y - 1.871\,38x^2 + 1.931\,71xy + 3.301\,43y^2,\end{aligned} \tag{26}$$

whose first integral is

$$\begin{aligned}H_4(x,y) = \frac{1}{54.173 - 16x + 19.657\,7y}\big(&16(-0.357\,63 - x - 0.908\,852y)^4 \\ &- 24(-0.357\,63 - x - 0.908\,852y)^2(3.573\,31 - x + 1.228\,61y) \\ &+ 9((-0.357\,63 - x - 0.908\,852y)^2 + (3.573\,31 - x + 1.228\,61y)^2)\big).\end{aligned} \tag{27}$$

The solutions to (24) satisfying $y < Y$ are

$$(y_1, Y_1) = (-1, 1), \quad (y_2, Y_2) = (-2, 3),$$

which provide the two limit cycles for the discontinuous differential piecewise systems (25) and (26) shown in Fig. 7.

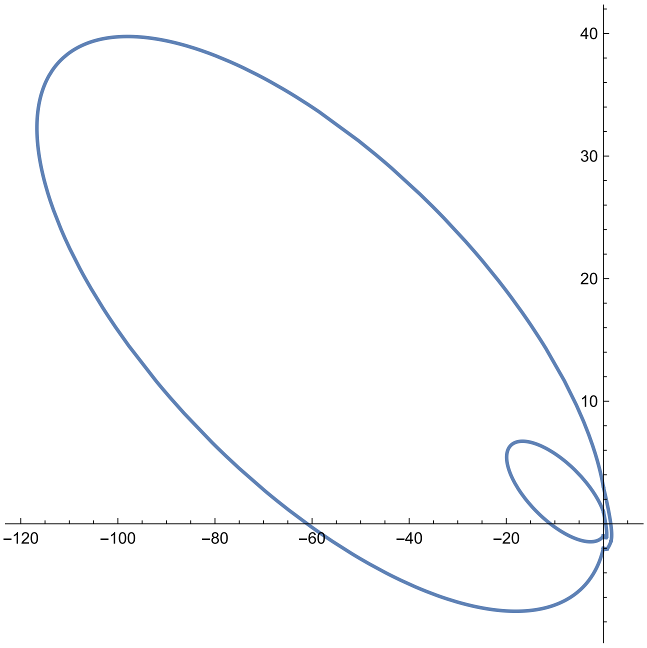

FIG. 7. The two limit cycles that exist for systems (25) and (26) of types (II)–(IV). They are travelled in counter-clockwise sense.

D. Proof of statement (d) of Theorem 2

We consider the quadratic polynomial differential system (18) with the first integral $\widetilde{H}_2(x,y)$ in the half-plane $x<0$, and for $x>0$, we take the isochronous differential system (6) whose first integral is $H_5(x,y)$.

If a limit cycle exists for the discontinuous piecewise differential system (18)–(6), then it has two different intersection points $(0,y)$ and $(0,Y)$ with the line $x=0$, satisfying the closing equations

$$\widetilde{H}_2(0,y)-\widetilde{H}_2(0,Y)=\frac{(Y-y)P_2(y,Y)}{(2\gamma_1+2\beta_1 y+1)(2\gamma_1+2\beta_1 Y+1)}=0, \quad (28)$$
$$H_5(0,y)-H_5(0,Y)=\frac{(Y-y)Q_5(y,Y)}{(8\gamma+8\beta y+3)(8\gamma+8\beta Y+3)}=0,$$

where both polynomials P_2 and Q_5 are of degree two and five, respectively. From the equation $P_2(y,Y)=0$, we get Y as a function of y, that is, $Y=f(y)$, and substituting this expression in the equation $Q_5(y,Y)=0$, we obtain an equation of order six in the variable y. Then, the maximum number of solutions of (28) is six, but because of the symmetry property as in the previous statements, systems (3)–(6) have three limit cycles at most. Next, we give an example of discontinuous piecewise differential system of types (3)–(6) having two limit cycles. On $x<0$, we consider the quadratic isochronous differential system

$$\begin{aligned}\dot{x}&=-4.243\,71+0.170\,949x+6.253\,88y+3.657\,48x^2-12.507\,8xy+8.040\,71y^2,\\ \dot{y}&=-2.200\,72-1.289\,15x+5.485\,91y+1.289\,15x^2-3.314\,96xy+0.579\,633y^2,\end{aligned} \quad (29)$$

with the first integral

$$\widetilde{H}_2(x,y)=\frac{14(1.5+x^2+x(0.414\,214-4.122\,84y)+y(-2.726\,59+4.851\,17y))}{-7+14x-18y},$$

and on $x>0$, we consider the quadratic isochronous differential system of type (6)

$$\begin{aligned}\dot{x}&=0.550\,125+9.009\,84x-1.248\,12y+6.059\,59x^2-0.600\,689xy-0.034\,14y^2,\\ \dot{y}&=3.905\,88+49.776\,5x-6.635\,34y+17.347\,6x^2+1.214\,15xy-0.491\,153y^2,\end{aligned} \quad (30)$$

whose first integral is

$$\begin{aligned}H_5(x,y)=&\frac{0.000\,244\,141}{(-0.450\,206+x-0.195\,584y)^4)}\big(9((-0.825\,206+x-0.195\,584y)^2\\ &+(0.178\,088+x-0.118\,725y)^2)+24(-0.825\,206+x-0.195\,584y)^3+16(-0.825\,206+x-0.195\,584y)^4\big).\end{aligned} \quad (31)$$

The solutions to (28) satisfying $y<Y$ that provide the two limit cycles for the discontinuous differential piecewise systems (29) and (30) are

$$(y_1,Y_1)=(0.1.,1.),\quad (y_2,Y_2)=(0.166\,666,0.833\,335),$$

as shown in Fig. 8.

V. PROOF OF THEOREM 3

A. Proof of statement (a) of Theorem 3

We consider the quadratic polynomial differential system (4) with first integral $H_3(x,y)$ in the half-plane $x<0$. By changing the parameters $(a,\alpha,b,\beta,c,\gamma)$ to $(a_1,\alpha_1,b_1,\beta_1,c_1,\gamma_1)$ in system (4) and in its first integral, we obtain a second isochronous quadratic

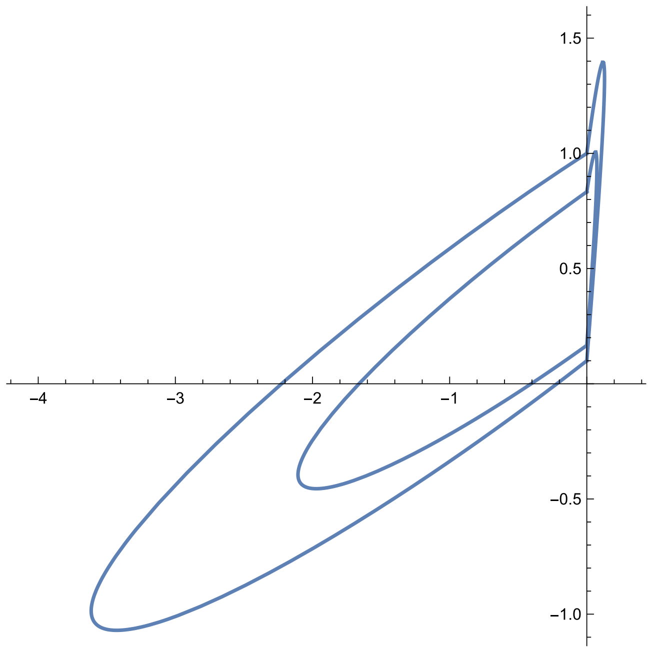

FIG. 8. The two limit cycles existing for system (29) and (30) of types (II)–(V). They are travelled in counter-clockwise sense.

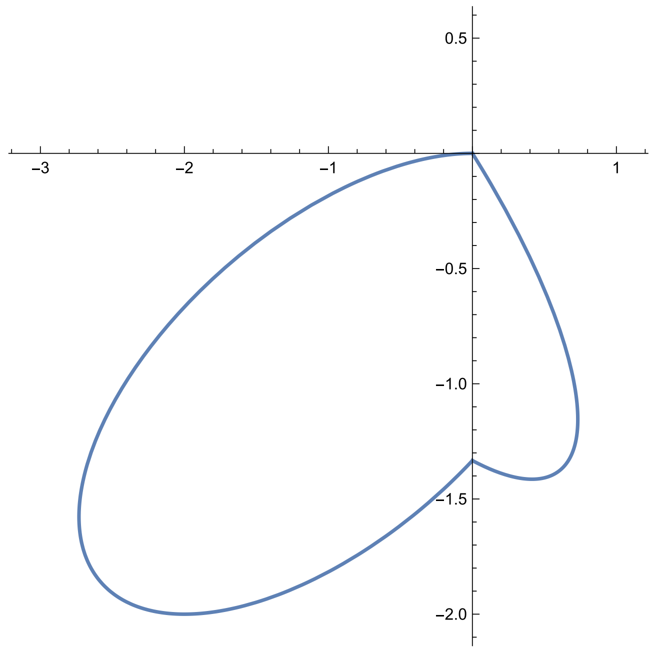

FIG. 9. The unique limit cycle that exists for systems (23)–(34) of types (III) and (III). It is travelled in counter-clockwise sense.

differential system of type (4) with the first integral $\widetilde{H}_3(x,y)$, namely,

$$\dot{x} = \frac{1}{b_1\alpha_1 - a_1\beta_1}\left(b_1^2 y(1 + x\alpha_1 + \gamma_1) + b_1(c_1 + a_1 x)(1 + x\alpha_1 - y\beta_1 + \gamma_1) + \beta_1(-c_1^2 - 2a_1c_1x - a_1^2x^2 + x\alpha_1 + y\beta_1 + \gamma_1)\right), \tag{32}$$

$$\dot{y} = \frac{1}{b_1\alpha_1 - a_1\beta_1}\left(\alpha_1(c_1^2 + 2b_1c_1y + b_1^2y^2 - x\alpha_1 - y\beta_1 - \gamma_1) - a_1^2x(1 + y\beta_1 + \gamma_1) - a_1(c_1 + b_1y)(1 - x\alpha_1 + y\beta_1 + \gamma_1)\right),$$

whose first integral is

$$\widetilde{H}_3(x,y) = \frac{(c_1 + a_1x + b_1y)^2 + (x\alpha_1 + y\beta_1 + \gamma_1)^2}{(1 + x\alpha_1 + y\beta_1 + \gamma_1)^2}.$$

If a limit cycle of the discontinuous piecewise differential systems (4)–(32) has two different intersection points $(0,y)$ and $(0,Y)$ with the line $x = 0$, then they must satisfy the system

$$H_3(0,y) - H_3(0,Y) = \frac{(Y-y)P_2(y,Y)}{(1 + y\beta + \gamma)^2(1 + Y\beta + \gamma)^2} = 0,$$
$$\widetilde{H}_3(0,y) - \widetilde{H}_3(0,Y) = \frac{(Y-y)Q_2(y,Y)}{(1 + y\beta_1 + \gamma_1)^2(1 + Y\beta_1 + \gamma_1)^2} = 0, \tag{33}$$

where both polynomials P_2 and Q_2 are of degree two. From equation $Q_2(y,Y) = 0$, we get Y as a function of y, that is, $Y = f(y)$, and after substituting this expression in the equation $P_2(y,Y) = 0$, we obtain a quadratic polynomial equation in the variable y. Then, the maximum number of solutions of (33) is two, namely, (y_1, Y_1) and (y_2, Y_2); but in fact, these two solutions represent the same limit cycle because they are symmetric as usual. Thus, systems (4)–(4) have at most one limit cycle.

Now we give an example of a discontinuous piecewise differential system of type (4)–(4) having one limit cycle. On $x < 0$, we consider the quadratic isochronous differential system (23), and on $x > 0$, we take the quadratic isochronous differential system of type (4)

$$\dot{x} = -2 - 2y + x^2 + xy, \quad \dot{y} = 2 + 2x + 3y + xy + y^2, \tag{34}$$

with a first integral

$$H_3(x,y) = \frac{(1+y)^2 + (1+x+y)^2}{(2+y)^2}.$$

The solution to (33) satisfying $y < Y$ is $(y,Y) = \left(-\frac{4}{3}, 0\right)$, which provides the limit cycle for the discontinuous differential piecewise systems (23)–(34) shown in Fig. 9.

B. Proof of statement (b) of Theorem 3

We consider again the quadratic polynomial differential system (32) with first integral $\widetilde{H}_3(x,y)$ in the half-plane $x < 0$, and for $x > 0$, we take the isochronous differential system (5) whose first integral is $H_4(x,y)$.

If there exists a limit cycle of the discontinuous piecewise differential systems (32)–(5), then it has two different intersection points

$(0, y)$ and $(0, Y)$ with line $x = 0$, which satisfy the systems

$$\widetilde{H}_3(0,y) - \widetilde{H}_3(0,Y) = \frac{(Y-y)P_2(y,Y)}{(1+y\beta_1+\gamma_1)^2(1+Y\beta_1+\gamma_1)^2)} = 0, \quad (35)$$
$$H_4(0,y) - H_4(0,Y) = \frac{(Y-y)Q_4(y,Y)}{(-3+16y\beta+16\gamma)(-3+16Y\beta+16\gamma)} = 0,$$

where the polynomials P_2 and Q_4 have degree two and four, respectively. From equation $P_2(y, Y) = 0$, we get Y as a function of y, that is, $Y = f(y)$, and substituting this expression in equation $Q_4(y, Y) = 0$, we obtain a polynomial equation of degree six in the variable y. Then, the maximum number of solutions of (35) is six. But due to the symmetry of the solutions, the systems (4) and (5) have at most three limit cycles.

Now, we show an example of discontinuous piecewise differential system of types (4) and (5) having two limit cycles. On $x < 0$, we consider the quadratic isochronous differential system

$$\begin{aligned}\dot{x} &= -5.925\,92 - 47.912\,7x - 58.660\,6y + x^2 - 0.101\,841xy,\\ \dot{y} &= -246.99 + 9.522\,4x - 26.911y + xy - 0.101\,841y^2,\end{aligned} \quad (36)$$

with the first integral

$$\widetilde{H}_3(x,y) = \frac{(24.941\,2 - x + 0.101\,841y)^2 + (602.838 - x + 61y)^2}{(603.838 - x + 61y)^2},$$

and on $x > 0$, we consider the quadratic isochronous differential system (26) of type (5) whose first integral is (27). The solutions to (35) satisfying $y < Y$ that provide the two limit cycles for discontinuous differential piecewise system (36)–(26) are

$$(y_1, Y_1) = (-1, 1), \quad (y_2, Y_2) = (-2, 3),$$

as shown in Fig. 10.

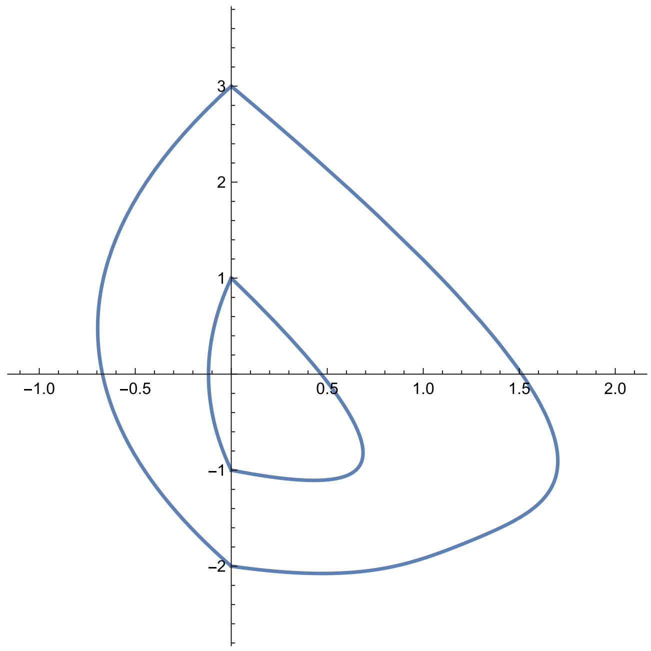

FIG. 10. The pair of limit cycles that exists for systems (36)–(26) of types (III)–(IV). They are travelled in counter-clockwise sense.

C. Proof of statement (c) of Theorem 3

We take the quadratic polynomial differential system (32) with the first integral $\widetilde{H}_3(x, y)$ in the half-plane $x < 0$, and for $x > 0$, we take the isochronous differential system (6) with the first integral $H_5(x, y)$.

If there exists a limit cycle of the discontinuous piecewise differential systems (32)–(6), then it has two different intersection points $(0, y)$ and $(0, Y)$ with the separation line $x = 0$, satisfying the system

$$\widetilde{H}_3(0,y) - \widetilde{H}_3(0,Y) = \frac{(Y-y)P_2(y,Y)}{(1+y\beta_1+\gamma_1)^2(1+Y\beta_1+\gamma_1)^2)} = 0, \quad (37)$$
$$H_5(0,y) - H_5(0,Y) = \frac{(Y-y)Q_5(y,Y)}{(3+8y\beta+8\gamma)^4(3+8Y\beta+8\gamma)^4} = 0,$$

where the polynomials P_2 and Q_5 are of degrees two and five, respectively. From equation $P_2(y, Y) = 0$, we obtain Y as a function of y, that is, $Y = f(y)$, and putting this expression in the second equation $Q_5(y, Y) = 0$, we obtain a polynomial equation of degree six in the variable y. Then, the maximum number of solutions to (37) is six; but due to the symmetry, there are at most three solutions of system (37) satisfying $y < Y$. Thus, systems (4)–(6) have at most three limit cycles.

Next, we give an example of discontinuous piecewise differential system of types (4)–(6) having two limit cycles. On $x < 0$, we consider the quadratic isochronous differential system

$$\begin{aligned}\dot{x} &= 6781.71 + 363.957x - 15\,585.4y + x^2 - 45.267\,4xy,\\ \dot{y} &= 140.782 + 7.249\,45x - 308.767y + xy - 45.267\,4y^2,\end{aligned} \quad (38)$$

with the first integral

$$\widetilde{H}_3(x,y) = \frac{(-70.432\,8 + x - 57.244\,8y)^2 + (18.396\,7 + x - 45.267\,4y)^2}{(-69.432\,8 + x - 57.244\,8y)^2},$$

and on $x > 0$, we consider the quadratic isochronous differential system (30) of type (6) whose first integral is (31). The solutions to (37) satisfying $y < Y$ that provide the two limit cycles for the discontinuous differential piecewise systems (38)–(30) are

$$(y_1, Y_1) = (0.1, 1), \quad (y_2, Y_2) = (0.166\,666, 0.833\,335),$$

as shown in Fig. 11.

VI. PROOF OF THEOREM 4

A. Proof of Theorem 4 for systems (IV)–(IV)

We consider the quadratic polynomial differential system (5) with the first integral $H_4(x, y)$ in the half-plane $x < 0$. By changing the parameters $(a, \alpha, b, \beta, c, \gamma)$ to $(a_1, \alpha_1, b_1, \beta_1, c_1, \gamma_1)$ in system (5)

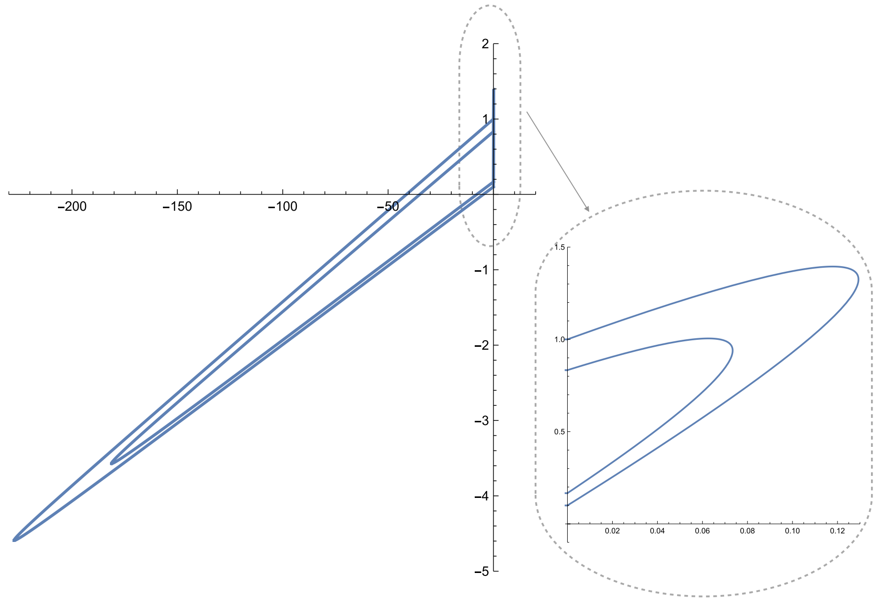

FIG. 11. The two limit cycles existing for systems (38)–(30) of types (III)–(V). They are travelled in counter-clockwise sense.

and its first integral, we obtain a second isochronous quadratic differential system of type (5) with the first integral $\widetilde{H}_4(x,y)$, namely,

$$\begin{aligned}\dot{x} &= \frac{1}{3a_1\beta_1 - 3b_1\alpha_1}\big(b_1(c_1+a_1x)(-3+16x\alpha_1+8y\beta_1+16\gamma_1)+b_1^2y(-3+16x\alpha_1+12y\beta_1+16\gamma_1)\\&\quad -\beta_1(4c_1^2+8a_1c_1x+4a_1^2x^2+3(x\alpha_1+y\beta_1+\gamma_1)\big),\\ \dot{y} &= \frac{1}{3b_1\alpha_1-3a_1\beta_1}a_1(c_1+b_1y)\big(-3+8x\alpha_1+16y\beta_1+16\gamma_1)+a_1^2x(-3+12x\alpha_1+16y\beta_1+16\gamma_1)\\&\quad -\alpha_1(4c_1^2+8b_1c_1y+4b_1^2y^2+3(x\alpha_1+y\beta_1+\gamma_1)\big),\end{aligned} \tag{39}$$

whose first integral is

$$\begin{aligned}\widetilde{H}_4(x,y) &= \frac{1}{16(\gamma_1+\alpha_1x+\beta_1y)-3}\big(-24(a_1x+b_1y+c_1)^2(\gamma_1+\alpha_1x+\beta_1y)\\&\quad +9\left((a_1x+b_1y+c_1)^2+(\gamma_1+\alpha_1x+\beta_1y)^2\right)+16(a_1x+b_1y+c_1)^4\big).\end{aligned}$$

If there exists a limit cycle of the discontinuous piecewise differential systems (5)–(39), then it has two different intersection points $(0,y)$ and $(0,Y)$ with separation line $x=0$, satisfying the system

$$\begin{aligned}\widetilde{H}_4(0,y)-\widetilde{H}_4(0,Y) &= \frac{(Y-y)P_4(y,Y)}{(-3+16y\beta_1+16\gamma_1)(-3+16Y\beta_1+16\gamma_1)}=0,\\ H_4(0,y)-H_4(0,Y) &= \frac{(Y-y)Q_4(y,Y)}{(-3+16y\beta+16\gamma)(-3+16Y\beta+16\gamma)}=0,\end{aligned} \tag{40}$$

where both polynomials P_4 and Q_4 are of degree four. From equation $P_4(y,Y)=0$, we obtain Y as a function of y, that is, $Y=f(y)$, and substituting it in the second equation $Q_4(y,Y)=0$, we obtain a polynomial equation of degree six in the variable y. Then, the maximum number of solutions to (40) is six; but due to symmetry, there are at most three solutions to system (40) satisfying $y<Y$. Thus, systems (5)–(5) has at most three crossing limit cycles.

Next, we give an example of discontinuous piecewise differential system of types (5)–(5) having two limit cycles. On $x<0$, we consider the quadratic isochronous differential system

$$\begin{aligned}\dot{x} &= -0.593\,985+6.893\,77x-6.770\,32y+3.034\,14x^2-0.667\,964xy-1.899\,16y^2,\\ \dot{y} &= -4.772\,86+7.534\,17x-9.754\,81y+1.871\,38x^2+1.931\,71xy-3.301\,43y^2,\end{aligned} \tag{41}$$

with the first integral

$$\widetilde{H}_4(x,y) = \frac{1}{-3+16(3.573\,31+x+1.228\,61y)}\big(16(-0.357\,63+x-0.908\,852y)^4$$
$$-24(-0.357\,63+x-0.908\,852y)^2(3.573\,31+x+1.228\,61y)$$
$$+9((-0.357\,63+x-0.908\,852y)^2+(3.573\,31+x+1.228\,61y)^2)\big), \qquad (42)$$

and on $x > 0$, we consider the quadratic isochronous differential system

$$\dot{x} = -4.352\,74+17.666x-44.116\,7y+1.652\,12x^2-2.115\,58xy-4.001\,74y^2,$$
$$\dot{y} = 279.554+16.398\,2x+89.623\,6y-0.305\,758x^2+4.695\,75xy+5.228\,3y^2, \qquad (43)$$

with the first integral

$$H_4(x,y) = \frac{1}{-3+16(47.304-x+12.039\,6y)}\big(16(-13.411\,2-x-1.042\,63y)^4$$
$$-24(-13.411\,2-x-1.042\,63y)^2(47.304-x+12.039\,6y)$$
$$+9((-13.411\,2-x-1.042\,63y)^2+(47.304-x+12.039\,6y)^2)\big).$$

The solutions to (40) satisfying $y < Y$ that provide the two limit cycles for the discontinuous differential piecewise system (41)–(43) are

$$(y_1, Y_1) = (-1, 1), \quad (y_2, Y_2) = (-2, 3),$$

as shown in Fig. 12.

B. Proof Theorem 4 for systems (IV)–(V)

We take the quadratic polynomial differential system (39) with the first integral $\widetilde{H}_4(x,y)$ in the half-plane $x < 0$, and for $x > 0$, we take the isochronous differential system (6) with the first integral $H_5(x,y)$.

If there exists a limit cycle of the discontinuous piecewise differential systems (39)–(6), then there exist two different intersection points $(0,y)$ and $(0,Y)$ with the separation line $x = 0$, satisfying the system

$$\widetilde{H}_4(0,y) - \widetilde{H}_4(0,Y) = \frac{(Y-y)P_4(y,Y)}{(-3+16y\beta_1+16\gamma_1)(-3+16Y\beta_1+16\gamma_1)} = 0,$$
$$H_5(0,y) - H_5(0,Y) = \frac{(Y-y)Q_5(y,Y)}{(3+8y\beta+8\gamma)^4(3+8Y\beta+8\gamma)^4} = 0, \qquad (44)$$

where the polynomials P_4 and Q_5 are of degrees four and five, respectively. Again, we obtain Y as a function of y, that is, $Y = f(y)$, from the equation $P_4(y,Y) = 0$, and putting this expression in the second equation $Q_5(y,Y) = 0$, we obtain a polynomial equation of degree seven in the variable y. Then, the maximum number of solutions of (44) is seven; but due to symmetry, there are at most three solutions to system (44) that satisfy the condition $y < Y$. Then, systems (5)–(6) have at most three limit cycles.

Now, we show an example of discontinuous piecewise differential system of types (5)–(6) having two limit cycles. On $x < 0$, we consider again the quadratic isochronous differential system (41) whose first integral is (42), and on $x > 0$, we consider the quadratic isochronous differential system

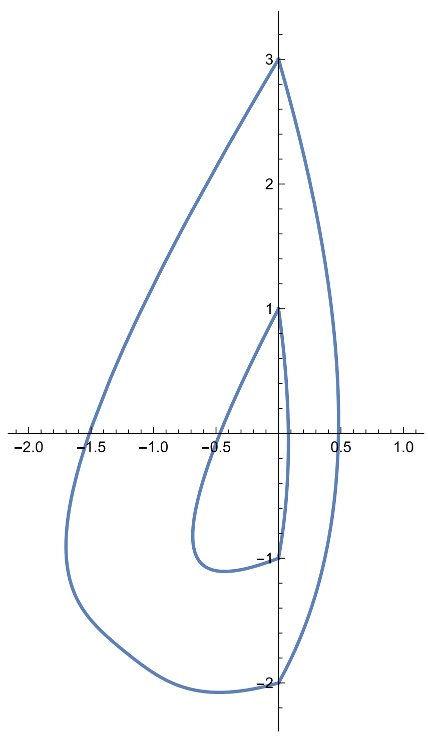

FIG. 12. The pair of limit cycles that exists for systems (41)–(43) of types (IV)–(IV). They are travelled in counter-clockwise sense.

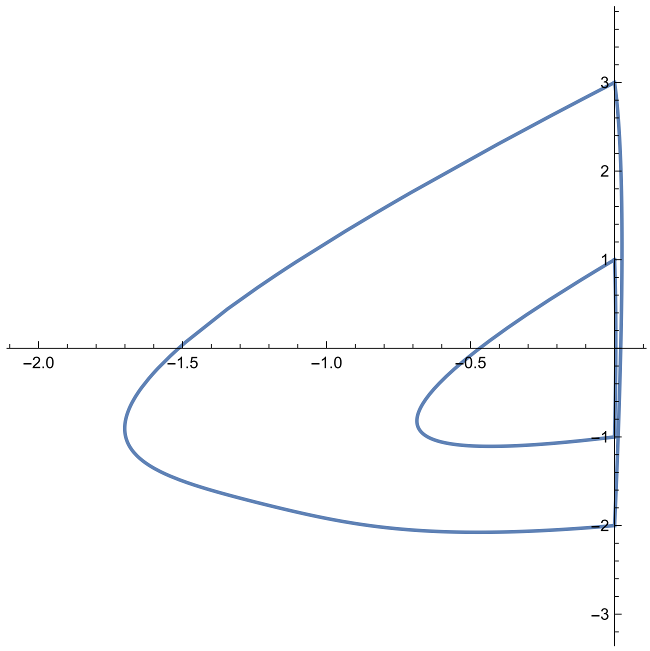

FIG. 13. The two limit cycles existing for systems (41)–(45) of types (IV) and (V). They are travelled in counter-clockwise sense.

$$\begin{aligned}\dot{x} &= -0.013\,519\,9 + 6.448\,12x - 0.134\,872y + 4.000\,57x^2 + 0.155\,285xy - 0.004\,530\,93y^2,\\ \dot{y} &= 15.137\,5 + 132.575x - 1.281\,87y + 22.887\,2x^2 + 5.332\,2xy - 0.077\,808\,3y^2,\end{aligned} \tag{45}$$

with the first integral

$$\begin{aligned}H_5(x,y) = &\frac{1}{(3+8(-1.241\,93+x-0.058\,281\,6y))^4}\big(9((-1.241\,93+x-0.058\,281\,6y)^2\\ &+(0.387\,469+x-0.000\,024\,863\,2y)^2)+24(-1.241\,93+x-0.058\,281\,6y)^3+16(-1.241\,93+x-0.058\,281\,6y)^4\big).\end{aligned}$$

The solutions to (44) satisfying $y < Y$ that provide the two limit cycles for the discontinuous differential piecewise systems (41)–(45) are

$$(y_1, Y_1) = (-1, 1), \quad (y_2, Y_2) = (-2, 3),$$

as shown in Fig. 13.

VII. PROOF OF THEOREM 5

We consider the quadratic polynomial differential system (6) with the first integral $H_5(x, y)$ in the half-plane $x < 0$. If we change the parameters $(a, \alpha, b, \beta, c, \gamma)$ to $(a_1, \alpha_1, b_1, \beta_1, c_1, \gamma_1)$ in system (6) and its first integral, we obtain a second isochronous quadratic differential system of type (6) with the first integral $\widetilde{H}_5(x, y)$, namely,

$$\begin{aligned}\dot{x} = &\frac{1}{3(\alpha_1 b_1 - a_1\beta_1)}\big(4\beta_1\gamma_1^2 + 8b_1\gamma_1c_1 + 3b_1c_1 + 3\beta_1\gamma_1 - 16\beta_1c_1^2 + (8a_1b_1\gamma_1 + 8\alpha_1\beta_1\gamma_1 + 3a_1b_1 + 3\alpha_1\beta_1 - 32a_1\beta_1c_1 + 8\alpha_1b_1c_1)x\\ &+ (8b_1^2\gamma_1 + 8\beta_1^2\gamma_1 + 3b_1^2 + 3\beta_1^2 - 24\beta_1b_1c_1)y + 4(\alpha_1^2\beta_1 - 4a_1^2\beta_1 + 2\alpha_1a_1b_1)x^2 + 8(\alpha_1\beta_1^2 - 3a_1\beta_1b_1 + \alpha_1b_1^2)xy - 4\beta_1y^2(2b_1^2 - \beta_1^2)y^2\big),\\ \dot{y} = &\frac{1}{3(\alpha_1 b_1 - a_1\beta_1)}\big(16\alpha_1c_1^2 - 4\alpha_1\gamma_1^2 - 8a_1\gamma_1c_1 - 3a_1c_1 - 3\alpha_1\gamma_1 - (8a_1^2\gamma_1 + 8\alpha_1^2\gamma_1 + 3a_1^2 + 3\alpha_1^2 - 24\alpha_1a_1c_1)x\\ &- (8a_1b_1\gamma_1 + 8\alpha_1\beta_1\gamma_1 + 3a_1b_1 + 3\alpha_1\beta_1 + 8a_1\beta_1c_1 - 32\alpha_1b_1c_1)y + 4\alpha_1(2a_1^2 - \alpha_1^2)x^2 + 8(a_1^2(-\beta_1) - \alpha_1^2\beta_1 + 3\alpha_1a_1b_1)xy\\ &- 4(\alpha_1\beta_1^2 + 2a_1\beta_1b_1 - 4\alpha_1b_1^2)y^2\big),\end{aligned} \tag{46}$$

whose first integral is

$$H_5(x,y) = \frac{1}{(8(\gamma_1+\alpha_1 x+\beta_1 y)+3)^4}\big(9\left((a_1x+b_1y+c_1)^2 + (\gamma_1+\alpha_1x+\beta_1y)^2\right) + 16(\gamma_1+\alpha_1x+\beta_1y)^4 + 24(\gamma_1+\alpha_1x+\beta_1y)^3\big).$$

If there exists a limit cycle of the discontinuous piecewise differential systems (6)–(46), then it has two different intersection points $(0,y)$ and $(0,Y)$ with the separation line $x=0$, satisfying the system

$$\widetilde{H}_5(0,y) - \widetilde{H}_5(0,Y) = \frac{(Y-y)P_5(y,Y)}{(3+8y\beta_1+8\gamma_1)^4(3+8Y\beta_1+8\gamma_1)^4} = 0,$$
$$H_5(0,y) - H_5(0,Y) = \frac{(Y-y)Q_5(y,Y)}{(3+8y\beta+8\gamma)^4(3+8Y\beta+8\gamma)^4} = 0, \tag{47}$$

where both polynomials P_5 and Q_5 are of degree five. Then, by Bézout theorem, the maximum number of solutions $(y,Y)\in\mathbb{C}$ of Eq. (47) is 25. Since our solutions appear in pairs because of symmetry, the maximum number of solutions satisfying $y<Y$ is 12.

Next, we give an example of discontinuous piecewise differential system of types (6)–(46) having two limit cycles.

On $x<0$, we consider the quadratic isochronous differential system (48)

$$\begin{aligned}\dot{x} &= 0.103\,814 + 0.025\,571\,7x - 0.097\,864\,9y - 0.036\,889\,7x^2 + 0.218\,646xy - 0.322\,863y^2,\\ \dot{y} &= 0.030\,51 + 0.017\,037\,6x - 0.057\,329\,8y - 0.015\,380\,9x^2 + 0.090\,517\,3xy - 0.132\,811y^2,\end{aligned} \tag{48}$$

with the first integral

$$\widetilde{H}_5(x,y) = \frac{1}{(3+8(-4.153\,32+x-2.616\,06y))^4}\big(9((-3.794\,77+2x-5.613y)^2 + (-4.153\,32+x-2.616\,06y)^2) + 24(-4.153\,32+x-2.616\,06y)^3 + 16(-4.153\,32+x-2.616\,06y)^4\big),$$

and on $x>0$, we consider the quadratic isochronous differential system

$$\begin{aligned}\dot{x} &= 0.000\,069\,679\,2 + 0.001\,141\,19x - 0.000\,158\,087y + 0.000\,767\,513x^2 - 0.000\,076\,083\,7xy - 0.000\,004\,324\,2y^2,\\ \dot{y} &= 0.000\,494\,721 + 0.006\,304\,72x - 0.000\,840\,437y + 0.002\,197\,27x^2 + 0.000\,153\,785xy - 0.000\,062\,209\,8y^2,\end{aligned} \tag{49}$$

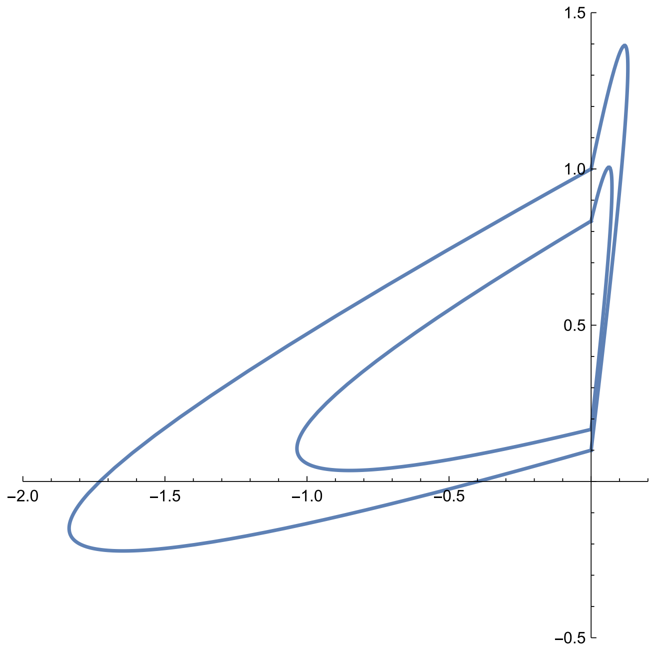

FIG. 14. The two limit cycles existing for systems (48) and (49) of types (V) and (V). They are travelled in counter-clockwise sense.

with the first integral

$$H_5(x,y) = \frac{1}{(3+8(-0.825\,206+x-0.195\,584y))^4} \times \big(9((-0.825\,206+x-0.195\,584y)^2 + (0.178\,088+x-0.118\,725y)^2) + 24(-0.825\,206+x-0.195\,584y)^3 + 16(-0.825\,206+x-0.195\,584y)^4\big).$$

The solutions to (47) satisfying $y<Y$ that provide the two limit cycles for the discontinuous differential piecewise systems (48) and (49) are

$$(y_1,Y_1) = (0.1,1),\quad (y_2,Y_2) = (0.166\,666, 0.833\,335),$$

as shown in Fig. 14.

Remark. Equation (47) can have at most 12 solutions using the Bézout theorem, but numerical evidences obtained given arbitrary values to the parameters of the class of discontinuous piecewise differential systems of types (V) and (V), we could only find at most eight real solutions, but these solutions in general do not provide crossing limit cycles. In all the particular piecewise differential systems studied numerically, we could only find at most two crossing limit cycles.

VIII. CONCLUSIONS

There is a unique family of linear isochronous centers and four families of quadratic isochronous centers. Considering all the possibilities of choosing two pairs of these isochronous centers, eventually repeated, we obtained 15 pairs. Therefore, we have 15 classes of discontinuous piecewise differential systems formed by two differential systems separated by a straight line when these two differential systems are linear isochronous centers or quadratic isochronous centers.

For these 15 classes of discontinuous piecewise differential systems, we provide an upper bound for the maximum number of limit cycles that they can exhibit, i.e., for these classes of differential systems, we have solved the 16th Hilbert problem. Moreover, for 7 of these 15 classes of discontinuous piecewise differential systems, the upper bound on the maximum number obtained is reached.

More precisely, it was known that discontinuous piecewise differential systems formed by two linear isochronous centers separated by a straight line cannot have limit cycles, see Ref. 27. If one of the systems is a linear isochronous center and the other is a quadratic isochronous center, then their maximum number of limit cycles is studied in Theorem 1. While if the two systems are quadratic isochronous centers, then their maximum number of limit cycles is studied in Theorems 2, 3, 4, and 5.

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DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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