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Single-facility huff location problems on networks

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Abstract Huff location problems have been extensively analyzed within the field of competitive continuous location.

In this work, two Huff location models on networks are addressed, by considering that users go directly to the facility or they visit the facility in their way to a destination.

Since the problems are multimodal, a branch and bound algorithm is proposed, in which two different bounding strategies, based on Interval Analysis and DC optimization, are used and compared. Computational results are given for the two bounding procedures, showing that problems of rather realistic size can be solved in reasonable time.

Keywords Huff location models · Location on networks · DC optimization · Interval analysis

1 Introduction

In this paper we address two competitive location models (Blanquero and Carrizosa 2009a; Drezner and Drezner 2004; Fernández et al. 2007; Huff 1964, 1966; Plastria 2001) on a

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network (Dearing and Shier 1983; Labbé et al. 1995). Let $N=(V,E)$ be a network, with node set V and edge set E. The length of the edge $e\in E$ is given, and it is denoted by l_e. The distance between two nodes a_i, a_j is calculated as the shortest path (Labbé et al. 1995) from a_i to a_j. For each $e\in E$, with end nodes a_i, a_j, we identify each $x\in[0,l_e]$ with the point in the edge e at distance x from a_i and distance l_e-x from a_j. This way, we obtain that, for any vertex $a_k\in V$, the distance $d(x,a_k)$ from x to a_k is, as a function of x, a concave piecewise linear function, given by:

$$\begin{aligned} &d(x,\ a_k)=\min\{f_{i_{a_k}}(x),\ f_{j_{a_k}}(x)\} \\ &f_{i_{a_k}}(x)=x+d(a_i,a_k) \qquad f_{j_{a_k}}(x)=(l_e-x)+d(a_j,a_k) \end{aligned} \tag{1}$$

We refer the reader to Labbé et al. (1995) for a comprehensive introduction to location models on networks.

The remainder of this paper is structured as follows. Two different location models on networks are considered. The first model (Sect. 2) is the classic Huff location model, as addressed in Berman et al. (2011), in which customers perceive the facility attractiveness in terms of their distance to the facility, while the second model (Sect. 3) is new, called the Huff origin-destination (OD) trip model. In the OD trip model the facility attraction is a function of the length of the shortest path from the origin to the destination through the facility. In Sect. 4 a branch and bound algorithm is designed to solve both problems and two different procedures to obtain bounds are presented.

Computational results are given in Sect. 5, comparing the two bounding strategies implemented. Some concluding remarks are presented in Sect. 6.

2 Huff location model

In this model, the finite set V of vertices of the network represents users, asking for a certain service. Each user $a\in V$ has demand $\omega_a\geq 0$. Such demand is being patronized by different existing facilities, located at points $x_1,\dots,x_r$ on the network, so that the demand captured by facility at x_i from user a is inversely proportional to a positive nondecreasing function of the distance $d(a,x_i)$ from the user at a to the facility at x_i. In other words, the demand captured by the facility at x_i from the user at a is given by

$$\omega_a\frac{1/\varphi_a(d(a,x_i))}{\sum_{j=1}^r 1/\varphi_a(d(a,x_j))}. \tag{2}$$

The usual choice for each φ_a has the form $\varphi_a(d)=d^{\lambda_a}$. When $\lambda_a=2$ for all $a\in V$, we have the so-called gravitational model.

A new firm is entering the market, by locating one facility at some point on the network. This perturbs market share, since the new facility at x will capture a demand from $a\in V$ equal to

$$\omega_a\frac{1/\varphi_a(d(a,x))}{1/\varphi_a(d(a,x))+\sum_{j=1}^r 1/\varphi_a(d(a,x_j))}. \tag{3}$$

Here φ_a is assumed to be non-negative, non-decreasing and twice continuously differentiable in $\mathbb{R}_+$. Expression (3) must be carefully considered when $\varphi_a(0)=0$. Indeed, in such case, if some x_j exists with $x_j=a\in V$, then the full demand of a will be captured by x_j.

Hence, such a will not be taken into account, and thus we assume in what follows without loss of generality that $x_j \notin V$, $j = 1, \dots, r$.

The goal of the entering firm is the maximization of its market share. This is written as the following optimization problem:

$$\max_{x \in [0,l_e], e \in E} \sum_{a \in V} \omega_a \frac{1/\varphi_a(d(a,x))}{1/\varphi_a(d(a,x)) + \sum_{j=1}^{r} 1/\varphi_a(d(a,x_j))}. \tag{4}$$

Defining for each $a \in V$ the positive constant β_a,

$$\beta_a = \sum_{j=1}^{r} \frac{1}{\varphi_a(d(a,x_j))}, \tag{5}$$

it follows that problem (4) can be rewritten as

$$\max_{x \in [0,l_e], e \in E} F(x) \tag{6}$$

with F defined as

$$F(x) = \sum_{a \in V} \omega_a \frac{1}{1 + \beta_a \varphi_a(d(a,x))}. \tag{7}$$

This problem has been addressed in Berman et al. (2011), in which a branch and bound algorithm with interval analysis bounds is proposed.

3 Huff OD trip model

In this model, we have the set $\{\{a,b\}, a \in V, b \in V\}$, of origin-destination pairs. The trip from origin a to destination b will imply a demand $\omega_{ab} > 0$.

Consumers in their way from origin a to destination b will stop at one facility; they choose among the r existing facilities, $x_1, \dots, x_r$, and the new facility at x as in the model described in Sect. 2: the demand from a user in his way from a to b captured by each facility is inversely proportional to a positive nondecreasing function of the length of the path from the origin to the destination via the facility.

In other words, the demand captured by x from users in their trip from origin a to destination b is given by

$$\omega_{ab} \frac{1/\varphi_{ab}(d(a,x) + d(x,b))}{1/\varphi_{ab}(d(a,x) + d(x,b)) + \sum_{j=1}^{r} 1/\varphi_{ab}(d(a,x_j) + d(x_j,b))}. \tag{8}$$

As in the other model, the goal of the entering firm is the maximization of its market share. This is written as the following optimization problem:

$$\max_{x \in [0,l_e], e \in E} \sum_{a,b \in V} \omega_{ab} \frac{1/\varphi_{ab}(d(a,x) + d(x,b))}{1/\varphi_{ab}(d(a,x) + d(x,b)) + \sum_{j=1}^{r} 1/\varphi_{ab}(d(a,x_j) + d(x_j,b))}. \tag{9}$$

Defining for each $a, b \in V$ the positive constant β_{ab},

$$\beta_{ab} = \sum_{j=1}^{r} \frac{1}{\varphi_{ab}(d(a,x_j) + d(x_j,b))},$$

it follows that problem (9) can be rewritten as

$$\max_{x \in [0, l_e],\, e \in E} F(x)$$

with F defined as

$$F(x) = \sum_{a,b \in V} \omega_{ab} \frac{1}{1 + \beta_{ab}\varphi_{ab}(d(a,x) + d(x,b))}. \tag{10}$$

We see that both the location and OD trip models yield an optimization problem of the same form, namely, (6), with a rather similar function F, as given by (7) and (10) respectively. Both functions F can be written in the form

$$F(x) = \sum_{\delta \in \Delta} \omega_\delta \frac{1}{1 + \beta_\delta \varphi_\delta(d_\delta(x))}. \tag{11}$$

where

$$\Delta = V \quad \text{and, for } \delta \in \Delta, \quad d_\delta(x) = d(a,x)$$

for the location model, and

$$\Delta = \{\{a,b\}, a, b \in V\} \quad \text{and, for } \delta \in \Delta, \quad d_\delta(x) = d(a,x) + d(x,b)$$

for the OD trip model.

4 Solving the models

4.1 Multimodality

The optimization problems described in Sect. 2 and 3 are, in general, multimodal, and standard optimization methods get stuck at local optima. This can be seen in simple examples, even when the network is a segment and is illustrated in the following examples for the Huff location problem introduced in Sect. 2.

First, data for one problem with two users and $r = 1$ facility on a segment were randomly generated. The objective function F of such instance is plotted in Fig. 1 (left). One can see that the problem is bimodal. 100 runs of a local search procedure starting with a random point were performed.

In Fig. 1 (right) one can see the histogram of the objective values provided by the optimizer: below a 50 % of the runs yielded the global optimum, whereas the remaining runs stopped at the local not globally optimal solution.

Another instance, with 500 user and 100 facilities was also generated. In Fig. 2 the problem is shown to be multimodal, and just below a 14 % of the runs solved with the optimizer yielded the global optimum.

In the right part of Figs. 1–2 the x axis is normalized, so that 1 corresponds to the best found objective value, z^*, and a bar at $x \in [0, 1]$ indicates that an objective value $x \cdot z^*$ was found.

Since multimodality appears even in the simplest cases, and local search procedures can get stuck at very bad local optima, global optimization tools are needed if the global optimum is sought.

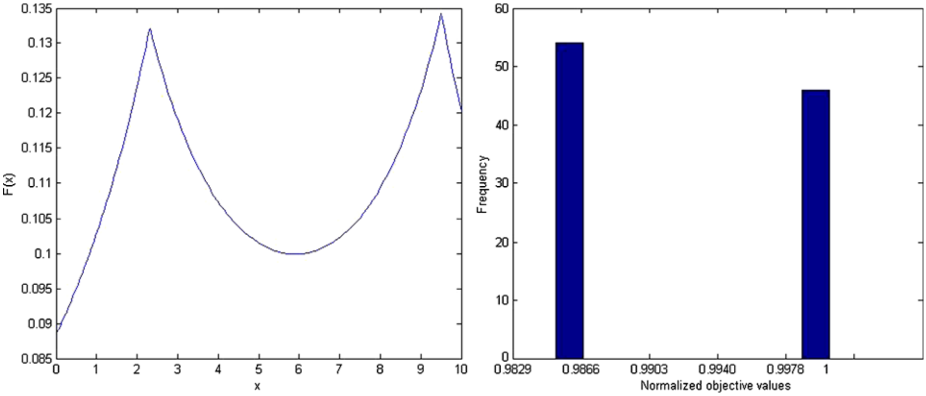

Fig. 1 Users = 2, facilities = 1, $\lambda_a = 1\ \forall a$

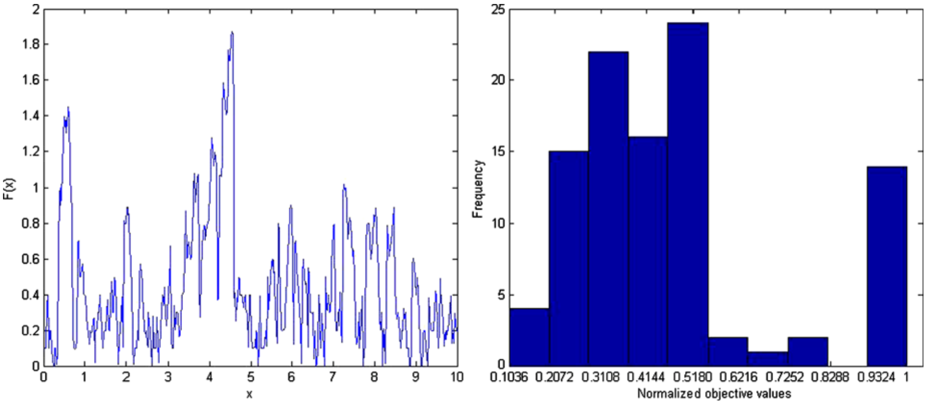

Fig. 2 Users = 500, facilities = 100, $\lambda_a = 20\ \forall a$

4.2 Algorithm

We will use a branch and bound algorithm (Drezner and Suzuki 2004; Hansen et al. 1985; Plastria 1992) to solve the problems under consideration. We first outline the algorithm that finds an optimal solution within a relative accuracy of ε, and later give the details on the bounding process, which will exploit monotonicity properties or the fact that the objective function is DC on each edge (Blanquero and Carrizosa 2009b; Horst and Thoai 1999; Tuy 1995; Tuy et al. 1995). We remind the reader that a function h is DC if it can be written as $h = h^+ - h^-$, where h^+, h^- are both convex; the expression $h^+ - h^-$ is called a DC decomposition of h. In the description of the algorithm, we have $\Delta = V$ for the Huff location model and $\Delta = \{\{a, b\}, a, b \in V\}$ for the Huff OD trip model.

Phase 1: Initialization

- Fix the required accuracy $\varepsilon > 0$.
- Set $LB = 0$.
- Compute the all-pairs distance matrix.
- Calculate β_δ, $\forall \delta \in \Delta$.

- Set the list H of remaining segments as empty.

Phase 2: Prepare the list of segments

- Consider the edge e as segment with its nodes as the segment vertices.
- The value of the objective function is evaluated at the segment midpoint. If the value is greater than LB, then LB is updated to such value and the midpoint stored as incumbent.
- Calculate an upper bound for the segment e, $UB(e)$.
- In case $UB(e) \geq LB \cdot (1+\varepsilon)$ insert e into H.

Phase 3: Branch and bound process.
Repeat as long as no stop is reached:

- Select from H the highest upper bound segment, $UB_{\max}$, with LB as ε-optimal value and the incumbent as an ε-optimal solution.
  If $UB_{\max} \leq LB \cdot (1+\varepsilon)$, stop the algorithm with LB as the solution.
- The highest upper bound segment, $UB_{\max}$, is selected for a split at its midpoint into two smaller segments.
- The value of the objective function at the midpoint of the two small segments is calculated.
  If any of these values is greater than LB, then LB is updated and all segments from H whose upper bound is lower than LB are discarded.
- An upper bound for each small segment is calculated.
- All segments whose upper bound is greater than $LB \cdot (1+\varepsilon)$ are added to H.

In the algorithm above, the segment midpoint yielding the best upper bound is given as ε-optimal solution. Observe that the full list of segments in H contain all ε-optimal solutions, and can thus be used in a two-phase process, as suggested in the GBSSS algorithm (Plastria 1992).

Let us detail the algorithm steps previously outlined. To calculate the all-pairs distance matrix we use the Floyd algorithm (Bertsekas 1998), which uses the edge length matrix to build recursively the distance matrix.

The computation of the bounds requires more detail. The algorithm needs the calculation of an upper bound, $UB(s)$, for each segment s. We present two procedures, one based on Interval Analysis, and the other in properties of DC functions.

4.3 Interval analysis bound

When the distance $d_\delta(x)$ decreases, the market share as given by the objective function (11) increases. Hence we obtain an upper bound on the market share $F(x)$ for any location x on a segment $s=[x_0,x_1]\subset e\in E$ by replacing $d_\delta(x)$ by the lowest possible of these distances on the segment. Defining therefore such lowest distance for the location model as in Berman et al. (2011) $d_\delta(s)=\min\{d(a,x_0),d(a,x_1)\}$ for $\delta=a$, and for the OD trip model as $d_\delta(s)=\min\{d(a,x_0)+d(b,x_0),d(a,x_1)+d(b,x_1)\}$ for $\delta=\{a,b\}$, we obtain the Interval Analysis bound

$$UB^{IA}(s)=\sum_{\delta\in\Delta}\omega_\delta\frac{1}{1+\beta_\delta\varphi_\delta(d_\delta(s))} \tag{12}$$

4.4 DC bound

An upper bound obtained making use of the fact that the objective function is DC on each edge exploits the following properties:

Proposition 4.1 *Let $I \subset \mathbb{R}$ be an interval. Let $d : I \to \mathbb{R}$ be a concave function on I, and let $g : \mathbb{R} \to \mathbb{R}$ be DC, with a DC decomposition given by $g(x) = g^+(x) - g^-(x)$, with both g^+ and g^- non-increasing functions. Then, the function $f : I \to \mathbb{R}$ defined as $f(x) = g(d(x))$ is DC on I and a DC decomposition is given by $f(x) = f^+(x) - f^-(x)$, where $f^+(x) = g^+(d(x))$ and $f^-(x) = g^-(d(x))$.*

Proof The proof follows directly from the fact that the compositions $g^+(d(x))$ and $g^-(d(x))$ are also convex functions. □

Remark 4.1 Proposition 4.1 makes use of a function g which can be written as the difference of two convex functions, g^+ and g^-. Since such convex functions are also non-increasing, it turns out that g belongs to a subclass of DC functions, namely DCM functions, as introduced in Blanquero and Carrizosa (2009a), which are those functions expressed as the difference of two convex monotonic functions, as g^+, g^- are. See Blanquero and Carrizosa (2009a) for further properties.

We are now in position to give a bound for F on an edge exploiting the fact that F is DC. Let $d_\delta(x)$ be the concave function given in (1). Assuming $\varphi_\delta(d) = d_\delta^\lambda$, (11) can be rewritten as

$$F(x) = \sum_{\delta \in \Delta} \omega_\delta \frac{1}{1 + \beta_\delta d_\delta^\lambda(x)} \tag{13}$$

Let us define a simpler function:

$$g(t) = \frac{1}{1 + \beta t^\lambda} \tag{14}$$

The following DC decomposition is known for function g (Bello et al. 2011):

$$c = \left(\frac{\lambda - 1}{(\lambda + 1)\beta}\right)^{1/\lambda}$$

$$g^+(t) = \begin{cases} g(c) + g'(c)(t - c) & \text{if } t \le c \\ g(t) & \text{if } t > c \end{cases}$$

$$g^-(t) = \begin{cases} g(c) + g'(c)(t - c) - g(t) & \text{if } t \le c \\ 0 & \text{if } t > c \end{cases}$$

That means that

$$g(t) = g^+(t) - g^-(t).$$

Applying Proposition 4.1 we have that a DC decomposition for F as defined in (13) is:

$$\begin{aligned} F_\delta^+(x) &= \omega_\delta g^+\big(d_\delta(x)\big) \\ F_\delta^-(x) &= \omega_\delta g^-\big(d_\delta(x)\big) \\ F(x) &= \sum_{\delta \in \Delta} F_\delta^+(x) - \sum_{\delta \in \Delta} F_\delta^-(x) = \sum_{\delta \in \Delta} \big(F_\delta^+(x) - F_\delta^-(x)\big) \end{aligned} \tag{15}$$

To construct an upper bound, UB^{DC}, first we obtain a convex minorant of $F_\delta^-(x)$ as in Blanquero and Carrizosa (2009a):

$$
\begin{aligned}
&F_\delta^-(x) \geq F_\delta^-(x_0) + \xi_\delta(x - x_0) \\
&F(x) \leq \sum_{\delta \in \Delta} \big(F_\delta^+(x) - F_\delta^-(x_0) - \xi_\delta(x - x_0)\big) \quad \text{for } \xi_\delta \in \partial F_\delta^-(x_0),
\end{aligned} \tag{16}
$$

where $\partial F_\delta^-(x_0)$ denotes the set of subgradients of F_δ^- at x_0 (Tuy 1995).

Define for each $\delta \in \Delta$ the function from (16):

$$
U(x) = \sum_{\delta \in \Delta} \big(F_\delta^+(x) - F_\delta^-(x_0) - \xi_\delta(x - x_0)\big) \quad \text{for } \xi_\delta \in \partial F_\delta^-(x_0) \tag{17}
$$

An upper bound for a segment s is obtained:

$$
UB^{DC}(s) = \max\big\{U(v_1), U(v_2)\big\}
$$

v_1, v_2 being vertices of s.

5 Computational results

The algorithm described in Sect. 4 was programmed in an Intel Fortran Compiler XE 12.0 and executed with an Intel Core i7 computer with 8.00 Gb of RAM memory at 2.8 Ghz. The solutions were found within an accuracy of 10^{-10}.

The Huff location model was first tested on the 55-node and 134-edge Swain's (1971) network (Serra et al. 1999; Marianov and Serra 2002), see the Appendix, Table 6, with both bounding strategies.

Several instances of the problem were generated using different values for the number r of existing facilities, ranging from low-saturated markets ($r = 10$ % of the number of edges of the network, $|E|$) to high-saturated markets ($r \leq 90$ %$|E|$). For each value of r, 10 different problems were solved. Each problem is obtained by randomly and independently generating the demands (each vertex of the network is assumed to have a demand uniformly distributed in the interval (0, 1)) and the location of the existing facilities. To generate the r facility locations, r edges are randomly chosen with replacement; on each edge, the facility location is generated following a uniform distribution. The results are shown in Table 1.

The percentage r of existing facilities is shown in the first column. Then, it is reported the minimum, maximum, mean and standard deviation (std) for the number of iterations, i.e., the number of executions of Phase 3, the branch and bound (B& B) list size and the CPU time. The branch and bound list size is the maximum size of the data structure used for storage reached during the algorithm execution.

In all cases, the best solution is found in less than 0.02 seconds for DC bounds, and 0.75 seconds for Interval Analysis bounds. It is remarkable that DC bounds lead to a very stable procedure, as can be seen for all values of r in memory requirements as well as computational time. On the other hand, when using Interval Analysis bounds one can see very extreme cases. There is always a huge difference between the minimum and maximum for the number of iterations, branch and bound list size and time. This means that in the ten runs that are solved for each value of r, Interval Analysis bounds are quite erratic for problems of the same difficulty. Therefore, we have an algorithm that when using DC bounds

Table 1 Results of the 55-node and 134-edge Swain's network for the Huff location model

	$r(\%\|E\|)$	Iterations				B&B list size				Time			
		min	max	mean±	std	min	max	mean±	std	min	max	mean±	std
DC	10	111	196	144.40±	34.68	9	61	31.50±	17.25	0.00	0.02	0.00±	0.00
	20	80	193	140.10±	34.32	15	83	42.70±	22.76	0.00	0.00	0.00±	0.00
	30	51	209	117.80±	47.86	25	96	66.70±	22.42	0.00	0.00	0.00±	0.00
	40	50	256	176.40±	55.33	15	107	84.80±	30.51	0.00	0.02	0.00±	0.00
	50	81	290	172.20±	67.40	74	126	107.50±	14.92	0.00	0.02	0.00±	0.00
	60	121	317	166.50±	60.54	72	121	110.10±	16.09	0.00	0.02	0.00±	0.00
	70	94	323	190.00±	64.79	113	131	123.80±	6.36	0.00	0.02	0.00±	0.00
	80	145	332	209.70±	58.43	117	133	123.70±	5.62	0.00	0.02	0.00±	0.01
	90	137	296	211.40±	55.51	111	129	121.40±	5.66	0.00	0.02	0.00±	0.00
Interval	10	158	318	223.70±	50.81	26	120	64.20±	31.72	0.00	0.02	0.00±	0.00
	20	185	178915	18100.40±	56504.50	29	58170	5890.10±	18369.30	0.00	0.59	0.06±	0.19
	30	203	70911	17779.60±	25278.44	75	25465	6219.20±	8904.76	0.00	0.20	0.05±	0.07
	40	243	243922	24639.40±	77048.06	96	84425	8551.40±	26659.27	0.00	0.75	0.08±	0.24
	50	187	37594	9132.40±	13971.49	118	13306	3143.50±	4794.67	0.00	0.11	0.03±	0.05
	60	186	29111	7734.70±	12130.03	118	9927	2703.30±	4174.80	0.00	0.09	0.03±	0.04
	70	212	10748	3029.00±	4148.73	138	3489	1038.00±	1349.47	0.00	0.03	0.01±	0.02
	80	172	1950	866.60±	677.23	127	648	310.10±	192.99	0.00	0.02	0.00±	0.00
	90	224	22520	4243.00±	7225.49	147	7248	1442.20±	2338.18	0.00	0.09	0.02±	0.03

spends the same resources (computing time) than when using Interval Analysis bounds, but is much more stable and reliable.

Then the algorithm was tested for both models on a battery of test instances of larger dimensions (up to 1000 nodes and 3083 edges) to analyze the dependence of running time and memory requirements with respect to the size of the network and the number of facilities for both bounding strategies.

To attain this end, the models were tested on 43 test networks obtained from Corberán and Sanchis (2007), Reinelt (1991).[1] For $r = 10\ \%|E|$ and $r = 90\ \%|E|$ of existing facilities, 10 instances of the problem were solved. Results for the comparison between the two methods for obtaining bounds are shown in the following tables. Results for the Huff location model are shown in Tables 2 and 3 comparing both bounding procedures applied in Phase 2 and Phase 3 of the algorithm. Results for the Huff OD trip model are shown in Tables 4 and 5. Results from Table 1 show that DC bounds seem to be sharper than Interval Analysis bounds but more computationally expensive; therefore, it seems desirable to delete segments in the initialization (Phase 2 of the algorithm) using easy bounds (Interval Analysis), and then use DC bounds for the rest of the segments (Phase 3). Hence, let us note that for the OD trip model, Interval Analysis bounds are used always in Phase 2 of the algorithm, and then, in Phase 3, the two different bounding procedures are compared. This strategy was also tested for the Huff location model with no improvement and therefore not used.

The name of the network, number of nodes and edges are reported in the first columns of Tables 2–5. The remaining columns have the same meaning than those in Table 1. In Table 2, when dealing with $r = 10\ \%|E|$ facilities in the Huff location model, both bounding methods are comparable, but when dealing with a saturated market, Table 3, DC bounds clearly outperform Interval Analysis bounds. This is due to the simplicity of the Interval Analysis bounds and because DC bounds are sharper. One can see that, when using DC bounds, the branch and bound size and the number of iterations is much smaller than with the other bounds, which means that DC bounds are harder to calculate, but sharper.

For the Huff OD trip model, as the number of computations of the bounds increases, both methods become comparable, as there is an equilibrium between the computational cost of bounds and how sharp they are.

One can see that, although the OD trip model is, in terms of computing time, much harder to solve than the location model, the number of iterations is relatively smaller, while the size of the branch and bound tree is comparable for most networks. Note that the relevant increase of the computing time for the OD trip model with respect to the Huff location model is due to the complexity of the objective function from the model, which involves evaluating many more terms. In both cases all problems can be solved in reasonable time.

6 Concluding remarks

In this paper we have addressed two Huff models on a network, by considering that the users choose the facility according to the distance from their location (Huff location model—Berman et al. 2011) or according to the length of the shortest path from the origin to the destination visiting the facility (Huff OD trip model), which is a new model. Since the problems are shown to be multimodal, in order to obtain a global optimum, a branch and bound algorithm based on two different bounding procedures, Interval Analysis and DC optimization, is proposed.

[1] Data instances for arc routing problems. www.uv.es/corberan/instancias.htm.

Table 2 Results of test instances for the Huff location model with 10 $\%|E|$ facilities

Network	Nodes	Edges	DC bounds						Interval analysis bounds					
			Iterations		B&B list size		Time		Iterations		B&B list size		Time	
			mean±	std	mean±	std	mean±	std	mean±	std	mean±	std	mean±	std
KROB150G	150	296	136.00±	18.38	28.60±	11.64	0.01±	0.01	256.80±	53.90	71.70±	25.31	0.01±	0.01
KROA150G	150	297	147.30±	21.31	22.50±	9.17	0.01±	0.01	257.40±	60.95	62.30±	19.75	0.01±	0.01
PR152G	152	296	104.20±	46.91	34.40±	7.92	0.01±	0.01	51433.40±	98074.47	16950.10±	32330.65	0.40±	0.76
RAT195G	195	336	110.00±	22.32	46.60±	21.96	0.02±	0.00	254.00±	54.23	130.40±	51.90	0.01±	0.01
KROB200G	200	386	127.90±	39.50	31.70±	14.60	0.01±	0.00	12954.80±	26959.17	4329.50±	9047.62	0.15±	0.27
KROA200G	200	392	142.70±	28.06	29.70±	11.83	0.02±	0.00	249.60±	37.98	78.80±	25.49	0.01±	0.01
TS225G	225	306	104.90±	15.20	30.40±	9.65	0.02±	0.01	182.40±	31.70	67.10±	20.12	0.02±	0.00
UR532	298	597	114.80±	36.14	21.90±	9.28	0.04±	0.01	44770.00±	140751.59	16124.10±	50692.35	0.72±	2.17
UR542	343	862	131.30±	50.06	44.10±	16.29	0.06±	0.00	40444.00±	81734.28	13882.90±	28532.14	0.77±	1.45
UR552	388	1135	173.00±	30.49	48.50±	13.94	0.10±	0.01	365.40±	145.20	197.30±	94.94	0.09±	0.01
UR562	416	1403	204.30±	35.22	49.90±	14.58	0.12±	0.01	388.30±	113.50	262.30±	116.58	0.11±	0.00
UR732	452	915	99.30±	29.19	36.00±	13.69	0.11±	0.01	25906.60±	81051.80	9207.20±	28797.07	0.71±	1.89
UR535	458	812	122.00±	17.33	32.20±	12.95	0.12±	0.01	214.80±	42.78	74.50±	32.52	0.11±	0.00
UR545	476	1104	173.60±	21.97	55.20±	22.23	0.15±	0.01	268.90±	41.01	140.40±	35.71	0.14±	0.01
UR555	490	1305	169.50±	68.40	57.40±	22.45	0.17±	0.01	12713.60±	39128.99	4248.30±	12756.97	0.48±	1.01
UR537	493	868	130.00±	27.01	41.30±	9.44	0.14±	0.00	225.40±	46.83	82.30±	29.69	0.13±	0.01
UR565	496	1513	155.40±	56.02	60.70±	31.20	0.18±	0.01	26442.70±	54904.79	9154.30±	18533.07	0.85±	1.42
UR547	498	1112	148.70±	35.15	52.20±	27.28	0.16±	0.01	283.50±	44.66	158.50±	84.24	0.15±	0.01
UR557	498	1310	168.00±	47.13	71.90±	32.18	0.18±	0.01	16840.90±	52028.01	5471.30±	16692.26	0.61±	1.40
UR567	499	1426	180.60±	65.51	72.00±	17.67	0.18±	0.01	8613.60±	25838.69	3126.20±	8980.20	0.39±	0.68
UR742	538	1325	166.30±	69.15	48.40±	13.79	0.22±	0.01	5654.30±	16741.63	2084.40±	5937.85	0.37±	0.51
UR752	580	1735	179.60±	50.92	57.20±	12.93	0.29±	0.01	401.10±	137.60	285.80±	94.01	0.28±	0.01

Table 2 (*Continued*)

Network	Nodes	Edges	DC bounds						Interval analysis bounds					
			Iterations		B&B list size		Time		Iterations		B&B list size		Time	
			mean±	std	mean±	std	mean±	std	mean±	std	mean±	std	mean±	std
UR762	593	2089	187.00±	34.16	88.00±	26.85	0.33±	0.01	511.90±	129.44	554.50±	131.18	0.31±	0.01
UR132	605	1122	126.60±	22.12	26.10±	11.78	0.27±	0.00	224.20±	44.83	95.00±	67.85	0.26±	0.01
UR735	662	1200	116.70±	34.47	30.20±	12.49	0.33±	0.01	47379.90±	149070.53	16336.00±	51404.94	2.22±	6.04
UR142	709	1815	171.70±	40.81	40.50±	14.43	0.46±	0.01	300.30±	77.67	161.10±	62.28	0.44±	0.01
UR745	713	1616	133.40±	40.35	47.20±	12.93	0.44±	0.00	22245.80±	69367.84	7879.60±	24392.62	1.36±	2.96
UR755	724	1966	177.60±	49.02	61.70±	22.51	0.49±	0.01	8604.90±	25951.73	3014.50±	8640.13	0.83±	1.06
UR765	741	2278	185.80±	61.95	90.50±	39.18	0.58±	0.01	388.30±	159.45	373.50±	185.09	0.55±	0.01
UR737	744	1315	146.20±	47.17	36.70±	13.47	0.44±	0.01	6668.80±	20303.00	2269.40±	6874.07	0.72±	0.90
UR747	745	1659	137.80±	38.54	49.90±	16.42	0.50±	0.02	22822.50±	71347.15	8003.30±	24891.27	1.47±	3.14
UR757	748	1969	172.10±	27.42	75.40±	20.79	0.55±	0.00	342.90±	109.03	306.70±	121.76	0.53±	0.01
UR767	749	2314	186.10±	26.79	55.60±	21.85	0.58±	0.02	432.40±	82.81	335.20±	135.51	0.56±	0.01
UR152	766	2390	205.10±	70.77	68.00±	21.48	0.63±	0.01	472.50±	163.57	395.90±	149.69	0.61±	0.03
UR162	802	2897	182.60±	98.28	63.80±	25.69	0.78±	0.01	29666.00±	61694.27	10082.00±	20547.08	2.13±	2.93
UR135	892	1619	124.60±	21.78	35.20±	13.51	0.77±	0.03	207.90±	32.32	87.80±	29.58	0.73±	0.01
UR145	929	2117	137.80±	44.58	45.40±	23.59	0.96±	0.01	902.00±	2002.12	362.00±	672.69	0.98±	0.11
UR155	975	2680	173.90±	36.48	49.60±	14.58	1.24±	0.03	451.60±	424.32	223.70±	46.83	1.19±	0.03
UR137	980	1744	127.10±	22.33	31.50±	15.04	1.00±	0.03	208.30±	31.13	88.00±	42.40	0.97±	0.00
UR165	980	3068	189.90±	57.79	117.10±	22.39	1.32±	0.02	27238.10±	84692.99	9609.40±	28744.20	2.86±	4.95
UR147	996	2254	144.00±	30.24	69.20±	13.28	1.13±	0.03	296.70±	94.75	222.10±	109.46	1.10±	0.01
UR157	1000	2690	157.70±	27.09	92.30±	32.34	1.29±	0.03	392.80±	179.80	384.40±	174.79	1.24±	0.02
UR167	1000	3083	200.20±	52.30	78.30±	28.60	1.32±	0.02	487.50±	270.66	425.40±	213.96	1.29±	0.03

Table 3 Results of test instances for the Huff location model with 90 $\%|E|$ facilities

Network	Nodes	Edges	DC bounds						Interval analysis bounds					
			Iterations		B&B list size		Time		Iterations		B&B list size		Time	
			mean±	std	mean±	std	mean±	std	mean±	std	mean±	std	mean±	std
KROB150G	150	296	124.70±	114.90	111.80±	56.52	0.01±	0.01	122523.30±	86721.97	42553.00±	30707.86	0.92±	0.65
KROA150G	150	297	57.80±	34.26	95.40±	39.59	0.01±	0.01	122187.70±	71891.54	41687.50±	24061.80	0.92±	0.54
PR152G	152	296	84.20±	62.10	146.90±	38.55	0.01±	0.01	125552.60±	116399.32	43041.20±	40982.46	0.97±	0.89
RAT195G	195	336	193.40±	75.58	220.00±	47.86	0.02±	0.00	23516.10±	20715.10	8296.90±	7377.71	0.25±	0.21
KROB200G	200	386	55.80±	29.67	120.60±	76.71	0.02±	0.00	133369.40±	87608.49	45817.30±	30144.88	1.35±	0.88
KROA200G	200	392	92.20±	57.67	134.30±	62.82	0.02±	0.00	85623.70±	83432.90	29232.40±	28551.07	0.87±	0.84
TS225G	225	306	114.30±	44.14	157.60±	57.11	0.02±	0.01	54545.00±	68301.12	19091.30±	24086.07	0.63±	0.76
UR532	298	597	78.70±	37.54	116.50±	67.46	0.04±	0.01	118298.20±	96988.07	40879.50±	34653.07	1.82±	1.45
UR542	343	862	73.50±	26.35	267.30±	168.52	0.07±	0.01	96761.40±	66415.73	33537.40±	23163.56	1.77±	1.18
UR552	388	1135	92.30±	38.03	335.20±	180.15	0.10±	0.01	80499.70±	70906.95	27384.10±	24108.84	1.69±	1.43
UR562	416	1403	102.50±	112.03	348.30±	283.73	0.12±	0.01	155678.90±	101564.05	54765.40±	36433.31	3.45±	2.17
UR732	452	915	52.30±	14.19	83.70±	53.48	0.12±	0.01	146740.60±	84982.85	49974.80±	28853.93	3.51±	1.98
UR535	458	812	58.80±	25.82	90.80±	56.03	0.12±	0.01	124581.70±	75273.00	42193.70±	24984.67	3.06±	1.78
UR545	476	1104	81.50±	36.51	187.10±	57.12	0.15±	0.01	133664.90±	120595.19	46760.60±	43088.04	3.44±	3.02
UR555	490	1305	137.80±	97.07	385.00±	174.05	0.18±	0.01	115100.50±	101518.18	39688.50±	35018.62	3.12±	2.62
UR537	493	868	64.90±	51.69	162.20±	93.81	0.14±	0.00	122309.80±	77142.00	41193.00±	26456.18	3.30±	2.01
UR565	496	1513	85.90±	68.70	227.20±	238.52	0.19±	0.01	156940.30±	123176.94	54343.30±	43896.68	4.25±	3.20
UR547	498	1112	73.40±	31.36	382.70±	185.60	0.17±	0.01	85720.80±	81049.90	29284.20±	27294.53	2.38±	2.11
UR557	498	1310	100.50±	41.85	422.40±	197.55	0.18±	0.01	88274.50±	68354.12	30293.50±	23842.36	2.47±	1.79
UR567	499	1426	69.40±	30.50	535.90±	265.81	0.19±	0.01	176179.80±	141980.77	61802.30±	50933.74	4.77±	3.69
UR742	538	1325	91.10±	64.65	291.10±	236.64	0.22±	0.01	106046.20±	76582.35	37192.30±	27427.96	3.29±	2.27

Table 3 (*Continued*)

Network	Nodes	Edges	DC bounds						Interval analysis bounds					
			Iterations		B&B list size		Time		Iterations		B&B list size		Time	
			mean±	std	mean±	std	mean±	std	mean±	std	mean±	std	mean±	std
UR752	580	1735	94.20±	79.80	379.00±	242.29	0.30±	0.01	165162.90±	111967.65	57192.20±	39367.44	5.73±	3.72
UR762	593	2089	145.80±	85.85	468.60±	344.81	0.34±	0.01	162436.40±	79937.30	57279.50±	28859.45	5.72±	2.64
UR132	605	1122	62.10±	20.57	179.70±	106.36	0.27±	0.00	145409.40±	119728.83	50053.60±	40595.93	5.33±	4.24
UR735	662	1200	70.40±	36.73	127.10±	78.70	0.33±	0.01	133680.00±	78116.64	46433.90±	27209.94	5.52±	3.13
UR142	709	1815	90.10±	56.57	207.40±	166.81	0.47±	0.02	178459.20±	112835.73	62576.90±	40798.83	7.72±	4.46
UR745	713	1616	78.70±	60.75	170.80±	128.32	0.45±	0.01	101158.10±	34508.89	34965.60±	12181.32	4.51±	1.36
UR755	724	1966	76.80±	45.70	450.70±	206.22	0.51±	0.01	142853.50±	72731.98	49434.30±	25240.04	6.52±	3.12
UR765	741	2278	107.40±	120.99	683.90±	321.98	0.59±	0.02	139391.60±	95422.39	48571.10±	33389.80	6.60±	4.16
UR737	744	1315	50.60±	10.00	137.40±	64.96	0.46±	0.01	171229.60±	88598.53	59189.20±	31967.97	7.54±	3.57
UR747	745	1659	100.00±	69.53	436.30±	190.51	0.50±	0.01	103101.70±	86649.55	35734.40±	30112.29	4.88±	3.72
UR757	748	1969	111.40±	122.22	565.50±	253.79	0.57±	0.02	123438.30±	55123.42	42512.80±	18497.15	5.86±	2.27
UR767	749	2314	106.30±	51.87	402.80±	282.60	0.59±	0.01	114120.10±	77627.96	39036.10±	27210.09	5.49±	3.49
UR152	766	2390	183.70±	215.90	666.50±	256.88	0.65±	0.02	146234.20±	95509.84	50240.20±	33865.44	7.10±	4.20
UR162	802	2897	222.00±	190.88	949.20±	277.23	0.81±	0.02	150684.40±	74064.68	52525.50±	25731.91	7.88±	3.51
UR135	892	1619	48.10±	20.44	166.90±	82.96	0.76±	0.01	123350.90±	59285.41	42688.70±	20694.34	7.17±	3.07
UR145	929	2117	66.30±	31.51	237.90±	183.17	0.98±	0.01	163381.20±	83966.41	56867.10±	29219.51	9.97±	4.67
UR155	975	2680	147.00±	208.69	764.30±	242.78	1.25±	0.02	112298.20±	69895.72	39005.50±	25084.91	7.66±	4.07
UR137	980	1744	91.00±	104.47	342.10±	185.67	1.04±	0.03	93091.80±	49939.66	31886.30±	17344.81	6.33±	2.80
UR165	980	3068	114.10±	62.51	688.40±	247.01	1.34±	0.02	173681.80±	107225.32	59126.40±	35851.23	11.62±	6.26
UR147	996	2254	110.10±	95.34	381.20±	244.67	1.14±	0.01	96126.40±	102777.49	33813.10±	36415.99	6.73±	6.10
UR157	1000	2690	73.20±	24.66	538.00±	136.81	1.28±	0.01	161473.50±	99420.21	56795.40±	34944.47	10.67±	5.86
UR167	1000	3083	87.80±	34.37	615.20±	256.40	1.34±	0.01	108227.60±	112302.86	37786.50±	40089.84	7.64±	6.57

Table 4 Results of test instances for the Huff OD trip model with 10 %$|E|$ facilities

Network	Nodes	Edges	DC bounds						Interval analysis bounds					
			Iterations		B&B list size		Time		Iterations		B&B list size		Time	
			mean±	std	mean±	std	mean±	std	mean±	std	mean±	std	mean±	std
KROB150G	150	296	97.20±	21.93	46.60±	6.82	0.19±	0.03	131.00±	9.66	46.60±	6.82	0.17±	0.01
KROA150G	150	297	124.20±	29.94	39.50±	8.44	0.21±	0.03	156.90±	17.88	39.50±	8.44	0.18±	0.01
PR152G	152	296	108.30±	27.86	46.70±	7.76	0.20±	0.03	124.90±	11.58	46.70±	7.76	0.17±	0.01
RAT195G	195	336	82.50±	11.04	83.60±	8.93	0.32±	0.02	117.10±	16.02	83.60±	8.93	0.29±	0.01
KROB200G	200	386	121.40±	16.03	34.00±	10.26	0.43±	0.03	141.70±	8.06	34.00±	10.26	0.36±	0.01
KROA200G	200	392	94.00±	27.53	30.10±	3.28	0.38±	0.05	131.30±	6.60	30.30±	3.53	0.35±	0.01
TS225G	225	306	97.50±	14.49	55.40±	9.81	0.43±	0.04	123.50±	3.78	55.40±	9.81	0.38±	0.01
UR532	298	597	120.80±	13.26	54.50±	8.75	1.29±	0.06	157.30±	12.56	54.60±	8.95	1.11±	0.03
UR542	343	862	163.70±	28.17	131.70±	21.41	2.45±	0.16	228.50±	24.53	133.00±	20.74	2.19±	0.07
UR552	388	1135	127.30±	22.23	95.80±	5.98	3.72±	0.18	212.50±	12.03	95.80±	5.98	3.64±	0.06
UR562	416	1403	290.90±	59.12	194.20±	12.87	7.02±	0.60	372.10±	35.92	207.00±	19.54	6.04±	0.26
UR732	452	915	95.80±	18.45	42.20±	4.05	4.45±	0.21	131.00±	6.45	42.20±	4.05	4.27±	0.08
UR535	458	812	130.20±	8.66	36.50±	8.97	4.69±	0.11	125.00±	21.75	36.60±	9.06	4.02±	0.19
UR545	476	1104	150.10±	30.86	88.60±	8.30	6.68±	0.43	148.80±	6.29	88.60±	8.30	5.90±	0.24
UR555	490	1305	160.70±	27.28	115.90±	25.43	8.22±	0.42	208.00±	13.23	115.90±	25.43	7.83±	0.40
UR537	493	868	83.60±	15.09	39.80±	12.24	5.23±	0.22	125.20±	13.96	39.80±	12.24	5.04±	0.12
UR565	496	1513	204.70±	53.16	120.00±	11.77	10.17±	0.87	289.20±	34.91	125.70±	11.66	9.31±	0.31
UR547	498	1112	110.50±	8.38	80.90±	7.50	6.88±	0.16	147.70±	7.90	80.90±	7.50	6.49±	0.07
UR557	498	1310	190.50±	34.95	120.10±	22.69	9.08±	0.52	258.10±	27.00	120.10±	22.69	8.38±	0.31
UR567	499	1426	130.60±	43.59	92.80±	9.21	8.68±	0.68	186.00±	26.19	92.80±	9.21	8.22±	0.24
UR742	538	1325	188.40±	46.01	148.90±	9.42	11.16±	0.89	223.40±	20.24	148.90±	9.42	9.94±	0.28

Table 4 (*Continued*)

Network	Nodes	Edges	DC bounds						Interval analysis bounds					
			Iterations		B&B list size		Time		Iterations		B&B list size		Time	
			mean±	std	mean±	std	mean±	std	mean±	std	mean±	std	mean±	std
UR752	580	1735	232.10±	33.86	160.20±	11.69	17.04±	0.79	330.80±	70.44	160.20±	11.69	16.15±	1.04
UR762	593	2089	266.80±	15.37	202.00±	13.52	21.13±	0.40	489.40±	32.67	202.00±	13.52	21.10±	0.35
UR132	605	1122	120.50±	54.90	24.60±	2.17	11.48±	1.48	186.00±	28.42	24.60±	2.17	10.93±	0.41
UR735	662	1200	78.90±	13.30	40.00±	1.49	13.37±	0.42	120.20±	4.89	40.00±	1.49	12.79±	0.15
UR142	709	1815	115.00±	16.57	94.70±	13.98	23.59±	0.65	179.30±	30.44	94.70±	13.98	22.73±	0.62
UR745	713	1616	146.90±	31.56	76.30±	2.91	23.02±	1.27	172.90±	26.59	76.30±	2.91	20.52±	0.47
UR755	724	1966	213.60±	28.51	252.60±	24.49	30.11±	1.20	290.40±	53.88	252.60±	24.49	27.37±	1.11
UR765	741	2278	213.70±	9.41	111.20±	3.77	35.85±	0.29	250.60±	11.48	111.20±	3.77	31.92±	0.33
UR737	744	1315	107.60±	17.78	49.40±	7.18	20.64±	0.81	137.70±	8.43	49.40±	7.18	19.15±	0.22
UR747	745	1659	122.60±	31.26	56.00±	2.31	25.11±	1.31	163.90±	15.77	56.00±	2.31	23.22±	0.33
UR757	748	1969	155.90±	28.57	165.10±	16.54	30.83±	1.30	256.00±	22.63	165.10±	16.54	29.70±	0.50
UR767	749	2314	274.60±	58.70	236.60±	21.26	39.65±	2.59	333.50±	22.87	236.60±	21.26	34.84±	0.51
UR152	766	2390	208.30±	39.61	155.40±	11.81	39.98±	1.88	268.30±	16.64	155.40±	11.81	36.43±	0.38
UR162	802	2897	407.20±	107.72	288.50±	31.36	62.94±	5.80	504.60±	39.82	288.50±	31.36	55.42±	1.04
UR135	892	1619	49.80±	9.35	33.00±	0.82	34.40±	0.69	132.50±	16.64	33.00±	0.82	36.79±	0.61
UR145	929	2117	136.40±	21.10	115.20±	6.73	57.26±	1.59	243.80±	18.41	116.30±	6.11	57.94±	0.74
UR155	975	2680	174.70±	31.59	195.80±	21.34	83.70±	2.69	242.80±	25.37	195.80±	21.34	85.42±	4.06
UR137	980	1744	94.60±	13.78	48.90±	5.99	53.14±	1.17	120.30±	3.09	48.90±	5.99	54.90±	0.28
UR165	980	3068	155.10±	26.17	181.40±	18.46	93.71±	2.34	236.00±	37.40	181.40±	18.46	98.32±	1.94
UR147	996	2254	89.00±	13.47	132.30±	7.53	69.48±	1.34	169.50±	10.26	132.30±	7.53	71.75±	2.24
UR157	1000	2690	154.00±	21.55	139.70±	12.14	83.01±	1.90	256.30±	13.62	139.70±	12.14	82.30±	0.99
UR167	1000	3083	229.10±	33.85	254.70±	24.43	100.38±	3.03	313.70±	42.59	254.70±	24.43	98.70±	2.65

Table 5 Results of test instances for the Huff OD trip model with 90 $\%|E|$ facilities

Network	Nodes	Edges	DC bounds						Interval analysis bounds					
			Iterations		B&B list size		Time		Iterations		B&B list size		Time	
			mean±	std	mean±	std	mean±	std	mean±	std	mean±	std	mean±	std
KROB150G	150	296	88.50±	3.66	62.70±	4.30	0.22±	0.00	137.80±	3.71	62.70±	4.30	0.21±	0.01
KROA150G	150	297	92.70±	19.97	41.70±	6.00	0.22±	0.02	167.20±	10.12	41.70±	6.00	0.22±	0.01
PR152G	152	296	71.90±	3.00	58.50±	9.69	0.20±	0.00	122.60±	2.01	58.50±	9.69	0.20±	0.01
RAT195G	195	336	55.90±	4.82	93.90±	6.42	0.36±	0.01	119.10±	11.12	93.90±	6.42	0.38±	0.02
KROB200G	200	386	90.90±	22.34	34.10±	3.96	0.47±	0.04	144.90±	5.63	34.10±	3.96	0.45±	0.01
KROA200G	200	392	78.60±	3.27	37.30±	1.89	0.45±	0.01	129.60±	2.17	37.30±	1.89	0.44±	0.01
TS225G	225	306	65.80±	2.49	57.10±	3.28	0.46±	0.02	120.20±	0.42	57.10±	3.28	0.46±	0.01
UR532	298	597	97.40±	4.01	59.10±	5.22	1.53±	0.02	155.60±	4.38	59.10±	5.22	1.41±	0.01
UR542	343	862	161.40±	8.77	150.30±	6.55	3.11±	0.04	240.00±	14.00	150.30±	6.55	2.83±	0.05
UR552	388	1135	113.80±	4.32	99.40±	4.48	4.74±	0.06	211.60±	3.75	99.40±	4.48	4.60±	0.03
UR562	416	1403	204.00±	10.14	195.20±	8.47	7.69±	0.09	356.40±	13.94	199.10±	12.56	7.29±	0.09
UR732	452	915	79.90±	3.38	45.10±	2.42	5.44±	0.08	130.20±	2.15	45.10±	2.42	5.22±	0.04
UR535	458	812	81.60±	4.53	44.40±	4.62	5.19±	0.08	137.80±	9.73	44.40±	4.62	4.98±	0.09
UR545	476	1104	86.80±	14.46	91.40±	4.35	7.42±	0.23	148.20±	8.11	91.40±	4.35	7.21±	0.10
UR555	490	1305	156.00±	29.30	133.00±	23.67	10.30±	0.48	223.20±	18.90	133.00±	23.67	9.49±	0.17
UR537	493	868	66.60±	2.63	39.10±	6.40	6.44±	0.15	127.20±	3.33	39.10±	6.40	6.25±	0.03
UR565	496	1513	153.50±	9.62	128.70±	2.98	11.96±	0.19	281.70±	28.41	136.30±	5.58	11.43±	0.24
UR547	498	1112	79.40±	3.50	77.50±	3.92	8.31±	0.09	139.90±	4.36	77.50±	3.92	7.90±	0.04
UR557	498	1310	132.10±	10.12	122.90±	13.92	10.40±	0.18	270.80±	18.26	122.90±	13.92	10.26±	0.17
UR567	499	1426	112.00±	2.67	95.10±	2.77	10.84±	0.03	187.50±	5.25	95.10±	2.77	10.26±	0.04
UR742	538	1325	162.20±	13.79	162.30±	9.13	13.17±	0.29	246.00±	16.07	162.30±	9.13	12.39±	0.21

Table 5 (*Continued*)

Network	Nodes	Edges	DC bounds						Interval analysis bounds					
			Iterations		B&B list size		Time		Iterations		B&B list size		Time	
			mean±	std	mean±	std	mean±	std	mean±	std	mean±	std	mean±	std
UR752	580	1735	194.70±	11.72	160.10±	7.37	20.15±	0.36	361.80±	73.69	160.10±	7.37	19.55±	0.96
UR762	593	2089	303.80±	77.63	191.50±	15.41	27.38±	2.02	445.70±	38.44	191.50±	15.41	24.96±	0.57
UR132	605	1122	73.50±	13.33	26.10±	0.32	12.95±	0.37	185.20±	28.54	26.10±	0.32	13.41±	0.60
UR735	662	1200	72.80±	3.55	43.60±	1.51	16.70±	0.15	123.50±	4.17	43.60±	1.51	16.46±	0.81
UR142	709	1815	105.50±	3.89	94.60±	6.74	29.68±	0.23	157.00±	6.02	94.60±	6.74	27.89±	0.32
UR745	713	1616	127.60±	44.94	81.50±	2.59	28.40±	1.88	188.00±	4.69	81.50±	2.59	25.80±	0.18
UR755	724	1966	211.70±	16.17	259.10±	6.17	38.18±	0.69	308.20±	28.83	259.10±	6.17	34.46±	0.62
UR765	741	2278	135.80±	4.73	111.90±	3.96	41.83±	0.35	258.40±	15.54	111.90±	3.96	40.24±	0.58
UR737	744	1315	69.50±	2.22	54.00±	2.67	24.32±	0.25	142.20±	2.57	54.00±	2.67	24.75±	0.48
UR747	745	1659	86.50±	8.45	56.90±	2.08	30.30±	0.31	162.50±	14.03	56.90±	2.08	29.43±	1.03
UR757	748	1969	145.80±	8.12	172.90±	4.18	38.24±	0.35	288.10±	35.09	172.90±	4.18	37.58±	1.21
UR767	749	2314	195.00±	13.23	241.50±	10.74	46.02±	0.75	327.40±	22.74	241.50±	10.74	44.82±	1.23
UR152	766	2390	178.80±	26.39	160.30±	3.50	49.72±	1.31	267.00±	5.29	160.30±	3.50	46.27±	1.14
UR162	802	2897	328.10±	22.26	324.70±	14.35	74.00±	1.49	550.80±	38.55	324.70±	14.35	68.54±	1.19
UR135	892	1619	50.50±	10.22	33.50±	0.53	42.73±	1.64	136.60±	10.36	33.50±	0.53	45.95±	0.72
UR145	929	2117	133.10±	5.65	127.50±	3.69	67.30±	0.42	266.10±	24.08	127.50±	3.69	71.57±	1.10
UR155	975	2680	171.90±	7.31	204.00±	16.49	99.39±	0.66	266.40±	15.06	204.00±	16.49	98.18±	0.90
UR137	980	1744	72.80±	3.22	51.00±	4.08	61.46±	0.28	124.60±	4.20	51.00±	4.08	62.38±	0.52
UR165	980	3068	161.20±	8.42	199.10±	7.92	111.06±	0.75	235.40±	4.81	199.10±	7.92	111.63±	0.68
UR147	996	2254	91.40±	7.89	135.60±	4.48	82.76±	0.73	171.10±	3.21	135.60±	4.48	86.18±	0.51
UR157	1000	2690	136.50±	4.28	136.10±	2.18	98.28±	0.41	247.50±	7.53	136.10±	2.18	99.93±	0.80
UR167	1000	3083	216.80±	7.25	262.90±	9.26	118.59±	0.77	327.70±	19.31	262.90±	9.26	116.10±	1.34

The computational experience reported shows that large networks can be successfully handled with both bounding procedures where the DC procedure seems to be more stable in both time and memory requirements.

Extensions of these problems to the multifacility case deserve further study.

Appendix

Table 6 The Swain data set (Marianov and Serra 2002; Serra et al. 1999)

Initial node	Final node	Edge length	Initial node	Final node	Edge length	Initial node	Final node	Edge length
1	2	3.1623	13	47	4.4721	25	49	7.0000
1	5	2.0000	14	16	10.2956	26	36	8.6023
1	8	4.4721	14	22	12.0416	27	38	9.2195
1	13	2.2361	14	27	9.8489	27	54	7.2111
1	43	10.0000	15	18	6.4031	29	31	3.6056
1	44	4.1231	15	25	7.2801	30	33	4.0000
2	3	4.4721	15	31	7.0000	30	45	5.0000
2	4	3.0000	15	36	6.7082	32	33	5.0000
2	8	3.1623	15	41	6.3246	32	38	4.4721
2	42	2.8284	15	42	6.3246	32	45	4.0000
3	7	4.2426	16	22	6.7082	34	43	6.0828
3	8	3.1623	16	27	6.4031	34	45	3.0000
3	19	6.4031	16	32	6.3246	35	36	7.6158
3	30	5.0000	16	33	6.7082	35	48	7.6158
3	31	6.0828	16	38	7.2111	36	48	6.0000
3	34	5.3852	17	22	5.0990	37	47	10.0499
4	5	3.0000	17	23	5.3852	37	50	10.4403
4	9	2.0000	17	28	7.0000	37	53	7.8102
4	42	2.2361	18	23	9.0000	38	43	8.2462
5	11	1.4142	18	26	7.0711	38	45	7.2111
5	13	2.2361	18	29	5.3852	38	54	6.4031
6	9	3.6056	18	31	4.4721	38	55	7.8102
6	10	5.0000	18	36	8.2462	39	40	11.7047
6	15	5.8310	19	22	8.2462	39	54	6.0828
6	41	4.2426	19	23	5.3852	39	55	9.8489
6	42	5.0990	19	29	3.6056	40	46	8.9443
7	15	5.8310	19	30	5.0990	40	55	8.2462
7	31	3.6056	19	31	5.0990	43	44	7.2801
7	42	4.2426	20	21	4.4721	43	45	6.3246
8	34	3.6056	20	25	9.2195	43	46	5.0000
9	10	6.0000	20	41	8.2462	43	55	5.0000
9	11	4.1231	20	49	6.0000	44	46	7.2111
9	42	3.6056	20	51	9.2195	44	47	6.0000
10	20	8.0623	21	37	7.2111	46	47	11.6619

Table 6 (*Continued*)

Initial node	Final node	Edge length	Initial node	Final node	Edge length	Initial node	Final node	Edge length
10	21	9.0000	21	51	7.2801	46	52	15.6205
10	37	7.8102	22	33	4.2426	46	55	6.0000
10	41	6.0828	23	26	12.2066	47	52	16.1245
10	47	8.6023	23	28	7.0711	47	53	5.6569
11	13	2.2361	23	29	4.4721	48	49	6.4031
11	47	3.6056	24	26	7.2111	49	51	12.0416
12	14	10.6301	24	35	4.4721	50	52	8.6023
12	17	6.3246	24	36	9.0554	50	53	5.8310
12	22	8.6023	25	36	5.6569	52	53	12.1655
12	28	7.8102	25	41	4.1231	54	55	10.1980
13	44	2.8284	25	48	4.4721			

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