J Glob Optim (2010) 48:209–213
DOI 10.1007/s10898-009-9487-y

On the norm of a dc function

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Received: 22 May 2009 / Accepted: 27 October 2009 / Published online: 12 November 2009

Abstract It is well known that, if a vector-valued function can be written as difference of componentwise convex functions, the norm of such function inherits this property. In this note we show that, if the norm in use is monotonic in the positive orthant and the functions are non-negative, a sharper decomposition can be obtained.

Keywords Dc functions · Monotonic norms

1 Dc functions and monotonic norms

Let Ω be a convex set in $\mathbb{R}^d$. A function $\phi : \mathbb{R}^d \longrightarrow \mathbb{R}^n$ is said to be dc on Ω if $f = (f_1, \ldots, f_n)$, $g = (g_1, \ldots, g_n)$ exist such that

$$\phi(x) = f(x) - g(x) \quad \forall x \in \Omega \tag{1}$$

and the functions f_i, g_i are convex in Ω. A pair (f, g) satisfying the conditions above is said to be a dc decomposition of ϕ on Ω. See [5,7] for an introduction to dc optimization and also [3,4] for recent successful applications.

The class of dc functions includes, in particular, all C^2 functions. Moreover, the rich algebra of dc functions (the class is closed under most common operations) usually allows one to show that a given function is dc. However, knowing that a function is dc is not sufficient to use the optimization algorithms for dc functions, since the knowledge of a dc decomposition is required. Proposition 1.1 of [1] states that, for a function ϕ which is dc on a convex set Ω with dc decomposition $\phi = f - g$ and for any norm $\|\cdot\|$ in $\mathbb{R}^n$, the function $\|\phi\|$ is dc on Ω. Moreover, a dc decomposition of $\|\cdot\|$ is given by

Partially supported by grants MTM2008-3032, FQM329 Spain.

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$$\|f-g\| = \left(\|f-g\| + \sum_{i=1}^{n} M_i\,(f_i+g_i)\right) - \sum_{i=1}^{n} M_i\,(f_i+g_i)\,, \tag{2}$$

where, for each $i = 1, \ldots, n$, e_i is the unit vector with 1 in its i-th coordinate and zeroes everywhere else, and M_i is an arbitrary constant satisfying $M_i \geq \|e_i\|$.

In this note we obtain a dc decomposition for $\|\phi\|$, alternative to (2), for the case in which ϕ is non-negative on Ω and $\|\cdot\|$ is monotonic in the positive orthant $\mathbb{R}^n_+$. We recall that a norm $\|\cdot\|$ in $\mathbb{R}^n$ is said to be monotonic in $\mathbb{R}^n_+$ if

$$\|x\| \leq \|y\| \quad \forall x = (x_1, \ldots, x_n), \forall y = (y_1, \ldots, y_n)\,, 0 \leq x_i \leq y_i, i = 1, 2, \ldots, n. \tag{3}$$

Observe that, in particular, any linear combination with positive coefficients of ℓ_p norms is monotonic in $\mathbb{R}^n_+$.

For a norm $\|\cdot\|$, let $\|\cdot\|^\circ$ denote its dual norm, i.e.,

$$\|z\|^\circ = \max_u \{u^\top z : u \in \mathbb{R}^n, \|u\| \leq 1\} \tag{4}$$

We have

Lemma 1 *Let the norm $\|\cdot\|$ be monotonic in $\mathbb{R}^n_+$. Then,*

$$\|z\| = \max_u \{u^\top z : u \in \mathbb{R}^n_+, \|u\|^\circ \leq 1\} \quad \forall z \in \mathbb{R}^n_+. \tag{5}$$

Proof First observe that $(\|\cdot\|^\circ)^\circ = \|\cdot\|$, and thus, by definition of dual norm,

$$\|z\| = \max_u \{u^\top z : u \in \mathbb{R}^n, \|u\|^\circ \leq 1\} \quad \forall z \in \mathbb{R}^n,$$

the optimal value of the optimization problem being attained at any u, subgradient of $\|\cdot\|$ at z.

Hence, it suffices to show that for any $z \in \mathbb{R}^n_+$, there exists some non-negative subgradient, i.e.,

$$\partial\|z\| \cap \mathbb{R}^n_+ \neq \emptyset \quad \forall z \in \mathbb{R}^n_+. \tag{6}$$

Let $z \in \mathbb{R}^n_+$. Assume first z has all its components strictly positive. Let $p \in \partial\|z\|$. For any $i = 1, \ldots, n$, let e_i the unit vector with 1 in its i-th coordinate and zeroes everywhere else. For $\tau > 0$ sufficiently small, by the monotonicity of $\|\cdot\|$ and the definition of subgradient, one has

$$0 \geq \|z - \tau e_i\| - \|z\| \geq -\tau u^\top e_i. \tag{7}$$

Hence, $u \geq 0$ and (6) holds. Now, for an arbitrary $z \in \mathbb{R}^n_+$, take a sequence $\{z_k\}$ of componentwise strictly positive vectors converging to z, and, for each k, an arbitrary $u_k \in \partial\|z_k\|$. By the reasoning above, $u_k \geq 0$ for all k. Moreover, the sequence $\{u_k\}$ is contained in a compact set (the dual unit ball), so it contains a subsequence converging to some u, which, by construction, is a non-negative subgradient of $\|\cdot\|$ at z. Hence, (6) holds. □

We recall that a function $\gamma : \mathbb{R}^n \rightarrow \mathbb{R}$ is a gauge, [6], if there exists a closed convex set B, called the unit ball of γ, with the origin in its interior such that

$$\gamma(x) = \inf\{t > 0 : x \in tB\} \quad x \in \mathbb{R}^n.$$

In particular, norms are those gauges with compact unit ball symmetric with respect to the origin.

Moreover, given a non-empty convex set C, its polar set C° is given by $\{x : u^\top x \leq 1 \; \forall u \in C\}$, see page 125 of [6].

The following result appears in [1] as Proposition 1.2.

Lemma 2 *Let $\Omega \subset \mathbb{R}^d$ be a convex set. Let $\gamma : \mathbb{R}^n \to \mathbb{R}$ be a gauge in $\mathbb{R}^n$ with unit ball B, such that $B^0 \subset \mathbb{R}^n_+$. Let $\phi = (\phi_1, \ldots, \phi_n) : \Omega \to \mathbb{R}^n$ be a dc function, with dc decomposition known: $\phi_i = f_i - g_i$, with f_i, g_i convex. For any $i = 1, \ldots, n$, let $M_i \geq \gamma(e_i)$, where e_i is the i-th unit vector of $\mathbb{R}^n$. Then, $\gamma \circ \phi : \Omega \to \mathbb{R}$ is a dc function, and a dc decomposition for it is given by*

$$\gamma \circ \phi = \left(\gamma \circ \phi + \sum_{i=1}^{n} M_i \, g_i\right) - \left(\sum_{i=1}^{n} M_i \, g_i\right) \tag{8}$$

We are in position to construct a new dc decomposition for $\|\phi\|$.

Theorem 1 *Let the norm $\|\cdot\|$ be monotonic in $\mathbb{R}^n_+$, and let $\phi : \mathbb{R}^d \mapsto \mathbb{R}^n_+$ be a non-negative dc function on the convex set $\Omega \subset \mathbb{R}^d$, with dc decomposition $\phi = f - g$. Then, a dc decomposition on Ω for $\|\phi\|$ is given by*

$$\|f - g\| = \left(\|f - g\| + \sum_{i=1}^{n} \|e_i\| g_i\right) - \left(\sum_{i=1}^{n} \|e_i\| g_i\right), \tag{9}$$

where, for each $i = 1, \ldots, n$, e_i is the unit vector with 1 in its i-th coordinate and zeroes everywhere else.

Proof Let γ be the gauge in $\mathbb{R}^n$ defined as

$$\gamma(z) = \sup_u \{u^\top z : \|u\|^\circ \leq 1, u \in \mathbb{R}^n_+\}, \tag{10}$$

see [6]. Let $B_{\|\cdot\|}$ denote the unit ball of $\|\cdot\|$. Observe that

$$\begin{aligned}(B^\circ_{\|\cdot\|} \cap \mathbb{R}^n_+)^\circ &= \{x \in \mathbb{R}^n : u^\top x \leq 1 \, \forall u \in \mathbb{R}^n_+ \cap B^\circ_{\|\cdot\|}\} \\ &= \{x \in \mathbb{R}^n : 1 \geq \max_u \{u^\top x : u \in \mathbb{R}^n_+, \|u\|^\circ \leq 1\}\} \\ &= \{x \in \mathbb{R}^n : 1 \geq \gamma(x)\} \\ &= B_\gamma\end{aligned}$$

Hence $B^\circ_\gamma = (B^\circ_{\|\cdot\|} \cap \mathbb{R}^n_+)^{\circ\circ} = B^\circ_{\|\cdot\|} \cap \mathbb{R}^n_+ \subset \mathbb{R}^n_+$ and Lemma 2 applies. We conclude that $\gamma \circ \phi$ is dc with dc decomposition given by

$$\gamma \circ \phi = \left(\gamma \circ (f - g) + \sum_{i=1}^{n} \gamma(e_i) g_i\right) - \left(\sum_{i=1}^{n} \gamma(e_i) g_i\right). \tag{11}$$

By Lemma 1,

$$\|z\| = \gamma(z) \quad \forall z \geq 0, \tag{12}$$

and thus, since ϕ is assumed to be non-negative on Ω, $\|\phi\| = \gamma(\phi)$ on Ω, and thus $\|\phi\|$ is dc, with the dc decomposition on Ω as given in (9). □

2 A numerical example

As an illustration of the numerical advantages of using the new dc decomposition (9) against (2), we have considered an example in the context of Compromise Programming, [8]. We recall the reader that, given a multiple-objective problem,

$$\max_{x \in \Omega} (\phi_1(x), \ldots, \phi_n(x))$$

with $\Omega \subset \mathbb{R}^d$, in Compromise Programming one seeks the optimal solution of

$$\min_{x \in \Omega} \left\{ \sum_{i=1}^{n} w_i^p \left| \frac{\overline{\phi}_i - \phi_i(x)}{\overline{\phi}_i - \underline{\phi}_i} \right|^p \right\}^{1/p} \tag{13}$$

where w_i is a weight chosen by the decision maker which measures the relative importance of the i-th criterion, $\overline{\phi}_i = \max_{x \in \Omega} \phi_i(x)$ and $\underline{\phi}_i = \min_{x \in \Omega} \phi_i(x)$. Note that the objective in (13) is the ℓ_p-norm of the vector which i-th component is

$$w_i \frac{\overline{\phi}_i - \phi_i(x)}{\overline{\phi}_i - \underline{\phi}_i} \quad i = 1, \ldots, n$$

and thus, the resolution of (13) provides the feasible point closest in that sense to the so-called *ideal point*, whose coordinates are $(\overline{\phi}_1, \ldots, \overline{\phi}_n)$. If every function ϕ_i is dc with a dc decomposition available, $\phi_i = f_i - g_i$, one can easily obtain a dc decomposition for the overall problem by using Proposition 1.1 of [1] or Theorem 1 in this note.

We have considered a Compromise Programming problem with $n = 5, w_i = 1$ $(i = 1, \ldots, n)$ and functions:

$$\begin{aligned}
\phi_1(x, y) &= x + 10\cos(5\pi y) \\
\phi_2(x, y) &= -x^2 + 6x + y^2 \\
\phi_3(x, y) &= 10x + 7y \\
\phi_4(x, y) &= -x^2 + 2x + 4y \\
\phi_5(x, y) &= x^2 + y^2
\end{aligned}$$

The feasible region is the rectangle

$$\Omega = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 6, 0 \leq y \leq 4\}$$

The global maxima and minima of the separable functions ϕ_i above can easily be computed, yielding

$$\begin{array}{ll}
\overline{\phi}_1 = 16 & \underline{\phi}_1 = -10 \\
\overline{\phi}_2 = 25 & \underline{\phi}_2 = 0 \\
\overline{\phi}_3 = 88 & \underline{\phi}_3 = 0 \\
\overline{\phi}_4 = 17 & \underline{\phi}_4 = 0 \\
\overline{\phi}_5 = 52 & \underline{\phi}_5 = 0
\end{array}$$

Table 1 Computational results for the Compromise Programming example

Dc decomposition	Iterations			Vertices			Overall time (s)
	Min	Max	Ave	Min	Max	Ave	
(2)	93	202	183	188	406	368	0.1406
(9)	51	101	93	104	204	188	0.0625

Using the standard procedures for obtaining dc decompositions [7], we get the following dc representations for $\phi_i = f_i - g_i, \ i = 1, \ldots, 5$.

$$\begin{array}{ll} f_1(x, y) = x + 10\cos(5\pi y) + 125\pi^2 y^2 & g_1(x, y) = 125\pi^2 y^2 \\ f_2(x, y) = y^2 & g_2(x, y) = x^2 - 6x \\ f_3(x, y) = 10x + 7y & g_3(x, y) = 0 \\ f_4(x, y) = 4y & g_4(x, y) = x^2 - 2x \\ f_5(x, y) = x^2 + y^2 & g_5(x, y) = 0 \end{array} \tag{14}$$

Instead of choosing one value for the parameter p, a grid of 180 values in the interval [1,10] has been chosen for p, and for each of those values, the covering algorithm for dc optimization introduced in [2] has been used to solve the resulting problem. Such algorithm computes at each iteration a lower bound for the dc objective function by underestimating it with a piecewise-concave function. This way, an ε-optimal solution is obtained in a finite number of iterations. See [2] for further details.

The program code was written in Fortran, compiled by Intel Fortran 10.1 and ran on a 2.4 GHz computer under Windows XP. The solutions were found to a relative accuracy of 10^{-10}.

Each problem was solved twice, using the dc decompositions (2) and (9) for functions f and g in (14). The computational experience, summarized in Table 1, shows that the new dc representation clearly outperforms (2) in every computational aspects considered: number of iterations, memory usage (measured through the number of vertices of the final polytope in the execution of the covering algorithm) and CPU time.

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