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Locating a semi-obnoxious covering facility with repelling polygonal regions☆

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ABSTRACT

A facility is to be located in the Euclidean plane to serve certain sites by covering them closely. Simultaneously, a set of polygonal areas must be protected from the negative effects from that facility. The problem is formulated as a margin maximization model. Necessary optimality conditions are studied and a finite dominating set of solutions is obtained, leading to a polynomial algorithm. The method is illustrated on some examples.

1. Introduction

A facility is called semi-obnoxious, semi-desirable or push–pull (to use the expressive terminology introduced in [15]) when it gives a service to demand in its neighborhood but is also felt as obnoxious, potentially or outrightly noxious to the environment, or needs to be protected from potential threats. For instance hospitals, airports, train stations, radio or wireless stations and alarm sirens are examples of semi-obnoxious facilities, since they are useful and necessary for the community, but are also a source of negative effects, such as noise or electromagnetic energy, and therefore considered as NIMBY (*not in my backyard*) facilities.

Since the early work on modeling the optimal location in such situations as surveyed in general in [7] and in a discrete setting in [20], the last decade has known many more publications around this difficult topic. The research presented in this paper falls within this general framework but presents several distinctive features, as shortly described next, and more in detail in the next section.

In our problem, a semi-obnoxious facility must be located in the plane with regards to two different sets. On the one hand, we have the demand points served by the proximity of the facility that is therefore pulled towards them, so should be as close as possible to all of them. On the other hand, a *repelling* region should be protected from the (ob)noxious effects coming from the facility, or it is the facility that should be protected from possible threats out of this region. Therefore the facility should be as far as possible from all points of the region. As usual the region is described as in a Geographical Information System (GIS) as a union of polygons. So the demand points must lie well covered within a circle centered at the facility,

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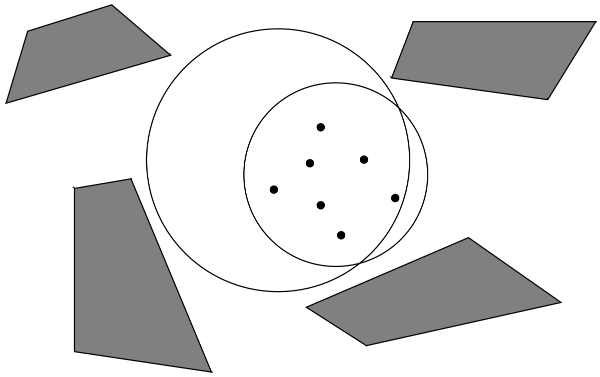

Fig. 1. Two possible separating balls.

while the region should remain well outside this same circle. Both the push and pull objectives are thus considered to be of Rawlsian nature [15] using Euclidean distances, yielding a new continuous covering location model of the kind reviewed in [29].

The three last examples cited higher typically present such features. The quality of connection with a wireless station sharply depends on the Euclidean distance to it, but is also considered as harmful when living too close by, so highly populated areas should be touched only at low level. Also if some countries neighboring a region form a possible threat of attack to the facility or may be a source of spying by listening in to the emissions of the station, the facility should lie at some largest possible Euclidean distance from it. Similarly an alarm siren should be well heard by the whole community, but should not trouble too much the wildlife in the nature reserves around, while noise decays uniformly in all directions.

As far as we are aware no earlier work seems to have been published with all the same characteristics. The major differences with mainstream work in semi-obnoxious models are as follows.

- Our model is about location in a continuous space, contrary to many studies in a discrete or a network environment [6,16,2,3].
- We use Euclidean distance, instead of other (less realistic) distance measures [23,18].
- At least the pull part of the objective is often considered of minsum type (Weber-like) see [14,9,31,34,19], whereas we use minmax.
- The problem is usually seen as biobjective and the efficient set or a finite approximation of it [24,4,26,25] is sought, but we seek one explicit solution.
- We consider repelling regions, as in [4], instead of repelling points (see [8,18,5,1,22,21]).

In the next section, we introduce a formulation of our problem using the popular quadratic margin maximization method of Support Vector Machine in Machine Learning. This formulation enables the construction of an exact global optimal solution, without resorting to the use of cumbersome global optimization methods that yield only approximations as in [28,32,13,5]. The structural properties derived in Section 3 lead to a finite dominating set, the enumeration of which yields a finite and polynomial method detailed in Section 4, that is tested on some artificial databases in Section 5.

2. The model

2.1. The basic aim

Consider G_+ and G_-, two groups of objects in the Euclidean plane, where G_+ is a finite set of points $G_+ = \{x_1, \ldots, x_n\} \subset \mathbb{R}^2$, and G_- is a set of compact convex polygonal areas $G_- = \{S_1, \ldots, S_m\} \subset \mathbb{R}^2$ (with $n, m \geq 3$). The points of G_+ represent individual customers to be serviced by the facility, while the polygons represent areas to be protected from the inconveniences of the semi-obnoxious facility to be located. The points of G_+ are assumed not to be contained in any element of G_-. Also the polygons in G_- are assumed to have pairwise disjoint interiors. Note that this is not a restriction because any (possibly disconnected) polygonal region can be decomposed into a finite set G_- which satisfies our assumptions.

Our aim is to locate a single semi-obnoxious facility, $x_0 \in \mathbb{R}^2$, which is as near as possible to the points of G_+ (*attracting* elements) in order to receive a high-quality service, and far from the polygons of G_- (*repelling* elements).

As explained in the Introduction, all distances are measured in the Euclidean way. Therefore the location of the facility will be decided through the construction of a ball $B(x_0, r)$, with $x_0 \in \mathbb{R}^2$ and $r \in \mathbb{R}_+$, such that every point of G_+ is deeply contained in the ball and every polygon of G_- lies far outside the ball.

In Fig. 1, an example of the problem is depicted. The black points represent the attracting points of G_+, whereas the gray-colored areas represent the repelling elements of G_-. Our problem is to build a ball containing all the points well inside while remaining well away from any polygon.

Different solutions may exist separating the elements in G_+ and G_-. For instance, in Fig. 1 two possible circles separating the two groups have been depicted. In order to single out one ball, we follow the strategy that is successfully used in Support Vector Machines, [10,33], and maximize a quadratic margin as defined in the next section. Following this strategy, the smallest circle in Fig. 1 will be preferred.

2.2. The optimization problem

Given the elements of the two groups, G_+ and G_-, the following constraints are desirable,

$$\|x_0 - x_i\|^2 < r^2 \quad \forall x_i \in G_+, \tag{1}$$

$$\min_{x \in S_j} \|x_0 - x\|^2 \geq r^2 \quad \forall S_j \in G_-, \tag{2}$$

where $\|\cdot\|$ is the Euclidean norm.

Constraints (1)–(2) are equivalent respectively to

$$r^2 - \|x_0 - x_i\|^2 > 0 \quad \forall x_i \in G_+ \Leftrightarrow \min_{x_i \in G_+} (r^2 - \|x_0 - x_i\|^2) > 0, \tag{3}$$

$$\min_{x \in S_j} (\|x_0 - x\|^2 - r^2) \geq 0 \quad \forall S_j \in G_- \Leftrightarrow \min_{S_j \in G_-} \min_{x \in S_j} (\|x_0 - x\|^2 - r^2) \geq 0. \tag{4}$$

We propose to maximize the (quadratic) margin Δ, which is the minimum of the two positive amounts described in (3)–(4). The optimization problem we have to solve is then

$$\max_{x_0, r} \Delta \tag{5}$$

where

$$\Delta = \min \left\{ \min_{x_i \in G_+} (r^2 - \|x_0 - x_i\|^2), \min_{S_j \in G_-} \min_{x \in S_j} (\|x_0 - x\|^2 - r^2) \right\}. \tag{6}$$

This may be rewritten as

$$\begin{array}{ll} \max\limits_{x_0, r, \Delta} & \Delta \\ \text{s.t.} & \Delta \leq \min\limits_{x_i \in G_+} (r^2 - \|x_0 - x_i\|^2) \\ & \Delta \leq \min\limits_{S_j \in G_-} \min\limits_{x \in S_j} (\|x_0 - x\|^2 - r^2) \end{array} \tag{7}$$

or equivalently,

$$\begin{array}{ll} \max\limits_{x_0, r, \Delta} & \Delta \\ \text{s.t.} & \Delta \leq r^2 - \|x_0 - x_i\|^2 \quad \forall x_i \in G_+ \\ & \Delta \leq \|x_0 - x\|^2 - r^2 \quad \forall x \in S_j, \ \forall S_j \in G_-. \end{array} \tag{8}$$

If we denote $r_+^2 = r^2 - \Delta$ and $r_-^2 = r^2 + \Delta$, the objective function of Problem (8) changes into $\Delta = \frac{r_-^2 - r_+^2}{2}$, and the problem can be reformulated as

$$\begin{array}{ll} \max\limits_{x_0, r_+, r_-} & r_-^2 - r_+^2 \\ \text{s.t.} & \|x_0 - x_i\| \leq r_+ \quad \forall x_i \in G_+ \\ & \|x_0 - x\| \geq r_- \quad \forall x \in S_j, \ \forall S_j \in G_- \\ & r_+, r_- \geq 0. \end{array} \tag{9}$$

and if needed one may recover r^2 as $\frac{r_-^2 + r_+^2}{2}$, if Δ is positive.

Indeed, Problem (9) is in fact more general since it also encompasses situations with negative optimal value, that correspond to the (frequent) situations of unfeasibility of previous constraints sets (1)–(2), (3)–(4), or those of problem (7). This new formulation is never unfeasible, and indicates what should be considered as optimal (or rather least undesirable) in these latter cases.

It will follow from Theorem 1 below that the optimal radii r_+ and r_- are easily defined as a function of the center x_0, and Problem (9) can be reformulated further with an objective depending on x_0 only, as follows

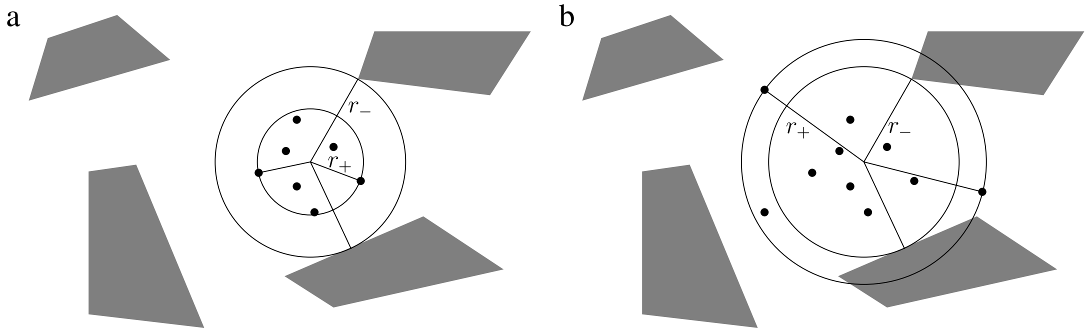

Fig. 2. The two separating concentric balls with maximum margin. (a) Feasible case, (b) unfeasible case.

$$\begin{aligned} \max_{x_0 \in \mathbb{R}^2} \quad & f(x_0) \\ \text{s.t.} \quad & f(x_0) = r_-^2(x_0) - r_+^2(x_0) \\ & r_+(x_0) = \max_{x_i \in G_+} \|x_0 - x_i\| \\ & r_-(x_0) = \min_{S_j \in G_-} \min_{x \in S_j} \|x_0 - x\|. \end{aligned} \tag{10}$$

Hence we may view our problem as obtaining two concentric balls $B(x_0, r_+)$ and $B(x_0, r_-)$, where the ball $B(x_0, r_-)$ does not contain strictly any points of the polygons of G_-, the ball $B(x_0, r_+)$ contains every point x_i belonging to G_+, and the difference between the squares of the first and second radius is as large as possible.

In case $r_- \geq r_+$, i.e. when the original problem is feasible, this geometrically means that the area of the annulus between the two circles is as large as possible (see Fig. 2(a)), showing that our problem is somewhat related to the so-called largest empty annulus problem [12]. (In that model, however, only a set of points is given and an annulus of maximal area without such points in its interior is sought, and no regions are considered.)

In case $r_- < r_+$, when the original problem is unfeasible, the area of the annulus should be as small as possible (see Fig. 2(b)).

Note that the use of the squared radii in the objective seems particularly suited to the main kinds of applications mentioned in the Introduction, since effects like noise and electro-magnetic energy decay quadratically with distance.

3. Necessary conditions for optimality

We now derive a number of necessary optimality conditions, based on the notion of active element as defined below. We will write (x_0, r_+, r_-) to either denote a finite feasible solution of Problem (9) or assume that $r_+ = r_+(x_0)$ and $r_- = r_-(x_0)$ as defined in Problem (10).

A point x_i from G_+ is an *active point* for the solution (x_0, r_+, r_-) iff the distance from x_i to the center x_0 is exactly r_+, that is, $d(x_0, x_i) = \|x_0 - x_i\| = r_+$. Thus, the set of active points of G_+, denoted by $A_+(x_0)$, consists of the points lying on the boundary of the ball $B(x_0, r_+)$.

In the same way, a polygon S_j from G_- is an *active polygon* for (x_0, r_+, r_-) iff the distance from S_j to x_0 is exactly r_-, that is, $d(x_0, S_j) = \min_{x \in S_j} \|x_0 - x\| = r_-$. We denote by $A_-(x_0)$ the set of active polygons from G_-.

When x_0 is clear from the context, we will simply write A_+ and A_-.

Showing that a feasible solution (x_0, r_+, r_-) is not optimal may be done in two ways: either find another feasible solution (x_0', r_+', r_-') with a better objective value or exhibit a direction of increase of f at x_0.

Theorem 1. *If (x_0, r_+, r_-) is an optimal solution, there exists at least one active element in each group G_+ and G_-, that is, the sets A_+ and A_- are non-empty.*

Proof. Suppose that A_+ is an empty set. Since (x_0, r_+, r_-) is a feasible solution of Problem (9), all the points of the group G_+ must be (due to the emptiness of A_+) contained strictly in the ball $B(x_0, r_+)$, that is, $\|x_0 - x_i\| < r_+, \forall x_i \in G_+$. Then, it is sufficient to take

$$r_+' = r_+(x_0) = \max_{x_i \in G_+} \|x_0 - x_i\|,$$

which is strictly smaller than r_+, and we obtain (x_0, r_+', r_-), a feasible solution improving strictly the value of the objective function.

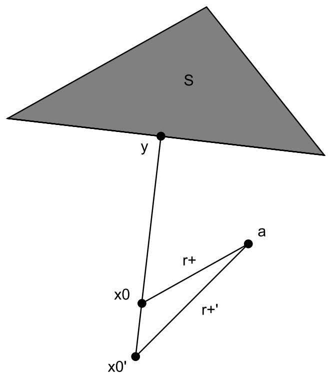

Fig. 3. Proof of Theorem 2, case $r_+ \leq r_-$.

On the other hand, suppose that A_- is empty. Due to the feasibility of (x_0, r_+, r_-), the distance from x_0 to every polygon of G_- is strictly greater than r_-, that is, $d(x_0, S_j) > r_-$, $\forall S_j \in G_-$. Thus, it is sufficient to consider

$$r'_- = r_-(x_0) = \min_{S_j \in G_-} d(x_0, S_j) = \min_{S_j \in G_-} \min_{x \in S_j} \|x_0 - x\|,$$

which is strictly greater than r_-, and the solution (x_0, r_+, r'_-) strictly improves the objective function.

In both cases, we conclude that the initial solution (x_0, r_+, r_-) cannot be optimal. □

Theorem 2. *For any optimal solution (x_0, r_+, r_-) one has*

1. *If $r_+ \leq r_-$, then there must exist at least two active polygons in G_-.*
2. *If $r_+ \geq r_-$, then there must exist at least two active points in G_+.*

Proof. By Theorem 1 there exists at least one active point a in G_+ and one active polygon S in G_-.

1. In case $r_+ \leq r_-$, suppose S is the only polygon in A_-. Let y be the projection of x_0 on S, i.e. the point in S such that $d(x_0, S) = \min_{x \in S} d(x_0, x) = d(x_0, y)$ and consider the direction $p = x_0 - y$. We will show that this vector p represents a direction of improvement for the objective function.
   If we move x_0 an amount $\epsilon > 0$, small enough (for not finding any new active element), in the direction $u = \frac{p}{\|p\|}$, we obtain that $x'_0 = x_0 + \epsilon u$ and $r'_- = r_- + \epsilon$.
   The other radius r'_+ must be measured as the maximum distance from x'_0 to the points belonging to $A_+(x_0)$.
   In case $r'_+ \leq r_+$, because the new center is closer to all the points in $A_+(x_0)$, the radii r'_+ and r'_- will have decreased and increased respectively, and consequently the objective function will also have strictly improved.
   Otherwise, the radius r'_+ will be the distance from x'_0 to the point a of $A_+(x_0)$ that is now the furthest one (see Fig. 3). Due to the triangle inequality on a, x_0 and x'_0, one has that $r'_+ \leq r_+ + \epsilon$, and the value of the objective function is strictly improved as soon as $r_+ < r_-$, since
   $$\begin{aligned} r'^2_- - r'^2_+ &\geq (r_- + \epsilon)^2 - (r_+ + \epsilon)^2 \\ &= r^2_- - r^2_+ + 2\epsilon(r_- - r_+) > r^2_- - r^2_+. \end{aligned}$$
   But also if $r_+ = r_-$ the points x'_0, a and y cannot be collinear since this would mean that $a = y$, contrary to our general assumption that no point of G_+ belongs to an element of G_-. Therefore x'_0, a and y are not collinear, so by the strict triangle inequality we have $r'_+ < r_+ + \epsilon$, and thus
   $$r'^2_- - r'^2_+ > (r_- + \epsilon)^2 - (r_+ + \epsilon)^2 = 0 = r^2_- - r^2_+.$$
2. When $r_+ \geq r_-$, suppose there is only one active point a in A_+. Then, the vector $p = a - x_0$ will be shown to represent a direction of improvement.
   If x_0 is moved an amount $\epsilon > 0$, small enough for not having new active elements, in the direction $u = \frac{p}{\|p\|}$, we obtain that $r'_+ = r_+ - \epsilon$. The radius r'_- will be the minimum distance from $x'_0 = x_0 + \epsilon u$ to the polygons in $A_-(x_0)$. If now $r'_- \geq r_-$, that is the new center is further from all the polygon candidates to become active, the two radii r'_+ and r'_- have improved and so also the objective function has. Any active polygon S for the center x_0 will also be the polygon closest to the new center x'_0, since there are no new active polygons in G_-. Let y be the projection of x_0 on S, i.e., the point of S such that $d(x_0, S) = \min_{x \in S} d(x_0, x) = d(x_0, y)$. Now the objective function can be expressed as follows:
   $$r^2_- - r^2_+ = \|x_0 - y\|^2 - \|x_0 - a\|^2.$$
   Three different situations must be considered.

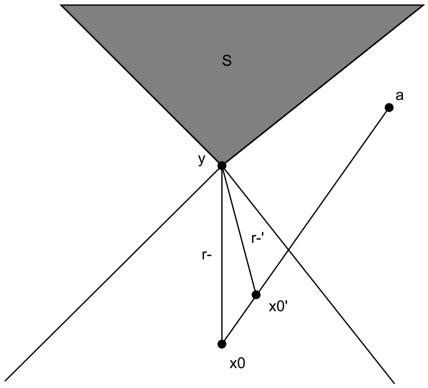

Fig. 4. Proof of Theorem 2, case $r_+ \geq r_-$: the distance to the polygon is measured in the vertex.

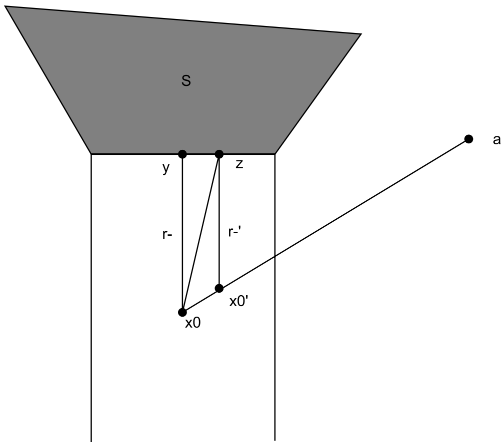

Fig. 5. Proof of Theorem 2, case $r_+ \geq r_-$: the distance to the polygon is measured in the edge.

- If y is a vertex of the polygon S and x_0 is strictly contained in the normal cone of S in y (denoted by $N_S(y)$), that is, x_0 satisfies $(x_0 - y)^t(y - s) > 0\ \forall s \in S$, then, for $\epsilon > 0$ small enough, x_0' will also be contained strictly in this normal cone, and $d(x_0', S) = \min_{x \in S} d(x_0', x) = d(x_0', y)$ (see Fig. 4).

  In that case, due to the triangle inequality, one has that $r_- \leq r_-' + \epsilon$ and consequently, $r_-' \geq r_- - \epsilon$, and the value of the objective function is improved in case $r_+ > r_-$, because

$$\begin{aligned} r_-'^2 - r_+'^2 &\geq (r_- - \epsilon)^2 - (r_+ - \epsilon)^2 \\ &= r_-^2 - r_+^2 + 2\epsilon(r_+ - r_-) > r_-^2 - r_+^2. \end{aligned} \tag{11}$$

  For $r_+ = r_-$, we know that $r_- < r_-' + \epsilon$, except for the case when x_0', y and a are collinear. But this situation is not possible for $r_+ = r_-$, because it would mean that $a \in S$, which is not allowed by assumption. Therefore,

$$r_-'^2 - r_+'^2 > (r_- - \epsilon)^2 - (r_+ - \epsilon)^2 = r_-^2 - r_+^2. \tag{12}$$

- If the point y is not a vertex of S it lies on (the relative interior of) an edge of S and the value $\epsilon > 0$ may be chosen small enough so that the point z of S closest to the new center lies on the same edge of the polygon (see Fig. 5).

  Using the definition of r_- and the triangle inequality on x_0, z and x_0' we obtain

$$r_- = \min_{x \in S} d(x_0, x) \leq d(x_0, z) \leq r_-' + \epsilon.$$

  Thus $r_-' \geq r_- - \epsilon$ and we obtain again the same inequalities as in (11) showing that the objective function is improved for $r_+ > r_-$.

  In case $r_+ = r_-$, we may again rule out x_0, y and a collinear because it would mean $a = y \in S$, contrary to assumptions. Therefore we obtain as above inequality (12), showing that the objective function is also improved.

- If y is a vertex of S and x_0 is on the boundary of the normal cone of S in y, then the projection of the new center x_0' on S will be either the same vertex y or a point z on an adjacent edge of S, depending on the position of a. Hence, one of the two arguments used previously applies to find a solution with a better value of the objective function. □

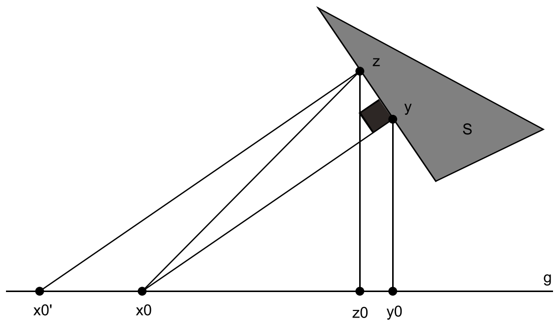

Fig. 6. Proof of Theorem 4, second part.

Remark 3. It can be proved that, when points of G_+ are allowed to lie in some polygon of G_-, Theorem 2 still holds in case of strict inequalities. However, when $r_+ = r_-$ one may only conclude existence of an optimal solution with two active elements in G_+ and of a (possibly different) optimal solution with two active elements in G_-.

Theorem 4. *For any optimal solution (x_0, r_+, r_-) the convex hulls $CH(A_+)$ and $CH(A_-)$ of the two groups of active elements intersect.*

Proof. Assume contrarily that $CH(A_+) \cap CH(A_-) = \emptyset$. Then a straight line h of equation $p^t x = c$ can be found which strictly separates these two convex hulls. We may choose p as a unit vector and $c \in \mathbb{R}$, such that the halfplane containing $CH(A_+)$ is defined by $\{p^t x > c\}$. Consider the straight line $g : \{x = x_0 + \lambda p,\ \lambda \in \mathbb{R}\}$. We now show that the objective function will be improved by moving x_0 along this straight line a certain amount $\epsilon > 0$.

Denote by S an active polygon from $A_-(x_0)$ which is the closest one to the new center $x'_0 = x_0 + \epsilon p$, and by a a point from $A_+(x_0)$ which maximizes the distance from x'_0 to $A_+(x_0)$. Denote by a_0 the orthogonal projection of a on g. Let y be the point of S such that $d(x_0, S) = \min_{x \in S} d(x_0, x) = d(x_0, y)$ and y_0 its orthogonal projection on the straight line g. With this notation, the objective function can be expressed as follows:

$$\begin{aligned} r_-^2 - r_+^2 &= \|x_0 - y\|^2 - \|x_0 - a\|^2 \\ &= \|x_0 - y_0\|^2 + \|y_0 - y\|^2 - \|x_0 - a_0\|^2 - \|a_0 - a\|^2. \end{aligned}$$

If we move x_0 to x'_0 along the straight line g, to measure the new radius r'_-, three different situations must be analyzed.

- In case the point y is a vertex of the polygon and x_0 is strictly contained in the normal cone of S in y, then, for an amount $\epsilon > 0$ small enough, the new center x'_0 will also be contained strictly in the normal cone, and the distance from x'_0 to S will continue being the distance from x'_0 to the vertex y, that is, $d(x'_0, S) = \min_{x \in S} d(x'_0, x) = d(x'_0, y)$.

  Then, since $p = \frac{a_0 - y_0}{\|a_0 - y_0\|}$ (observe that $a_0 \neq y_0$, because g is orthogonal to the separating hyperplane and hence, a_0 and y_0 are also separated by the straight line h), the following calculation shows that the objective function improves,

$$\begin{aligned} r'^2_- - r'^2_+ &= \|x_0 + \epsilon p - y_0\|^2 + \|y_0 - y\|^2 - \|x_0 + \epsilon p - a_0\|^2 - \|a_0 - a\|^2 \\ &= \|x_0 - y_0\|^2 + 2\epsilon (x_0 - y_0)^t p + \|y_0 - y\|^2 - \|x_0 - a_0\|^2 - 2\epsilon (x_0 - a_0)^t p - \|a_0 - a\|^2 \\ &= r_-^2 - r_+^2 + 2\epsilon (a_0 - y_0)^t \frac{a_0 - y_0}{\|a_0 - y_0\|} \\ &= r_-^2 - r_+^2 + 2\epsilon \|a_0 - y_0\| > r_-^2 - r_+^2. \end{aligned}$$

- If the point y is not a vertex of S it lies on (the relative interior of) an edge of S and the value $\epsilon > 0$ may be chosen small enough so that the point z of S closest to the new center x'_0 lies on the same edge of this polygon. (See Fig. 6.)

  Consider the orthogonal projection z_0 of z on the straight line g. Observe that $p = \frac{a_0 - z_0}{\|a_0 - z_0\|}$ ($a_0 \neq z_0$, because g is orthogonal to h and h separates a and z) and observe also that $x_0 - y$ and $z - y$ are orthogonal, because y is the projection of x_0 on the edge containing z. Therefore, by Pythagoras's Theorem, one has that

$$\begin{aligned} \|x_0 - z_0\|^2 + \|z_0 - z\|^2 &= \|x_0 - z\|^2 = \|x_0 - y\|^2 + \|y - z\|^2 \\ &\geq \|x_0 - y\|^2 = \|x_0 - y_0\|^2 + \|y_0 - y\|^2. \end{aligned} \tag{13}$$

  The objective function at x'_0 is then

$$\begin{aligned} r'^2_- - r'^2_+ &= \|x_0 + \epsilon p - z_0\|^2 + \|z_0 - z\|^2 - \|x_0 + \epsilon p - a_0\|^2 - \|a_0 - a\|^2 \\ &= \|x_0 - z_0\|^2 + 2\epsilon (x_0 - z_0)^t p + \|z_0 - z\|^2 - \|x_0 - a_0\|^2 - 2\epsilon (x_0 - a_0)^t p - \|a_0 - a\|^2 \end{aligned}$$

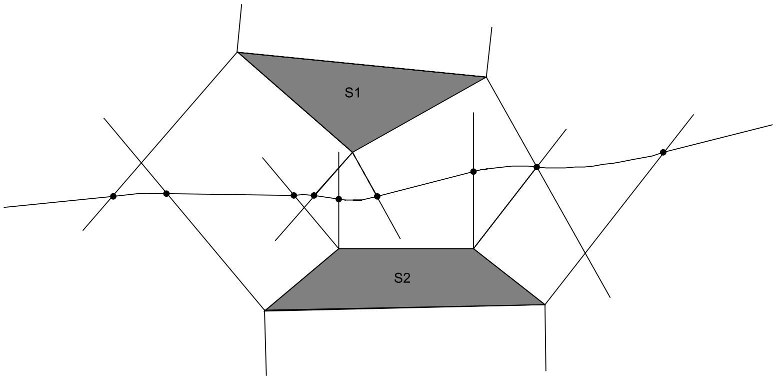

Fig. 7. Bisector of two polygons S_1 and S_2. Breakpoints •.

and using inequality (13) we obtain

$$\begin{aligned} r_-'^2 - r_+'^2 &\geq \|x_0 - y_0\|^2 + \|y_0 - y\|^2 - \|x_0 - a_0\|^2 - \|a_0 - a\|^2 + 2\epsilon(a_0 - z_0)^t p \\ &= r_-^2 - r_+^2 + 2\epsilon\|a_0 - z_0\| > r_-^2 - r_+^2. \end{aligned}$$

- In case y is a vertex of S and x_0 is on the boundary of the normal cone of S in y, then the projection of x_0' on S will be either the vertex y or a point z on an edge of S, and, depending on the position of a, one of the two previous arguments applies to find a solution which improves the objective function. □

Remark 5. For the following theorem, we consider the data to be in general position. This means that the following exceptional situations do NOT appear.

1. One point of G_+ and two vertices of two polygons of G_- are collinear.
2. One point of G_+ and one vertex of a polygon of G_- define an orthogonal direction to an edge of a polygon of G_-.
3. Two points of G_+ and one vertex of a polygon of G_- are collinear.
4. Two points of G_+ define an orthogonal direction to an edge of a polygon of G_-.

Clearly these exceptional situations are non-generic in the sense of occurring with zero probability. Would they occur, any small random perturbation of the data would destroy their presence. Therefore general position may be quite safely assumed. In several cases, though, the effect of their occurrence will be discussed anyway.

Likewise, the concept of bisector for two convex polygons and breakpoints will be necessary for the proof of Theorem 7 (see [11,27] for a detailed description).

Definition 6. The *bisector* of two convex polygons S_1 and S_2 is the locus of points $x \in \mathbb{R}^2$ satisfying $d(x, S_1) = d(x, S_2)$. This bisector is a continuous unbounded curve consisting of linear and parabolic segments. The points at which two such segments meet will be called *breakpoints* (see Fig. 7).

Theorem 7. *When the data are in a general position one of the following situations arises for any optimal solution* (x_0, r_+, r_-)*:*

1. *there exist at least four associated active elements;*
2. *there exist at least three active elements, two polygons* $S_1, S_2 \in A_-$ *and one point* $a \in A_+$*, satisfying that* y_1, y_2 *and* a *are collinear, with* y_i *such that* $d(x_0, S_i) = \min_{x \in S_i} d(x_0, x) = d(x_0, y_i)$, $i = 1, 2$*;*
3. *there exist at least three active elements, two polygons* $S_1, S_2 \in A_-$ *and one point* $a \in A_+$*, and* x_0 *is a breakpoint.*

Proof. In case the two radii are equal Theorem 2 immediately yields four active elements. Therefore the following two cases remain.

1. When $r_+ < r_-$, by Theorems 1 and 2, we know that an optimal solution must have at least two distinct active polygons $S_1, S_2 \in A_-$, and one active point, $a \in A_+$. Suppose an optimal solution (x_0, r_+, r_-) has been obtained with only these three active elements.
   Since x_0 must be at the same distance from the two active polygons of A_-, it must be along their bisector which is composed of line segments and pieces of parabola. So x_0 is either a breakpoint or an 'inner point' of such a segment or piece.
   It will suffice to show that the latter case cannot happen because a new better solution could then be found. To this end the following different cases must be considered.

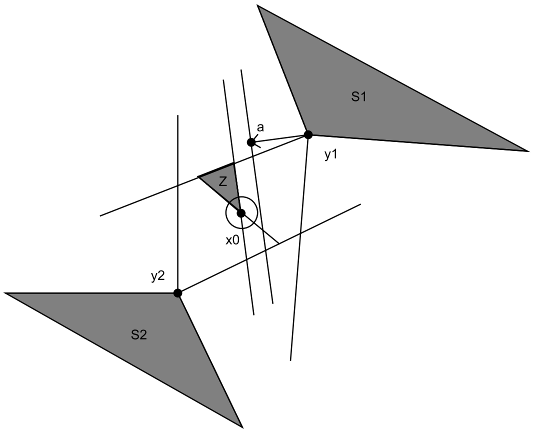

Fig. 8. Distances from x_0 to the polygons are the distances to two vertices.

- If the points of S_1 and S_2 closest to x_0 are vertices $y_1 \in S_1$ and $y_2 \in S_2 x_0$ lies on the mediatrix r of y_1 and y_2 (see Fig. 8). Suppose that the active point $a \in A_+$ is nearer to S_1 than to S_2 (the other case is analogous by symmetry).
Define R as the convex region determined by those points nearer to S_1 than to S_2 which are in the normal cone of S_1 at y_1, that is, $R = \{x : d(x, y_1) \leq d(x, y_2)\} \cap N_{S_1}(y_1)$. In this region define the function
$$g(x) = \|x - y_1\|^2 - \|x - a\|^2 = 2x^t(a - y_1) + \|y_1\|^2 - \|a\|^2. \quad (14)$$
One has that $f(x) = g(x), \forall x \in R$, with f the objective function of Problem (10), in particular, $f(x_0) = g(x_0)$.
In order to find a direction of improvement for the objective function in the neighborhood of x_0, we study the directional derivatives of the objective function f at this point. The function g is differentiable in the region R with gradient $\nabla g(x_0) = 2(a - y_1)$ at x_0. Since $f \equiv g$ in the convex region R the directional derivative along any vector $v = y - x_0$ with $y \in R$ equals
$$\nabla_v f(x_0) = \nabla_v g(x_0) = \nabla g(x_0)^t \cdot v = 2(a - y_1)^t v \quad (15)$$
and to obtain a direction of improvement of f, it is sufficient to choose such a vector v for which the scalar product $(a - y_1)^t v$ is strictly positive.
Consider the straight line $r : (a - y_1)^t(x - x_0) = 0$ through x_0 and orthogonal to the vector $(a - y_1)$ and construct the region Z determined by those points in R that are also in the positive halfplane defined by the straight line r: $Z = R \cap \{x : (a - y_1)^t(x - x_0) > 0\}$. Except in case the straight line coincides with the mediatrix, for some small enough $\epsilon > 0$ the intersection $Z \cap B(x_0, \epsilon)$ is not empty, and moving x_0 in direction of any point in this intersection will improve the objective.
The exceptional case where r coincides with the mediatrix is only possible if y_1, y_2, and a are collinear, hence corresponding to the non-generic situation number 1 in Remark 5 that was ruled out. One may note, however, that in this exceptional case, if we move x_0 along the mediatrix, the value of the objective function remains constant and the optimal solution is not unique.
- If the points of S_1 and S_2 closest to x_0 are a vertex $y_1 \in S_1$ and a point y_2 lying on the relative interior of an edge of S_2 then x_0 is not a breakpoint of the bisector of S_1 and S_2 and lies on a parabolic piece of it, the parabola being the bisector between the vertex y_1 and the edge of S_2 (see Fig. 9).
Suppose that the active point $a \in A_+$ is nearer to S_1 than to S_2 (for the other situation, see the reasoning described for the next case, with y_1 and y_2 lying on the edges of the polygons). Then a, y_1 and the parabola lie on the same side of the mediatrix of y_1 and y_2, and this mediatrix is tangent to the parabola at x_0.
Define the convex set $R = N_{S_1}(y_1) \cap \{x : d(x, y_1) \leq d(x, S_2)\}$ and the function g as above in (14). One has that $f(x) = g(x), \forall x \in R$ sufficiently close enough to x_0 not to introduce new active elements, and hence, the expression (15) for the directional derivative of f at x_0 along any vector $v = y - x_0$ with $y \in R$ remains valid and the objective improves along any such direction v for which $(a - y_1)^t v$ is strictly positive
Consider then the straight line r containing x_0 and orthogonal to $(a - y_1)$, so of equation $(a - y_1)^t(x - x_0) = 0$. It cannot be tangent to the parabola since this would mean it coincides with the mediatrix of y_1 and y_2, so that y_1, a, x_0 and y_2 are collinear, and hence that $a - y_1$ would be orthogonal to the edge containing y_2, a situation we ruled out as non-generic case number 2 of Remark 5 (however, in that case a local optimum has been found).
Therefore the mediatrix of y_1 and y_2 and the line r cross at x_0, implying we may find some z different from (but close to) x_0 such that $(a - y_1)^t(z - x_0) > 0$ and lying on the same side of the parabola as a, so for which we have $d(x, y_1) < d(x, S_2)$, thus lying in the interior of R (the set of points satisfying these two constraints is shown as Z in Fig. 9). Hence the direction $v = z - x_0$ improves the objective strictly.
- If the points y_1 of S_1 and y_2 of S_2 closest to x_0 are both on the relative interior of respective edges of S_1 and S_2, x_0 lies on the bisectrix of the angle formed by these two edges, and this latter is the bisector of S_1 and S_2 within some neighborhood of x_0 (see Fig. 10).

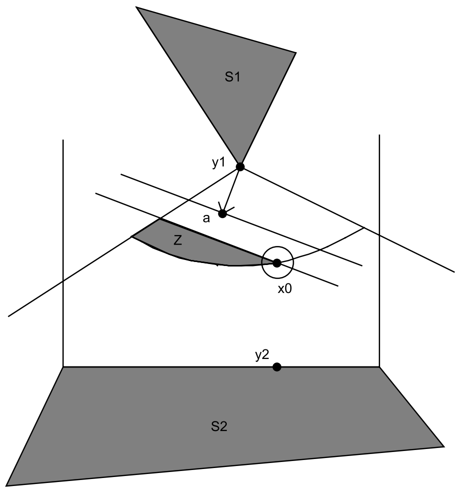

Fig. 9. Distances from x_0 to the polygons are the distances to a vertex and to an edge.

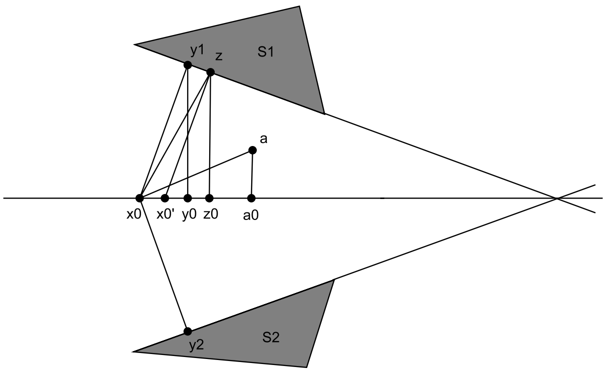

Fig. 10. Distances from x_0 to the polygons are the distances to two edges.

Suppose that the active point $a \in A_+$ is nearer to S_1 than to S_2 (by symmetry, the other case is analogous).

Denote by a_0 and y_0 the orthogonal projections of a and y_1 on the bisectrix. Then, the objective function in x_0 can be written as

$$\begin{aligned} r_-^2 - r_+^2 &= \|x_0 - y_1\|^2 - \|x_0 - a\|^2 \\ &= \|x_0 - y_0\|^2 + \|y_0 - y_1\|^2 - \|x_0 - a_0\|^2 - \|a_0 - a\|^2. \end{aligned}$$

We proceed to show that the vector $p = a_0 - y_0$, if non-zero (the opposite case is handled separately afterwards), is a direction of improvement of the objective function.

Let us move x_0 along the direction p by an amount $\epsilon > 0$. Denote by z be the orthogonal projection of the new center $x_0' = x_0 + \epsilon p$ on the edge of S_1 and by z_0 its orthogonal projection on the bisectrix. The new value of the objective function in x_0' is

$$\begin{aligned} r_-'^2 - r_+'^2 &= \|x_0 + \epsilon p - z_0\|^2 + \|z_0 - z\|^2 - \|x_0 + \epsilon p - a_0\|^2 - \|a_0 - a\|^2 \\ &= \|x_0 - z_0\|^2 + 2\epsilon(x_0 - z_0)^t p + \|z_0 - z\|^2 - \|x_0 - a_0\|^2 - 2\epsilon(x_0 - a_0)^t p - \|a_0 - a\|^2. \end{aligned}$$

Note that $y_1 \neq z$, so by Pythagoras's Theorem we obtain

$$\begin{aligned} \|x_0 - z_0\|^2 + \|z_0 - z\|^2 &= \|x_0 - z\|^2 = \|x_0 - y_1\|^2 + \|y_1 - z\|^2 \\ &> \|x_0 - y_1\|^2 = \|x_0 - y_0\|^2 + \|y_0 - y_1\|^2. \end{aligned}$$

It then follows that the objective function improves by

$$\begin{aligned} r_-'^2 - r_+'^2 &= \|x_0 - z_0\|^2 + \|z_0 - z\|^2 - \|x_0 - a_0\|^2 - \|a_0 - a\|^2 + 2\epsilon(a_0 - z_0)^t p \\ &> \|x_0 - y_0\|^2 + \|y_0 - y_1\|^2 - \|x_0 - a_0\|^2 - \|a_0 - a\|^2 + 2\epsilon(a_0 - z_0)^t p \\ &= r_-^2 - r_+^2 + 2\epsilon(a_0 - z_0)^t(a_0 - y_0) \geq r_-^2 - r_+^2 \end{aligned}$$

where the last inequality holds because for $\epsilon > 0$ small enough the vectors $(a_0 - y_0)$ and $(a_0 - z_0)$ are parallel and in the same sense.

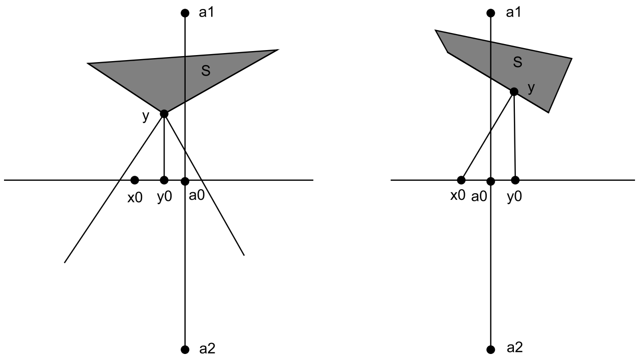

Fig. 11. Two situations when $r_+ > r_-$.

In case $a_0 = y_0$, the objective function cannot be improved and we have obtained a local optimal solution. But this corresponds to situation 2 of Theorem 7, that is, three active elements (two polygons S_1, S_2 and one point a) with the points y_1, a and y_2 being collinear.

2. When $r_+ > r_-$, by Theorems 1 and 2, we know that an optimal solution of the problem must have at least two active points $a_1, a_2 \in A_+$, and one active polygon, $S \in A_-$. Suppose an optimal solution (x_0, r_+, r_-) with only these three active elements has been obtained. A new solution will be found with a better value of the objective function.
   Denote by r the mediatrix between the two active points of A_+, by y the point belonging to S such that $d(x_0, S) = \min_{x \in S} d(x_0, x) = d(x_0, y)$, and by a_0 and y_0 the orthogonal projections of the points a_1 (or equivalently a_2) and y on the straight line r. We have to consider two situations (y is a vertex of the polygon or y lies on an edge of the polygon, see Fig. 11), which are exactly the same as those described in the proof of Theorem 4.
   With a similar reasoning, we derive that a new feasible solution can be obtained which improves the objective function, except when a_1, y, x_0 and a_2 are collinear. But this can only happen in the following two non-generic cases that were ruled out: either when a_1, a_2 and the vertex y of a polygon S are collinear (exception 3 in Remark 5) or when the two points a_1, a_2 define an orthogonal direction to an edge of a polygon S (exception 4 in Remark 5). □

The concepts of nearest and farthest-point Voronoi diagrams (see [27,30]) for a set of points or polygons will be necessary for the proof of Theorem 9.

Definition 8. Given the set of points $\{x_1, \ldots, x_n\}$ and the set of polygons $\{S_1, \ldots, S_m\}$, the farthest-point (resp. nearest-polygon) Voronoi cell associated to x_k (resp. S_l) denoted by V_k (resp. W_l) is defined as follows:

$$V_k = \bigcap_{i \in \{1,\ldots,n\} \setminus \{k\}} \{x : d(x, x_k) \geq d(x, x_i)\}, \tag{16}$$

$$W_l = \bigcap_{j \in \{1,\ldots,m\} \setminus \{j\}} \{x : d(x, S_l) \leq d(x, S_j)\}. \tag{17}$$

The sets $V = \bigcup_{k=1,\ldots,n} V_k$ and $W = \bigcup_{l=1,\ldots,m} W_l$ are called the *farthest-point and the nearest-polygon Voronoi diagrams*.

Theorem 9. *If the convex hulls of the two groups G_+ and G_- are disjoint, that is, $CH(G_+) \cap CH(G_-) = \emptyset$, then the solution is unbounded and the separating balls are transformed into straight lines.*

Proof. Since $CH(G_+) \cap CH(G_-) = \emptyset$, a straight line $h : \{p^t x = c\}$, with $p \in \mathbb{R}^2$ and $c \in \mathbb{R}$, strictly separating the two convex hulls can be found, in the same way as done in the proof of Theorem 4. Let $l : \{p^t x = c'\}$ be another straight line, parallel to h, such that every point $x_k \in G_+$ satisfies that $p^t x_k > c$ and $p^t x_k < c'$.

Construct the farthest-point and nearest-polygon Voronoi diagrams in the plane for G_+ and G_-, respectively, and the intersection of the two diagrams. Note that these diagrams and thus their intersection contain unbounded cells in every direction. Let V be such a cell that is unbounded in direction p. Then it contains a point x_0 with $\{p^t x_0 > c'\}$ for which the half-line $r : \{x = x_0 + \lambda p,\ \lambda \geq 0\}$ is completely included in the cell V.

Once x_0 is chosen, since it is inside a cell of the intersection of the two diagrams, the farthest point in G_+, say a, and the nearest polygon in G_-, say S, are known, that is, $a \in A_+$ and $S \in A_-$, and these two elements remain active for all the possible solutions in the cell V, in particular for all the possible solutions in r. Then, with a similar reasoning as in the proof of Theorem 4 (either when the point of S closest to x_0 is a vertex of S or on an edge of S), one always has that $f(x_0 + \lambda p)$ is linearly increasing in $\lambda > 0$.

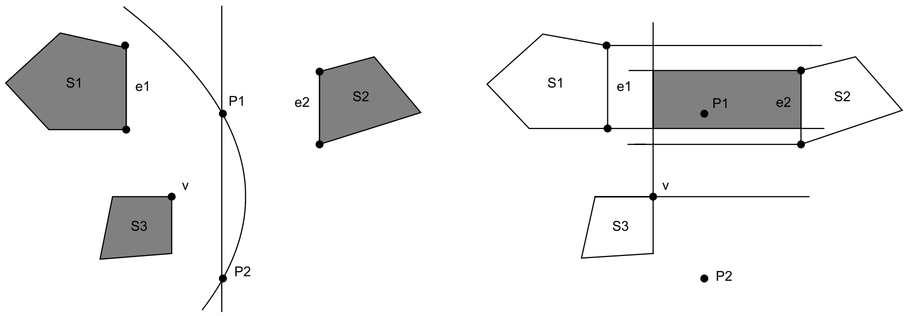

Fig. 12. Left: computing the intersection of a bisectrix and a parabola. Right: checking the feasibility of the two points.

It follows that the optimal solution is unbounded, or more precisely does not exist. In fact this rather means that the balls are transformed into two straight lines $\{p^t x = b\}$ and $\{p^t x = d\}$, with $b > d$, such that the closed halfplane $\{p^t x \geq b\}$ contains $CH(G_+)$ whereas $\{p^t x \leq d\}$ contains $CH(G_-)$. □

4. An algorithm to build the set of optimal solutions

With the necessary optimality conditions studied in the previous section a finite dominating set of solutions has been obtained. Once it has been checked that the data are in general position, a method to find an optimal solution is to perform a complete enumeration of all the candidate solutions, as will be described below.

We are going to study all the local optimal solutions, and we will compute the objective value for those points. The one with the highest value will be the global optimal solution. According to Theorems 1, 2 and 7, there must exist at least one active element in each set (A_+ and A_-), there must exist at least two active elements in the set associated to the biggest ball (that is, if $r_+ > r_-$, there will exist at least two active points in A_+, and if $r_- > r_+$, there will be at least two active polygons), and one of the situations described in Theorem 7 must be reached. The finite dominating set of solutions will therefore be formed by points x_0 whose configuration of associated active elements belongs to one of the following options:

1. three active polygons S_1, S_2, S_3 and one active point a (in this case, $r_- > r_+$);
2. two active polygons S_1, S_2 and two active points a_1, a_2 (no condition on the radii);
3. two active polygons S_1, S_2, one active point a and x_0 is a breakpoint of the bisector defined by S_1 and S_2 (in this case, $r_- > r_+$);
4. two active polygons S_1, S_2, one active point a and x_0 satisfies that y_1, y_2 and a are collinear, with y_i such that $d(x_0, S_i) = \min_{x \in S_i} d(x_0, x) = d(x_0, y_i)$, $i = 1, 2$ (in this case, $r_- > r_+$);
5. three active points a_1, a_2, a_3, and one active polygon S (in this case, $r_- < r_+$).

In the algorithm to construct all the candidates we will consider all the possible configurations. The solution x_0 is each time computed as the intersection of the corresponding bisectors of the sets A_+ and A_-. Since the bisector of two polygons consists of segments (for two vertices, the bisector is their mediatrix, and for two edges, the bisector is the bisectrix) and pieces of parabola (for one vertex and one edge), we will study each vertex and edge of a polygon as different active elements in the algorithm.

4.1. *Case 1:* $\text{card}(A_+) = 1$ *and* $\text{card}(A_-) = 3$

Let S_1, S_2 and S_3 be the three active polygons. As mentioned before every vertex and every edge of a polygon is studied as a possible active element. For a polygon S, considering a vertex v as the active element will mean that the closest point of S to the solution x_0 is v. Analogously, considering an edge e as the active element will mean that the point of S closest to x_0 lies on this edge e and is not one of the two vertices of e.

Thus x_0 is computed by the following strategies depending on the number of active vertices and edges:

- Three vertices: x_0 is the circumcenter of these three points (equivalently, x_0 is the intersection of the mediatrices for any pair of points).
- Two vertices and one edge: x_0 is the intersection of the mediatrix of the vertices and the parabola of any vertex and the edge.
- One vertex and two edges: x_0 is the intersection of the bisectrix of the two edges and the parabola of the vertex and any edge.
- Three edges: x_0 is the intersection of any two bisectrices.

Once x_0 is computed (in some cases, more than one solution can be obtained), the next step is to check if this solution is feasible, that is, given the three active elements, we must find out if x_0 belongs to the intersection of the normal cones of the polygon S_i at the vertex v_i or the edge e_i respectively, for $i = 1, 2, 3$, otherwise it must be rejected.

An example of this situation can be seen in Fig. 12. On the left there are two active edges e_1 and e_2, and one active vertex v (belonging, respectively, to the active polygons S_1, S_2 and S_3). The bisectrix for the two edges is computed, and as well the parabola which represents the bisector of the edge e_2 and the vertex v. There exist two points (P_1 and P_2) as the result of intersecting the bisectrix and the parabola. On the right we check the feasibility of these two possible solutions, and P_1 is accepted as a solution, because it belongs to the intersection of the normal cones of the three active elements (the shadowed rectangle in the picture) whereas P_2 is rejected because it is outside that rectangle.

If we obtain a solution x_0 with this combination of active elements, we define r_- as the distance from x_0 to any of these active elements. Observe that we must also check that the distance from x_0 to these active polygons S_i, $i = 1, 2, 3$, coincides with the distance to the active vertices or edges which have been considered, that is, the closest points from the polygons to x_0 must be the selected active vertices or must lie on the selected active edges (otherwise, the solution is not feasible).

Afterwards, we compute the distance from x_0 to the rest of polygons of G_-. If the minimum of these distances is larger than or equal to r_- (if this minimum was smaller, the polygons S_1, S_2 and S_3 could not belong to A_-), we compute r_+ as the maximum distance from x_0 to the points of G_+, and the point a whose distance to x_0 is r_+ will be the fourth active element.

In this case, r_+ must be smaller than r_- to have the guarantee of having obtained a local optimal solution (otherwise, according to Theorem 2 a better solution can be found in a neighborhood of x_0).

4.2. *Case 2:* $\text{card}(A_+) = 2$ *and* $\text{card}(A_-) = 2$

Let a_1 and a_2 be the active points. Let S_1 and S_2 be the active polygons (in this case, we choose immediately all four active elements). We compute the mediatrix of the two active points, and we compute the bisector of the two active elements in the polygons (it will be a mediatrix if we have two active vertices, a bisectrix if there are two active edges, or a parabola in case of a vertex and an edge). The intersection of the mediatrix and the bisector is computed and we check the feasibility of this solution as done in the previous case (that is, we check if the solution x_0 belongs to the intersection of the normal cones of the polygons at the corresponding vertex or edge, and we also check that each selected vertex or edge is really active, in the sense that it is or it contains the closest point from the corresponding polygon to x_0).

Once a solution x_0 is obtained, we compute r_+ as the distance from x_0 to one of the active points and r_- as the distance to one of the active polygons. Then, we compute the maximum distance from x_0 to the rest of the points of G_+ (x_0 is a candidate only if this maximum distance is smaller than or equal to r_+) and the minimum distance from x_0 to the rest of the polygons of G_- (x_0 is the only candidate to be optimal if this minimum distance is at least r_-).

4.3. *Case 3:* $\text{card}(A_+) = 1$, $\text{card}(A_-) = 2$ *and* x_0 *is a breakpoint*

Let S_1 and S_2 be the two active polygons. We will inspect all breakpoints along their bisector. This latter consists of alternating linear pieces and parabolic pieces, some of which may (accidentally) be reduced to a single point (see Fig. 7). So each breakpoint always has two active vertices, one of each polygon, and for these active vertices there are at most two corresponding breakpoints that form the endpoint(s) of the line-segment of the mediatrix of the vertices within the intersection of both normal cones of the respective polygons at these same vertices.

So for both polygons we have to consider only the vertices as active elements in the enumeration, and each pair will yield at most two breakpoints. For each such breakpoint x_0 we compute r_- as the distance from x_0 to their active vertices. Then, we compute the distances from x_0 to the remaining polygons, rejecting x_0 if some of these distances are less than r_-. We finally compute r_+ as the maximum distance from x_0 to the points in G_+, possibly rejecting this solution as soon as we find some distance (and thus r_+) to be at least r_-.

4.4. *Case 4:* $\text{card}(A_+) = 1$, $\text{card}(A_-) = 2$ *and* y_1, y_2 *and* a *are collinear*

Let a be the active point and S_1, S_2 be the two active polygons. Three cases are possible for y_1 and y_2: two vertices, one vertex and one on an edge, or both on an edge. The first case is ruled out for data in the general position (case 1 of Remark 5).

In order to construct possible instances of the second case we consider each point $a \in A_+$ in turn as active, and study if the line connecting a with some vertex y_1 of an $S_1 \in A_-$ cuts an edge e_2 of some $S_2 \in A_-$. In this case we call the cutpoint y_2. On the line through y_2 orthogonal to e_2 we construct the point x_0 at equal distance from y_2 and y_1. Such a case is illustrated in Fig. 13 on the left.

The instances of the third case are found as follows. Given a' and the edges e_1 and e_2, we study if there can exist two points y_1' and y_2' lying on the edges, such that the condition of collinearity is satisfied. If this is possible, we compute the bisectrix r of the two edges and the orthogonal straight line r' to the bisectrix containing the point a' (see Fig. 13). Let y_1' and y_2' be the intersection of r' with e_1 and e_2, respectively. Then, x_0' will be built as the intersection of the bisectrix with the orthogonal straight line to e_1 containing y_1' (see Fig. 13, right). Symmetrically, we can also do the same using e_2 and y_2'.

Once x_0 is built, we follow the same procedure to build the radii and to check feasibility as in case 2.

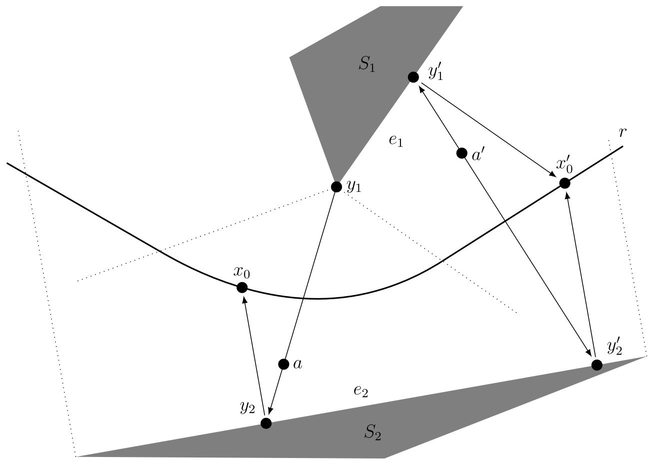

Fig. 13. Case 4. Constructing x_0.

4.5. *Case 5:* $\text{card}(A_+) = 3$ *and* $\text{card}(A_-) = 1$

Note first that this case can only arise for instances of the problem with negative optimal value. Hence, if we previously found a candidate solution with positive value, we do not need to consider this case at all. For this reason this case should better be kept till the end.

Let a_1, a_2, a_3 be the three active points, we compute x_0 as their circumcenter (equivalently, x_0 is the intersection of the mediatrices between these points), and $r_+ = d(x_0, a_i)$, for any $i = 1, 2, 3$.

Now, we compute the distance from x_0 to the remaining points of G_+. Only in case the maximum of these distances is smaller than or equal to r_+ (if it is bigger than r_+, the points a_i, $i = 1, 2, 3$, cannot be active), we compute r_- as the minimum distance from x_0 to the polygons of G_-. The polygon S whose distance to x_0 is equal to r_- will be the fourth active element.

Finally, r_+ must be bigger than r_- to ensure that we have a local optimal solution (otherwise, a better solution can be found in a neighborhood of x_0, according to Theorem 2).

4.6. *Computational considerations*

4.6.1. *Complexity*

Let us now study the total number of candidate points that are inspected during the algorithm. Denote by n the number of points in G_+, by m the number of polygons in G_- and by k the maximum number of vertices of these polygons.

For the candidate solutions of type 1, we need to study all the possible combinations of three polygons, and all the possible combinations of vertices and edges that define different active elements. This yields a set of $\mathcal{O}(k^3m^3)$ points.

For the candidates of type 2, we need to select two polygons and every possible combination of active elements (vertices and edges of these two polygons) and two vertices. We have then $\mathcal{O}(n^2k^2m^2)$ points.

Two polygons are needed to build each candidate of type 3. We have $\mathcal{O}(k^2m^2)$ such points. For the candidates of type 4, we need to consider all possible combinations of edges of two polygons and one point, yielding $\mathcal{O}(nk^2m^2)$. Finally, if we need to compute the candidates of type 5, we need to study all the combinations of three active points, and we have $\mathcal{O}(n^3)$ points.

The overall number of candidate points is therefore $\mathcal{O}((n + km)^3 + n^2k^2m^2)$.

Since in each case constructing and evaluating the candidate point is clearly polynomial (naively at most $\mathcal{O}(n + km)$), we obtain a polynomial algorithm at most of $\mathcal{O}((n + km)^4 + n^3(km)^2 + n^2(km)^3)$.

Note that if one knows in advance that the optimal value will be positive, i.e. that the original constraint set (1)–(2) is feasible, or if one is only interested in such cases and accepts a simple 'no' answer in case of non-feasibility, the number of cases to consider is reduced to $\mathcal{O}((km)^3 + (nkm)^2)$ and the overall complexity to $\mathcal{O}((n^2 + km)(n + km)(km)^2)$. We will call this 'the original problem'.

4.6.2. *Preprocessing*

The effort needed for the enumeration algorithm may often considerably be reduced in practice by some preprocessing that eliminates points, edges and/or vertices that can certainly not yield candidate optimal solutions. The following geometrical conditions are of this kind. The first is always applicable, the last two only for the original problem.

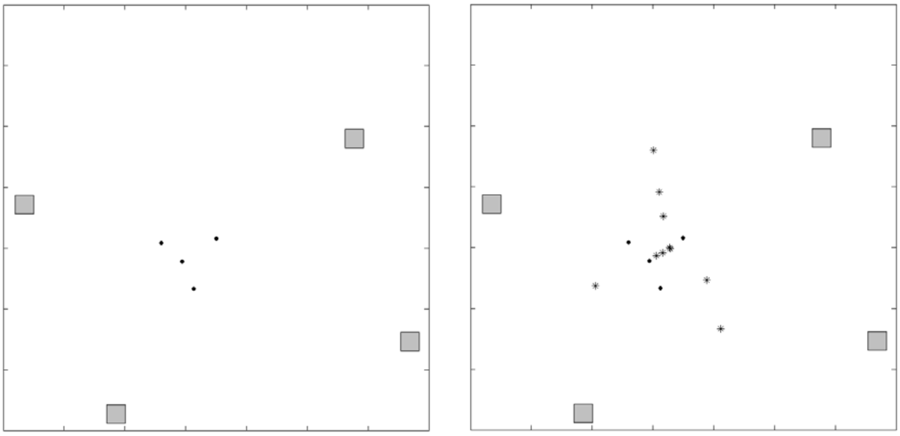

Fig. 14. Left: initial scenario. Right: set of candidate optimal solutions.

Points For any solution x_0 any active point in A_+ maximizes the distance $d(x, a)$ over $a \in G_+$, so by convexity maximizes also the distance $d(x, c)$ over all $c \in CH(G_+)$, the convex hull of G_+. Since this maximum will always be reached in some extreme point of $CH(G_+)$ it follows that the set G_+ may be reduced to the usually much smaller set EG_+ of such extreme points, without modifying the optimal solutions. This may be achieved easily as a preprocessing through some convex hull algorithm, see e.g. [11].

Vertices Let v be an active vertex of some polygon. This means that the circle $B(x_0, r_-)$ contains v on its boundary, but no other points of any polygon of G_- in its interior. If $r_+ \le r_-$ all points of G_+ lie within this circle and therefore, by convexity, the circle contains $CH(G_+ \cup \{v\})$. It follows that no point of the polygons of G_- lies on a half-open segment $]v, c]$ for any $c \in CH(G_+)$. We will say in this case that v 'sees' G_+.
Therefore when $r_+ \le r_-$ no vertex of a polygon that does not see G_+ can be active.
Such vertices can be identified a priori using visibility techniques from computational geometry, see e.g. [11], and then further ignored in the enumeration algorithm.

Edges Let now e be an active edge of some polygon of G_-, and let p be the touching point of e with the circle $B(x_0, r_-)$. The same reasoning as above shows that the point p must see G_+.
Therefore when $r_+ \le r_-$ and none of the points of an edge e of a polygon sees G_+ it follows that e cannot be active. Such fully 'blind' edges can be similarly identified a priori using visibility techniques from computational geometry, see e.g. [11], and then further ignored in the enumeration algorithm.

5. Illustrative examples

The algorithm described in the previous section to compute an optimal solution of our problem by complete enumeration of all the possible candidates (without preprocessing) has been implemented in Matlab 6.5. Some numerical tests have been performed with artificial databases, built at random, but always such that the original problem was feasible.

5.1. Small dataset: comparing areas for all the candidates

The first example is a small dataset (4 points and 4 squares) to illustrate the types of candidate solutions that can arise. We generated 4 points for the group G_+ (uniformly distributed $U(-5, 5)^2$). The polygons for the group G_- were chosen to be 4 squares of equal size 2×2, and random center points (also uniformly distributed $U(-20, 20)^2$, but outside the box enclosing the 4 first points). Our aim is to locate a single semi-obnoxious facility in a point $x_0 \in \mathbb{R}^2$, or equivalently, to compute two concentric balls such that $B(x_0, r_+)$ contains all the points and $B(x_0, r_-)$ does not intersect any squares. Fig. 14 (left) shows a picture of the artificial database.

All the candidate optimal solutions have been computed via the method described in Section 4, by taking into account all the possible combinations of active elements. Fig. 14 (right) shows this set of candidate locations, represented by stars.

In Fig. 15, we show the two candidates with a configuration of type 1 (according to the previous section), that is, there are three active polygons (squares) and one active point. In the picture, the active squares are the black ones, while the active point is encircled. These active elements (points and squares) lie on the boundary of the balls $B(x_0, r_+)$ and $B(x_0, r_-)$, respectively, where x_0 is represented by a star. Maximizing the objective function is equivalent to maximizing the area of the annulus defined by the boundaries of the two balls.

In Fig. 16, the three candidate solutions have two active points and two squares. In Fig. 17, we show five candidates with two active squares and one active point, and x_0, the location of the facility, is a breakpoint of the bisector defined by the two active squares. Observe that, although the active elements are the same for the three first pictures in this configuration, the

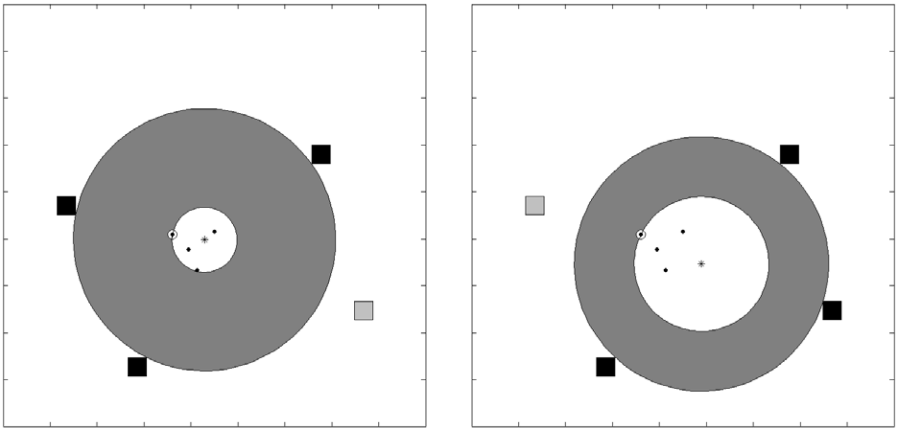

Fig. 15. Candidates type 1. Area of the annulus: 183.27 and 132.32, respectively.

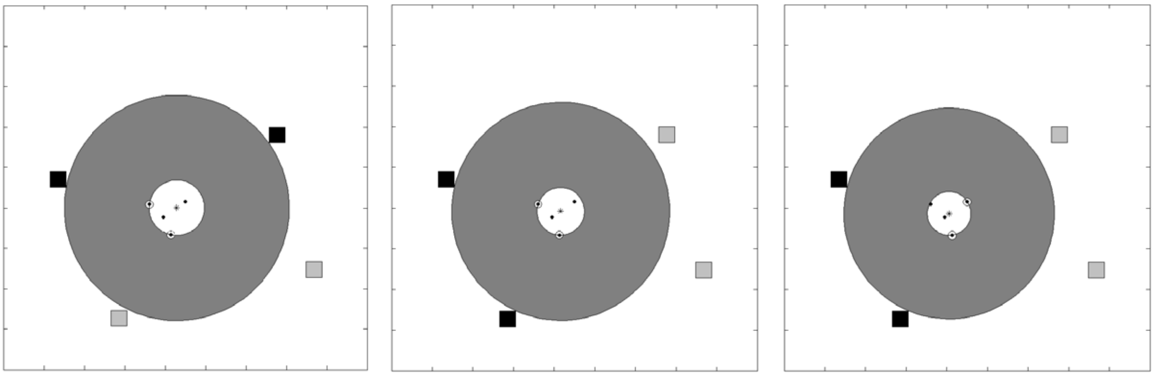

Fig. 16. Candidates type 2. Area of the annulus: 182.32, 171.45 and 160.97, respectively.

solutions are different because the centers of the balls are different breakpoints of the same bisector. Due to the definition of breakpoint, one of the active squares in this kind of solutions has a vertex as the active element, but the adjacent edge touches tangentially the ball $B(x_0, r_-)$. Hence, one can say that the two elements (the vertex and the edge) can be considered as active.

In this example, there are no candidate solutions with a configuration of type 4. Type 5 is excluded because we found feasible solutions with positive value.

If we compare the ten areas (that is, the values of the objective function), we find that the point x_0 illustrated in the first picture in Fig. 15 is optimal for our problem.

5.2. Medium dataset

Other larger databases have been similarly generated to test the algorithm. In the next example we generated at random 50 points for the group G_+ and 20 points as the centers of the squares of G_-. Fig. 18 illustrates the dataset.

By means of the method described in Section 4, all the candidate optimal solutions have been studied. The two pictures in Fig. 19 show (at different zoom levels) all 37 obtained candidate locations represented by stars. All the stars far from the set of squares and points represent local optima with a negative value of the objective function. These solutions have at least two active points associated (configurations of type 2 and 5), and represent local optima of formulation (10) with negative objective value. Since our dataset is spherically separable these cannot be global optima.

Fig. 20 shows the optimal solution for this dataset. The solution x_0, represented by a star, has two active points associated (encircled) and two active squares (in both of them, the point lying on the boundary of the ball is a vertex).

5.3. Large dataset

Finally, we show in Fig. 21 a larger random database with 100 points and 50 squares (smaller than above). In Fig. 22 we can see the finite dominating set on two different zoom levels. In total there are 42 candidate points, but most have negative objective value.

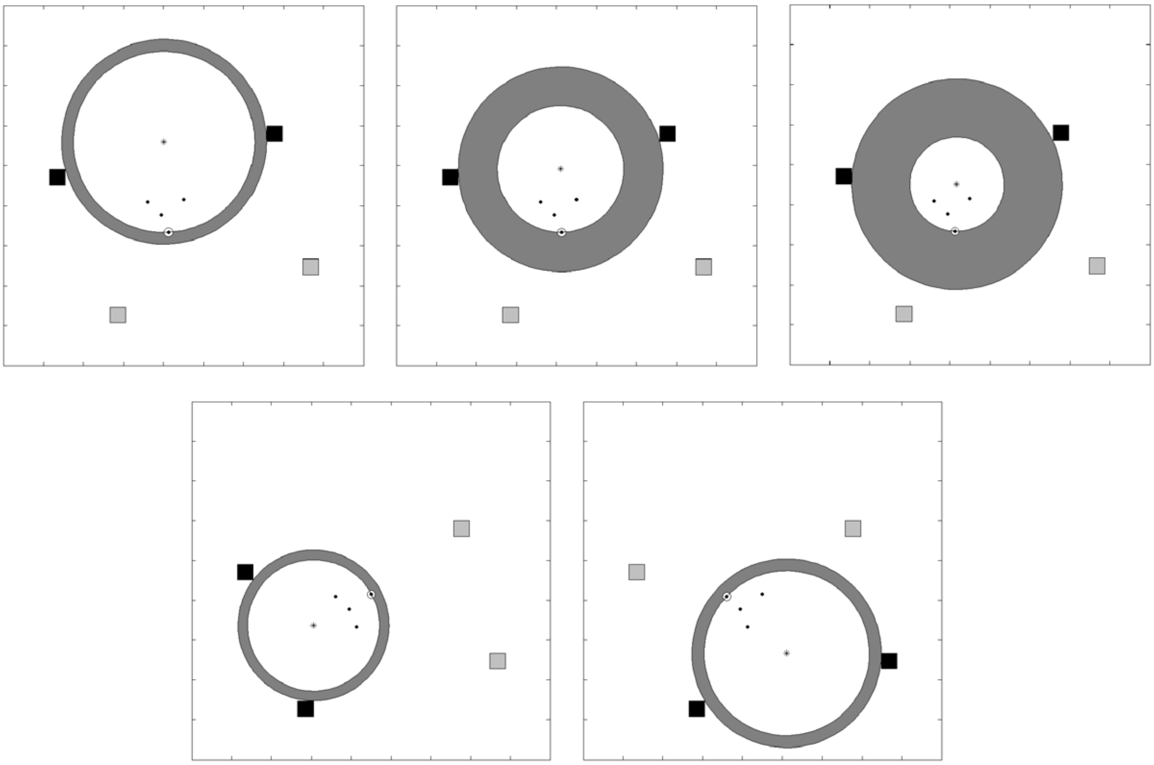

Fig. 17. Candidates type 3. Area of the annulus: 35.648, 101.54, 138.16, 21.53 and 33.932, respectively.

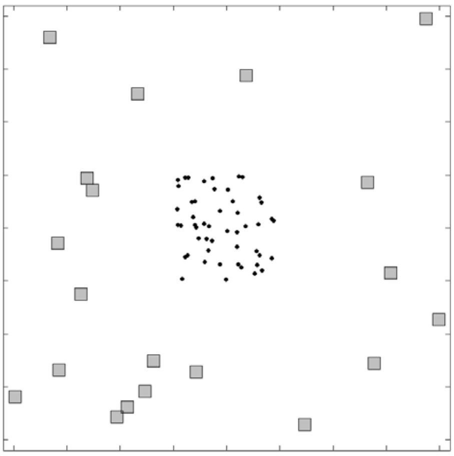

Fig. 18. Initial scenario (50 points and 20 squares).

However as the database is spherically separable we have some solutions with a positive value. All 7 candidates with positive value are depicted in Fig. 23. The optimal location of the facility and the separating balls are shown in Fig. 24. Again, the solution has four active elements associated with it, two active points and two active polygons. The ball of radius r_- touches these two squares at one vertex of each square.

Observe the quite small number of candidates in the finite dominating set, as compared to the cardinality of the number of cases considered as calculated in the complexity study in Section 4.6.1. This is due to the fact that most cases considered during its construction are found to be unsuitable and are therefore discarded. Some of this work will be avoided by the preprocessing.

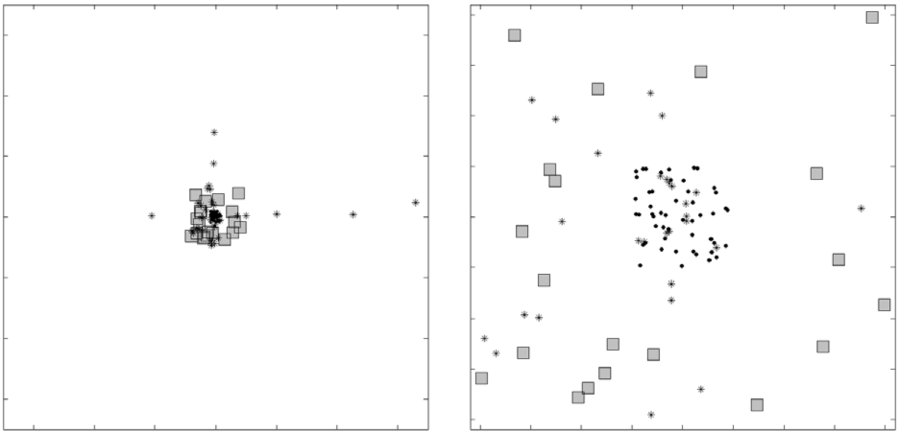

Fig. 19. Candidates to optimal solution.

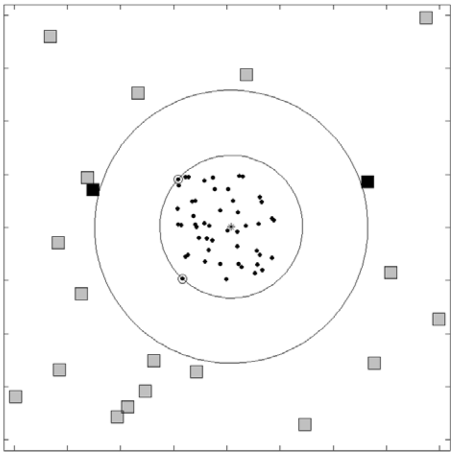

Fig. 20. Optimal solution.

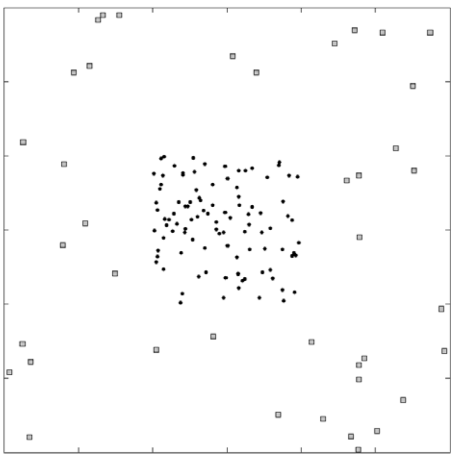

Fig. 21. Initial scenario (100 points and 40 squares).

6. Conclusion and extensions

In this work, the problem of locating a single semi-obnoxious facility in the Euclidean plane with repelling areas has been solved. The idea of maximizing a margin, as done in the field of Support Vector Machines, has been introduced to define the concept of solution.

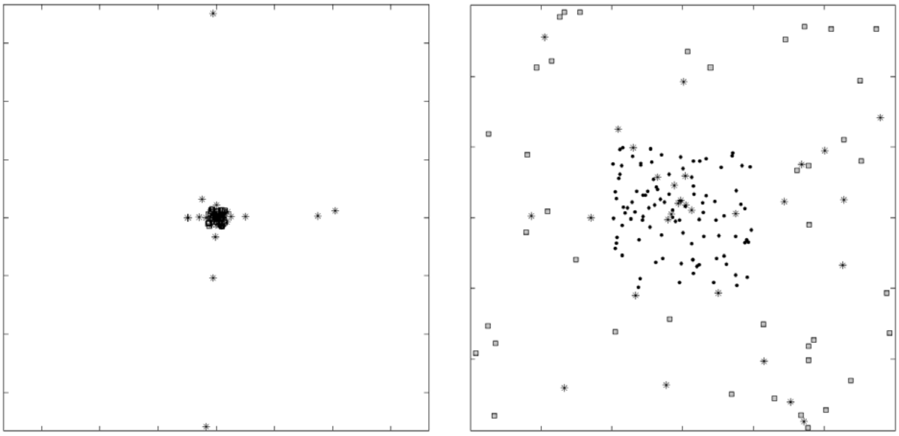

Fig. 22. Finite dominating set.

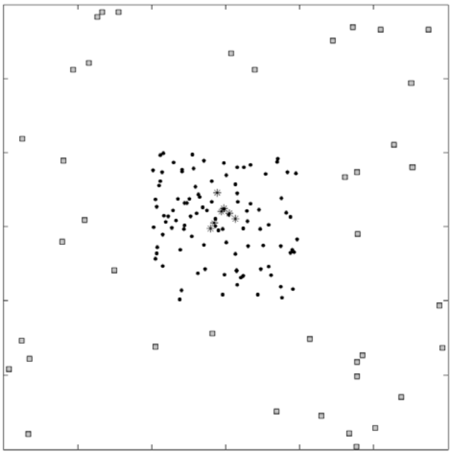

Fig. 23. Candidates with positive value of the objective function.

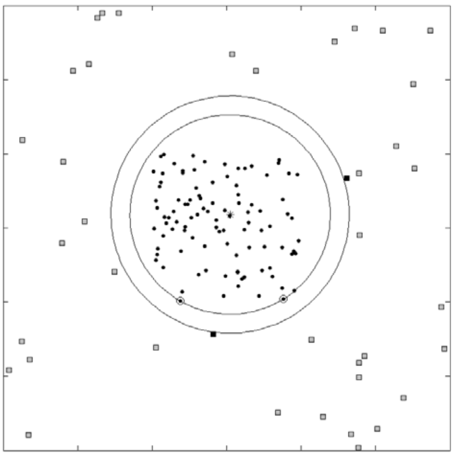

Fig. 24. Optimal solution.

The problem has been formulated as a nonlinear continuous optimization problem and necessary conditions for optimality have been deduced. These conditions state that every candidate solution must have at least four active elements (except for some special cases), two of them belonging to the group whose associated ball is larger and one of them belonging to the other group. Likewise, other conditions have been obtained by studying the intersection of the convex hulls of the sets of active elements and the sets of groups, respectively.

With these necessary conditions, it has been proved that a finite dominating set of solutions can be built in order to obtain an optimal solution. This dominating set of solutions has been constructed algorithmically and been implemented. Furthermore, some numerical results have been given.

More research, e.g. using more advanced algorithmic techniques of computational geometry, should allow to develop better preprocessing rules to avoid a lot of unnecessary work or even to reduce the size of the finite dominating set, and possibly fathoming rules may be developed that would avoid its full enumeration. However, it seems doubtful that its complexity could be reduced without a fundamentally different approach.

The concept of a solution for this problem can be extended by considering other types of balls such as ellipsoids. The problem may also be extended to higher dimensions, and/or to the location of several facilities, for which several of the necessary conditions developed here should still hold. For such extensions, as well as for large databases, the number of candidate solutions will become prohibitive and heuristic techniques will have to be used to obtain a solution within an acceptable amount of time. The use of metaheuristics such as Variable Neighborhood Search, see e.g. [17], would be a good candidate.

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