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LOCATING AN UNDESIRABLE FACILITY BY GENERALIZED CUTTING PLANES

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We address the problem of locating an undesirable facility within a compact set by minimizing a strictly decreasing boundedly lower subdifferentiable function of the squared Euclidean distances to a set of fixed points.

Using (generalized) cutting planes, the resolution of this problem is reduced to solving a sequence of maxmin problems. These maxmin problems have a clear geometrical interpretation, which enables to solve them sequentially by means of an on-line enumeration of the vertices of polyhedra in higher dimensions.

1. The model and motivations. Let A be a finite nonempty subset of $\mathbb{R}^d$ and let $\|\cdot\|$ be the Euclidean norm, $\|u\| = \sqrt{\langle u, u\rangle}$, where $\langle \cdot, \cdot \rangle$ stands for the usual scalar product. Define the function

$$D: \mathbb{R}^d \to \mathbb{R}_+^{|A|}$$

$$x \mapsto D(x) = (\|x - a\|^2)_{a\in A}.$$

Given a nonempty compact subset S of $\mathbb{R}^d$ and a decreasing function $\varphi: \mathbb{R}_+^{|A|} \to \mathbb{R}$, consider the problem (P),

$$(P) \qquad \min_{x\in S} \varphi(D(x)).$$

This problem has a clear interpretation in Planar Location Theory (see, e.g., Love, Morris, and Wesolowsky (1988) for an introduction and Plastria (1995a) for a comprehensive presentation of the field). Suppose that an undesirable facility (e.g., a nuclear plant) is to be located at some point x within a region $S \subset \mathbb{R}^2$; the facility will affect existing population, which is assumed to be concentrated at the points of A. The (risk of) damage caused to population is modeled as a decreasing function φ of the vector $D(x)$ of (squared) Euclidean distances separating x from the population. We stress that, since distances are used here to measure effects such as pollution, heat, noise, magnetic waves, etc., which propagate homogeneously in space, the use of the Euclidean norm seems to be judicious, whilst other distance measures such as polyhedral norms, e.g., Love, Morris, and Wesolowsky (1988) or Plastria (1995a), so popular for locating desirable facilities, are of little interest in our context.

Moreover, the noxious character of the facility to be located makes transportation costs negligible. In other words, the facility should be located at the optimal solution to some problem of type (P).

In order to make (P) tractable we need to impose some assumptions on the feasible region S and the function φ, as discussed below.

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1.1. The feasible region. Concerning the shape of S, in practice one does not lose much generality by imposing further assumptions on S apart from compactness. Indeed, one may assume that S is given as a union of polygonal regions (which is the typical representation in Geographical Information Systems) with extra constraints reflecting legal or environmental considerations on the facility location; for instance, due to the undesirable character of the facility, there may exist protection areas around population centers, leading to reverse-convex constraints of the form

$$\|x - a\| \geq r_a, \tag{1.1}$$

Plastria (1995b). For simplicity we make the following assumption on S:

A1. S is a nonempty compact subset of $\mathbb{R}^d$, given as

$$S = \{x \in \mathbb{R}^d : h_j(x) \geq 0, \text{ for all } j = 1, 2, \ldots, p\}, \tag{1.2}$$

where each h_j is either of the form

$$h_j(x) = \langle v_j, x\rangle + \delta_j, \tag{1.3}$$

(thus the corresponding constraint defines a closed halfspace in $\mathbb{R}^d$), or

$$h_j(x) = \|x - v_j\|^2 - r_j, \tag{1.4}$$

defining the closed complement of a ball in $\mathbb{R}^d$.

Observe that more complicated regions S—in fact, most of those provided by Geographical Information Systems—can be constructed as finite unions of sets S_i in the form of (1.2), thus the resolution of the corresponding problem may be reduced to solving (a finite number of) problems with feasible region S_i verifying Assumption A1.

1.2. The objective function. We make the following assumption on φ.

A2. φ is *strictly decreasing* and *boundedly lower subdifferentiable* with BLSD bound M on the compact set $D(S)$.

By strictly decreasing we mean that

$$\varphi(u) > \varphi(v) \quad \text{whenever } u_i \leq v_i \text{ for all } i = 1, 2, \ldots, d, \text{ and } u \neq v.$$

We also recall that a function f is said to be *lower subdifferentiable* on the set K iff for every $u \in K$ there exists some nonempty set $\partial^- f(u)$ (the *lower subdifferential of f at u*) such that for all $\eta \in \partial^- f(u)$, it follows that

$$f(v) \geq f(u) + \langle \eta, v - u\rangle \quad \forall v \in K \text{ such that } f(v) < f(u).$$

Moreover, f is said to be *boundedly lower subdifferentiable* (BLSD) with BLSD-bound M on K if f is lower subdifferentiable, and

$$\forall u \in K\ \exists \eta_u \in \partial^- f(u) \text{ such that } \|\eta_u\| \leq M.$$

We refer the reader to Plastria (1983, 1985) and Martínez-Legaz (1988) for further details on this concept.

Assuming that φ is strictly decreasing expresses that the damage caused by the facility (or risk run by the population) is decreasing with distance; the BLSD character of φ is more technical and necessary for algorithmic purposes, but, as the following examples show, is general enough to enable us to cope with a wide range of minisum and minimax problems; see Hansen, Peeters, and Thisse (1981), Plastria (1995b).

EXAMPLE 1. One can address under Assumption A2 problems of the form

$$\min_{x \in S} \sum_{a \in A} f_a(\|x - a\|^2), \tag{1.5}$$

in which one minimizes the total damage caused by the facility to the population centers in A, assuming that the damage caused to each $a \in A$ is given by $f_a(\|x - a\|^2)$.

Indeed, if for each $a \in A$ the function $f_a : \mathbb{R}_+ \to \mathbb{R}$ is strictly decreasing and convex, then the function $\varphi : (z_a)_{a \in A} \mapsto \sum_{a \in A} f_a(z_a)$ is strictly decreasing and convex, thus BLSD on the compact set $D(S)$, Plastria (1985), thus φ verifies A2.

On the other hand, since the facility is assumed to be dangerous for population, it seems sensible to give a high penalty, tending to infinity in the limit, to locations close to population, concentrated at A. In particular, as an alternative to (1.5), one may consider problems of the form

$$\inf_{x \in S \setminus A} \sum_{a \in A} f_a(\|x - a\|), \tag{1.6}$$

where, for each $a \in A$, $f_a : (0, +\infty) \to \mathbb{R}$ is a strictly decreasing convex function such that $\lim_{t \downarrow 0} f_a(t) = +\infty$. For instance, taking $f_a(t) = w_a t^{-\lambda_a}$ for some $w_a, \lambda_a > 0$, similar to the way Erkut and Neuman (1992) build data for their discrete obnoxious location model, (1.6) becomes

$$\inf_{x \in S \setminus A} \sum_{a \in A} w_a \|x - a\|^{-\lambda_a}$$

which represents the minimization of the total damage, if each w_a reflects the population concentrated at a, and now the damage per inhabitant caused to the point a by a facility located at a distance d is assumed to be $1/d^{\lambda_a}$. In particular, for $\lambda_a = 2$ for all $a \in A$, one obtains the gravitational model previously considered in Melachrinoudis and Cullinane (1986).

Defining $g_a : \mathbb{R}_+ \to (\mathbb{R} \cup \{+\infty\})$ as

$$g_a(t) = \begin{cases} f_a(\sqrt{t}) & \text{if } t > 0, \\ +\infty & \text{if } t = 0, \end{cases}$$

and φ as

$$\varphi : (u_a)_{a \in A} \in \mathbb{R}_+^n \mapsto \sum_{a \in A} g_a(u_a) \in (\mathbb{R} \cup \{+\infty\}),$$

we see that (1.6) can be rewritten as

$$\inf_{x \in S \setminus A} \varphi(D(x)). \tag{1.7}$$

Furthermore, each g_a is strictly decreasing and convex (it is the composition of the strictly increasing concave function $\sqrt{\cdot}$ with the strictly decreasing convex function f_a), hence φ has the same properties. On the other hand, although φ is not BLSD on $S \backslash A$ (the subgradients have arbitrarily high norm near the hyperplanes $u_i = 0$), it is not difficult to construct an equivalent problem of type (P) by showing, as in Hansen, Peeters, and Thisse (1981), that points too close to A are not actual candidates to optima. Indeed, choose any $\bar{x} \in S \backslash A$ (such $\bar{x}$ exists unless $S \subset A$, which would lead to an infeasible problem), and let $z = \varphi(D(\bar{x}))$. In particular, $z \geq \inf_{x \in S \backslash A} \varphi(D(x))$.

For each $a \in A$, let $c_a > 0$ be such that $g_a(c_a) \geq z$. Observe that, since $\lim_{t \downarrow 0} g_a(t) = +\infty$, such c_a exists. It then follows that one can restrict the search of optimal solutions of (1.7) to the region

$$\{x \in S : \|x - a\| \geq c_a \ \forall a \in A\}. \tag{1.8}$$

In particular, this implies that solving (1.7) can be reduced to solving

$$\begin{aligned} &\inf \varphi(D(x)) \\ &\text{s.t.} \quad \|x - a\| \geq c_a \ \forall a \in A \\ &\qquad x \in S, \end{aligned} \tag{1.9}$$

whose constraints verify Assumption A1 and the objective function is BLSD.

Observe, however that, if S is a polyhedron, the polyhedral (and convex!) structure of S is lost after adding the reverse-convex constraints leading to (1.9). We now show that it is possible to obtain an equivalent problem to (1.7) with BLSD objective function and S as feasible region, thus avoiding the inclusion of reverse-convex constraints. Indeed, for any $a \in A$, let η_a be a subgradient of the convex function g_a at c_a, and define the function $\tilde{g}_a$ as

$$\tilde{g}_a(t) = \begin{cases} g_a(t) & \text{if } t \geq c_a, \\ g_a(c_a) + (t - c_a)\eta_a & \text{otherwise.} \end{cases}$$

In particular, for $f_a(t) = w_a t^{-\lambda_a}$, one obtains

$$g_a(t) = f_a(\sqrt{t}) = w_a t^{-\lambda_a/2}.$$

Hence one can take

$$\begin{aligned} c_a &= (w_a/z)^{2/\lambda_a} \\ \eta_a &= g'(c_a) \\ &= -\frac{\lambda_a w_a}{2} c_a^{-1-(\lambda_a/2)}. \end{aligned}$$

Let $\tilde{\varphi} : \mathbb{R}_+^{|A|} \to \mathbb{R}$ be defined as

$$\tilde{\varphi} : (u_a)_{a \in A} \mapsto \tilde{\varphi}((u_a)_{a \in A}) = \sum_{a \in A} \tilde{g}_a(u_a).$$

It is easily seen that $\tilde{\varphi}$ is BLSD with BLSD-constant $M = \sum_{a\in A} |\eta_a|$, and Problem (1.7)—thus also (1.6)—is equivalent to $\min_{x\in S} \tilde{\varphi}(D(x))$, as asserted.

EXAMPLE 2. For each $a \in A$, let $f_a : \mathbb{R}_+ \to \mathbb{R}$ be strictly decreasing and Lipschitz on compact subsets of $\mathbb{R}_+$, and consider the problem

$$\min_{x\in S} \max_{a\in A} f_a(\|x-a\|^2), \tag{1.10}$$

which seeks the location x for which the highest damage (assumed to be of the form $f_a(d^2)$ if a lies at distance d from x) to any of the points a is minimized.

Let $R = \max_{x\in S, a\in A}\|x-a\|^2$, which, as S is compact, is finite. Hence, f_a is monotonous (thus quasiconvex) and Lipschitz-continuous on $[0, R]$, thus (see Plastria (1985)) it is BLSD, and, consequently, the function $\varphi = \max_{a\in A} f_a$ (i.e. the objective function of (1.10)), is also BLSD.

The aim of the paper is to solve (P) under the following assumptions:

- The decision space is $\mathbb{R}^d$ (although, in practice, we always have $d = 2$)
- $\|\cdot\|$ is the Euclidean norm
- S is a nonempty compact subset of $\mathbb{R}^d$, defined as a finite intersection of closed halfspaces and closed complements of spheres.
- The utility function φ is strictly decreasing and boundedly lower subdifferentiable.

The rest of the paper is structured as follows. In §2 we propose a cutting-plane algorithm to solve (P). The actual implementation of such an algorithm requires, at each step, solving a nonlinear subproblem, the geometrical structure of which is described first in a particular case (§3) and then in the general case (§4), including reverse-convex constraints of type (1.1).

2. Solving (P). Extending the cutting-plane method of Plastria (1983, 1985) for BLSD optimization, Barros and Frenk (1995) have introduced in a recent paper a generalized cutting-plane method which reduces the resolution of problems of the form $\min_{x\in S} \Psi(f(x))$ for some BLSD function Ψ to solving a sequence of nonlinear minmax problems, see Barros and Frenk (1995) for further details. When particularized to problem (P), one obtains the following

ALGORITHM.

Initialization

Take $x_1 \in S$.
Set $UB_1 = \varphi(D(x_1))$ and $x^* = x_1$.
Set $r = 1$.

Iteration $r = 1, 2, \ldots$

Let $\eta_r \in \partial^-\varphi(D(x_r))$ such that $\|\eta_r\| \le M$.
If $\eta_r = 0$ then Stop.
Solve (Q_r),

$$z_r = \min_{x\in S} \max_{1\le k\le r} (\varphi(D(x_k)) + \langle \eta_k, D(x) - D(x_k)\rangle). \tag{Q_r}$$

Let x_{r+1} be an optimal solution to (Q_r).
If $UB_r > \varphi(D(x_{r+1}))$ then set $x^* = x_{r+1}$.
Set $UB_{r+1} = \min\{UB_r, \varphi(D(x_{r+1}))\}$.
If $z_r = UB_{r+1}$ then Stop. Else, go to iteration $r + 1$.

If the algorithm stops after a finite number of steps, it is clear that x^* is optimal to (P). On the other hand, one has

THEOREM 3 (Barros and Frenk 1995). *If the algorithm above does not stop after a finite number of steps, then*

(1) *the sequence* $\{z_r\}_r$ *converges monotonically from below to the optimal value of* (P);

(2) *the sequence* $\{UB_r\}_r$ *converges monotonically from above to the optimal value of* (P);

(3) *any accumulation point of the sequence* $\{x_r\}_r$ *is an optimal solution to* (P).

Two critical issues in the above algorithm are how to find lower subgradients η_r with bounded norm, and how to solve the sequence of problems (Q_r).

We refer the reader to Plastria (1983, 1985) for details on the construction of lower subgradients and lower-subdifferential calculus and focus here on solving the nonlinear subproblems (Q_r). Such subproblems are, as a rule, multimodal, so their resolution still involves global optimization; however, their structure is rich enough to enable the optimization in *finite* time; in fact, solving the sequence of problems $\{(Q_r)\}_r$ can be reduced to inspecting the vertices of a sequence $\{\Gamma_r\}_r$ of nested polyhedra in higher dimensions as soon as S is polyhedral, as will be shown in §§3 and 4. (Reduction to enumeration schemes is also possible for the subproblem (Q_r) if in Problem (P) we replace $D(x)$ by the vector $(\gamma(x-a))_{a\in A}$, γ being a polyhedral gauge. However, as described in the Introduction, the motivation for the problem forces us to use the Euclidean norm.)

Now we give an equivalent expression for (Q_r) which will enable us later to rewrite the problem in geometrical terms. First we need the following

LEMMA 4. *Let* $u \in \mathbb{R}^A_+$, *and let* $\eta = (\eta_a)_{a\in A} \in \partial^-\varphi(u)$. *Then*, $\eta_a < 0$ *for each* $a \in A$.

PROOF. For each $a \in A$, let e^a be the unit vector of $\mathbb{R}^A$ with 1 at its ath component and 0 everywhere else. As φ is strictly decreasing, $\varphi(u) > \varphi(u + te^a)$ for all $t > 0$ and we have

$$t\langle \eta, e^a\rangle \leq \varphi(u + te^a) - \varphi(u) < 0.$$

Hence $\eta_a = \langle \eta, e^a\rangle < 0$ for all $a \in A$, as asserted. $\square$

Defining, α_k, c_k and β_k for each k as

$$\alpha_k = \sum_{a\in A} -(\eta_k)_a,$$

$$c_k = \frac{1}{\alpha_k}\sum_{a\in A} -(\eta_k)_a a,$$

$$\beta_k = \left(-\sum_{a\in A}(\eta_k)_a\|a\|^2\right) - \varphi(D(x_k)) + \langle \eta_k, D(x_k)\rangle - \alpha_k c_k,$$

it follows after a few calculations that (Q_r) can also be written as

$$\max_{x\in S}\ \min_{1\leq k\leq r}\,(\alpha_k\|x - c_k\|^2 + \beta_k). \tag{2.11}$$

Since, by Lemma 4, each α_k is positive, each function $f_k := \alpha_k\|\cdot - c_k\|^2 + \beta_k$ is convex, thus the objective function in (2.11) is d.c., Horst and Tuy (1990), making it solvable by general-purpose d.c. techniques. However, as we show in the rest of this section, the structure of the problem can be used to reduce the resolution of (2.11) to evaluating a finite number of points, and discuss in the two remaining sections how to design such more direct procedures.

Now we show that, for a point x^* to be optimal for (Q_r), a sufficiently high number of functions f_k and constraints defining S must be active at x^*.

By Assumption A1, S is a nonempty compact set of the form

$$S = \{x \in \mathbb{R}^d : h_j(x) \geq 0, j = 1, 2, \dots, p\},$$

each h_j being either $h_j(x) = \langle v_j, x\rangle + \delta_j$ or $h_j(x) = \|x - v_j\|^2 - r_j$. For each $x^* \in S$ define $AC(x^*)$ (the set of active or binding constraints), $AF(x^*)$ (the set of active functions) and $T(x^*)$ as

$$AC(x^*) = \{j \in \{1, 2, \dots, p\} : h_j(x^*) = 0\},$$

$$AF(x^*) = \left\{k \in \{1, 2, \dots, r\} : f_k(x^*) = \min_{1 \leq j \leq r} f_j(x^*)\right\},$$

$$T(x^*) = \{x \in \mathbb{R}^d : h_j(x) \geq 0 \; \forall\, j \notin AC(x^*) \langle \nabla h_j(x^*), x - x^*\rangle \geq 0 \; \forall\, j \in AC(x^*)\},$$

and define $f = \min_{1 \leq k \leq r} f_k$.

The set $T(x^*)$ enjoys the following properties.

LEMMA 5. *For each $x^* \in S$ one has*

(1) $T(x^*) \subset S$;

(2) $x^* \in T(x^*)$;

(3) *the set of feasible directions of $T(x^*)$ at x^* is the polyhedral cone*

$$\{\eta \in \mathbb{R}^d : \langle \nabla h_j(x^*), \eta\rangle \geq 0 \; \forall\, j \in AC(x^*)\};$$

(4) *The nonzero vector $\eta \in \mathbb{R}^d$ is a feasible direction of $T(x^*)$ at x^* of strict ascent for f iff*

$$\langle \nabla h_j(x^*), \eta\rangle \geq 0 \quad \forall\, j \in AC(x^*) \tag{2.12}$$

$$\langle \nabla f_k(x^*), \eta\rangle \geq 0 \quad \forall k \in AF(x^*). \tag{2.13}$$

PROOF. To show 1, let $x \in T(x^*)$. We have, by construction of $T(x^*)$, that $h_j(x) \geq 0 \; \forall\, j \notin AC(x^*)$. Moreover, the convexity of each h_j implies for any $j \in AC(x^*)$ that

$$\begin{aligned} h_j(x) &= h_j(x) - h_j(x^*) \\ &\geq \langle \nabla h_j(x^*), x - x^*\rangle \geq 0, \end{aligned}$$

thus $x \in S$, showing 1. Parts 2 and 3 then immediately follow from the definition of $T(x^*)$.

For Part 4, observe that Part 3 of the lemma states the equivalence between (2.12) and $T(x^*)$-feasibility of η. Moreover, the strict convexity of each function f_k implies the equivalence between (2.13) with the strict ascent of each f_k in the direction η, from which the result follows. □

From Lemma 5, one immediately has

LEMMA 6. *If $x^* \in S$ is an optimal solution to (Q_r), then x^* is also optimal for problem*

$$\max_{x \in T(x^*)} f(x).$$

We are now in position to present the result which states that, at any optimal solution of (Q_r), a sufficiently high number of functions or constraints must be active:

THEOREM 7. *If x^* is an optimal solution for (Q_r), then*

$$d < |AC(x^*)| + |AF(x^*)|. \tag{2.14}$$

PROOF. Assume by contradiction that $|AC(x^*)| + |AF(x^*)| \leq d$ and let us prove that x^* is not an optimal solution to (Q_r). Defining

$$C = \{\nabla h_j(x^*) : j \in AC(x^*)\} \cup \{\nabla f_k(x^*) : k \in AF(x^*)\}$$

two cases may arise:

Case 1. C does not span $\mathbb{R}^d$, choose then any $c \neq 0$ orthogonal to C.

Case 2. C spans $\mathbb{R}^d$, then it is a basis of $\mathbb{R}^d$ because, by assumption it has at most d elements. In this case we choose $c = \sum_{j \in AC(x^*)} \nabla h_j(x^*) + \sum_{k \in AF(x^*)} \nabla f_k(x^*)$.

Consider now the system of linear inequalities and equalities

$$\sum_{j \in AC(x^*)} u_j(-\nabla h_j(x^*)) + \sum_{k \in AF(x^*)} v_k(-\nabla f_k(x^*)) = c, \tag{2.15}$$

$$u_j \geq 0 \quad \forall j \in AC(x^*), \tag{2.16}$$

$$v_k \geq 0 \quad \forall k \in AF(x^*), \tag{2.17}$$

and observe that it has no solution. Indeed, in the first case (2.15) cannot be satisfied, while in the second case the unique solution to (2.15) is given by $u_j = -1 \; \forall j$, $v_k = -1$ $\forall k$ which contradicts both (2.16) and (2.17).

By Farkas' lemma, there exists η^* satisfying

$$\langle \nabla h_j(x^*), \eta^* \rangle \geq 0 \; \forall j \in AC(x^*),$$

$$\langle \nabla f_k(x^*), \eta^* \rangle \geq 0 \; \forall k \in AF(x^*),$$

$$\langle c, \eta^* \rangle > 0.$$

Hence by Lemma 5, η^* is a feasible direction of $T(x^*)$ and is of strict ascent for f at x^*, showing that x^* cannot be an optimal solution to $\max_{x \in T(x^*)} f(x)$, and, by Lemma 6, also not for (Q_r). □

The search of an optimal solution of (Q_r) can be further reduced, since, as shown in Theorem 8 below, not only sufficiently many functions must be active or constraints binding, but also the level sets of the functions f_k must have a particular shape at the true candidates to optimality. We first introduce some notation: for each k, j, let V_{kj} be the set of points x at which $f_k(x) \leq f_j(x)$,

$$V_{kj} = \{x \in \mathbb{R}^d : \alpha_k \|x - c_k\|^2 + \beta_k \le \alpha_j \|x - c_j\|^2 + \beta_j\}, \tag{2.18}$$

and for any $r \ge k$, let $C_{k:r}$ be the set of points at which f_k is active for (Q_r), i.e.,

$$\begin{aligned} C_{k:r} &= \bigcap_{1 \le j \le r} V_{kj} \\ &= \left\{ x \in \mathbb{R}^d : \alpha_k \|x - c_k\|^2 + \beta_k = \min_{1 \le j \le r} \alpha_j \|x - c_j\|^2 + \beta_j \right\} \end{aligned} \tag{2.19}$$

Given a set X, let $\operatorname{conv}(X)$ denote the convex hull of X and define $\operatorname{ext}(X)$ as

$$\operatorname{ext}(X) = \{x \in X : x \notin \operatorname{conv}(X \setminus \{x\})\},$$

which coincides with the set of extreme points of X when X is convex.

THEOREM 8. *Let x^* be an optimal solution to (Q_r). Then, if $x^* \in \operatorname{conv}(C_{j:r} \cap S)$ it will follow that $x^* \in \operatorname{ext}(C_{j:r} \cap S)$.*

PROOF. Consider for some $j \le r$ any point

$$x^* \in \operatorname{conv}(C_{j:r} \cap S) \setminus \operatorname{ext}(C_{j:r} \cap S).$$

Then, by definition of $\operatorname{ext}(C_{j:r} \cap S)$ and Caratheodory's theorem, there would exist $y_1, y_2, \ldots, y_{d+1} \in C_{j:r} \cap S$ and $\lambda_1, \lambda_2, \ldots, \lambda_{d+1} \ge 0$, $\sum_{i=1}^{d+1} \lambda_i = 1$ such that $x^* = \sum_{i=1}^{d+1} \lambda_i y_i$, but $x^* \ne y_i$ for all i. Hence, by the strict convexity of each f_j, one would have

$$\begin{aligned} \min_{1 \le k \le r} f_k(x^*) &\le f_j(x^*) \\ &= f_j\left(\sum_{i=1}^{d+1} \lambda_i y_i \right) \\ &< \max_{1 \le i \le d+1} f_j(y_i) \\ &= f_j(y^*) \quad \text{for some } y^* \in C_{j:r} \cap S. \end{aligned}$$

But then

$$\min_{1 \le k \le r} f_k(x^*) < \min_{1 \le k \le r} f_k(y^*),$$

showing that x^* is not optimal for (Q_r). □

The search of an optimal solution for (Q_r) has now been reduced to the identification of those points verifying the conditions in Theorems 7 and 8. A naive procedure to do this might be to solve, for each $i = 0, 1, \ldots, d+1$, each subset $J_C \subset AC$ of i constraints h_j and each subset $J_F \subset AF$ of $d + 1 - i$ functions f_k, the nonlinear system of equations

$$\begin{cases} f_k(x) - f_l(x) = 0 \quad \forall k, l \in J_F, \\ h_j(x) = 0 \quad \forall j \in J_C, \\ x \in S. \end{cases} \tag{2.20}$$

However, most choices of (J_C, J_F) in system (2.20) lead to infeasible problems, while the feasible ones might (at first glance) lead to nonfinite solution sets, making this enumerative procedure rather inefficient.

We devote the rest of the paper to exploring the geometry of the subproblems (Q_r), which will enable us to identify exactly which choices of (J_C, J_F) make (2.20) feasible. It turns out that only a polynomial number of choices—the dimension d considered to be fixed—need to be considered.

We first address the (easiest) particular case in which all the constraints defining S are linear and all the lower subgradients used in the algorithm have equal norm, and postpone to §4 the discussion of the general case with nonconvex constraints of the form (1.1) and general lower subgradients.

3. Linear constraints and lower subgradients of equal norm. Further simplifications can be made in case all the lower subgradients of φ used in the Algorithm have equal L_1-norm. We first show that this can always be obtained thanks to the use of lower subgradients. More precisely, from any $x \in \mathbb{R}^n$ and $u \in \partial^- \varphi(D(x))$, with $\|u\| \le M$, let $\eta(u)$ be given by

$$\eta(u) = \frac{|A|\, M}{\sum_{a \in A} (-u_a)}\, u. \tag{3.21}$$

By Lemma 4, $u_a < 0$ for all a, thus

$$\begin{aligned} 0 < \sum_{a \in A} - u_a &= \sum_{a \in A} |u_a| \\ &\le |A| \max_{a \in A} |u_a| \\ &\le |A| \|u\| \\ &\le |A|\, M, \end{aligned}$$

thus

$$\frac{|A|\, M}{\sum_{a \in A} - u_a} \ge 1,$$

which implies that $\eta(u) \in \partial^- \varphi(D(x))$, Plastria (1985), and its L_1 norm equals $|A|\, M$, which is independent of x and the choice of the lower subgradient u.

If α denotes this common L_1 norm, it will follow that (2.11) (thus also (Q_r)) is equivalent to

$$\max_{x \in S} \min_{1 \le k \le r} \left\{ \|x - c_k\|^2 + \frac{\beta_k}{\alpha} \right\},$$

and also to

$$\max_{x \in S} \left[\|x\|^2 + \min_{1 \le k \le r} \left(-2\langle x, c_k \rangle + \|c_k\|^2 + \frac{\beta_k}{\alpha} \right) \right].$$

Hence, denoting by $\tilde{D}_r$ the set

$$(3.22)\quad \tilde{D}_r = \left\{ (x, z) : x \in S, z \leq \|c_k\|^2 + \frac{\beta_k}{\alpha} - 2\langle x, c_k\rangle \quad \text{for all } k,\ 1 \leq k \leq r \right\},$$

(Q_r) is equivalent to

$$(3.23)\qquad \max_{(x,z)\in \tilde{D}_r} \|x\|^2 + z.$$

When S is a polytope in $\mathbb{R}^d$, $\tilde{D}_r$ is a polyhedron in $\mathbb{R}^d \times \mathbb{R}$, and (3.23) is a convex quadratic maximization problem with linear constraints; hence, solving (Q_r) reduces to evaluating the vertices of $\tilde{D}_r \subset \mathbb{R}^d \times \mathbb{R}$. This task is rather difficult in general; in fact, problem (Q_r) contains as particular instances maxmin problems, which, for variable d, are known to be NP-complete, see Crama and Ibaraki (1995). Observe, however, that the most important case in practice is $d = 2$, which leads to vertex enumeration in $\mathbb{R}^3$. Observe also that each $\tilde{D}_{r+1}$ is obtained from $\tilde{D}_r$ by adding one linear constraint. Hence, on-line vertex enumeration procedures, like that suggested in Chen, Hansen and Jaumard (1991), seem to be most appropriate for the resolution of the sequence of problems $\{(Q_r)\}_r$.

Geometrically the family $\{C_{k:r}\}_{1\leq k\leq r}$ constitutes the *power diagram* associated with the sites $c_1, c_2, \ldots, c_r$, with additive weights $\beta_1/\alpha, \beta_2/\alpha, \ldots, \beta_r/\alpha$. Power diagrams are well studied concepts of Computational Geometry (see, e.g., Aurenhammer (1991) and Edelsbrunner (1987)), and can be constructed in polynomial time—the dimension d considered to be fixed, e.g., by incremental insertion. In particular, it is known that the family $\{C_{k:r}\}_{1\leq k\leq r}$ constitutes a polyhedral subdivision in $\mathbb{R}^d$, which can be seen as the projection of the faces of a certain lifted polyhedron in $\mathbb{R}^{d+1}$. See Aurenhammer (1991) for further details. When S is a polytope it is precisely this lifting construction which transforms the constrained power diagram $\{C_{k:r} \cap S\}_{1\leq k\leq r}$ into the polyhedron $\tilde{D}_r$ given in (3.22).

4. Nonconvex constraints and general lower subgradients. Including in the model protection areas around population centers, i.e., including in the definition of S reverse-convex constraints of the form (1.1), the polyhedral structure of the problem (3.23) in §3 is destroyed, making it harder to solve.

We will show in this section that such more realistic situation can also be handled by a more complex geometric approach which also allows for the use of lower subgradients of unequal norm.

With this we gain not only in realism of the model, but also in convergence speed of the procedure. Indeed, taking as lower subgradient always one with norm $|A|\,M$ (or any other fixed one) does not do much more than indicating a halfspace containing all decrease directions. As shown in Barros and Frenk (1995), this information is sufficient to ensure convergence of the cutting plane algorithm. It does not, however, give any real clue to the actual rates of decrease in these directions, except for the constant Lipschitz constant-bound. This means that on the level set at the current x_r the minmax subproblems (Q_r) approximate problem (P) from below but not really in shape and scale.

Intuitively it is clear that better convergence should be obtained when more information is exploited, i.e., by choosing lower subgradients which better reflect the behavior of φ at x_r, by having smallest possible norm. What may easily be shown is that using lower subgradients of smaller norm, the lower bounds z_r generated by the algorithm are increased. The effect in practice will then be that a same guaranteed precision will be reached with fewer iterations, a valuable property indeed when each iteration is quite costly, as happens here.

Doing so means, however, that the lower subgradients may not be assumed to have a fixed norm. This is what would happen anyway when in case of convex φ the standard subgradients would be used.

For these reasons, we address in this section the general case in which the lower subgradients η_k used in the algorithm are not constrained to have equal L_1 norm and at the same time we allow reverse-convex constraints of the form (1.1) in the definition of S.

In this case, the family $\{C_{k:r}\}_{1\le k\le r}$ constitutes the *weighted power diagram* associated with the sites $c_1, c_2, \ldots, c_r$, with multiplicative weights $\alpha_1, \alpha_2, \ldots, \alpha_r$ and additive weights $\beta_1, \beta_2, \ldots, \beta_r$. Although not necessarily polyhedral, the sets $C_{k:r}$ have rather manageable shapes. Indeed, one obtains after some algebra that, given j, k,

• if $\alpha_j = \alpha_k > 0$, then

$$V_{kj} = \left\{ x \in \mathbb{R}^d : \langle x, c_k - c_j \rangle \ge \frac{1}{2\alpha_k}\left(-\beta_j + \beta_k - \alpha_j\|c_j\|^2 + \alpha_k\|c_k\|^2\right) \right\}, \tag{4.24}$$

which is a closed halfspace in $\mathbb{R}^d$ unless $c_k = c_j$, which leads to the degenerate cases $V_{kj} = \mathbb{R}^d$ or $V_{kj} = \emptyset$;

• if $\alpha_k > \alpha_j > 0$, then

$$V_{kj} = \left\{ x \in \mathbb{R}^d : \left\| x - \frac{\alpha_k c_k - \alpha_j c_j}{\alpha_k - \alpha_j} \right\|^2 \le \frac{\alpha_k\alpha_j\|c_k - c_j\|^2 - (\alpha_k - \alpha_j)(\beta_k - \beta_j)}{(\alpha_k - \alpha_j)^2} \right\}, \tag{4.25}$$

which is a ball unless the righthand side above is negative, (leading to $V_{kj} = \emptyset$);

• if $\alpha_j > \alpha_k > 0$, then $V_{kj} = \mathbb{R}^d \setminus \operatorname{int}(V_{jk})$ as defined above.

Denote by Δ_1 the set of closed balls in $\mathbb{R}^d$, by Δ_2 the family of closed complements of nondegenerate balls in $\mathbb{R}^d$, and denote by Δ_3 the set of closed halfspaces in $\mathbb{R}^d$.

Joining (4.24) and (4.25), one obtains the following

PROPERTY 9. *For any $j, k, j \le k$ the set V_{kj} is either empty, the whole space $\mathbb{R}^d$, a ball, the closed complement of a ball, or a halfspace*:

$$V_{kj} \in \{\mathbb{R}^d, \emptyset\} \cup (\Delta_1 \cup \Delta_2 \cup \Delta_3). \tag{4.26}$$

We will say that a nonempty set in $\mathbb{R}^d$ is a *linear-spherical boundary-set* (*lsb-set*, for short), if it can be expressed as a finite intersection of sets in $\Delta_1 \cup \Delta_2 \cup \Delta_3$. Observe that, by Assumption A1, S is an lsb-set.

Property 9 and the definition (2.19) of the sets $C_{k:r}$ thus show the following

PROPOSITION 10. *Each set $C_{k:r} \cap S$ is either empty, $\mathbb{R}^d$, a point, or an lsb-set.*

The results of §2 would be useful as soon as one could describe the sets $C_{j:r} \cap C_{k:r} \cap S$ (cf. Theorem 7) or $\operatorname{ext}(C_{k:r} \cap S)$ (cf. Theorem 8). This goal can be attained using standard tools of Computational Geometry. Indeed, as shown, e.g., in Aurenhammer (1987), sets of these shapes can be transformed via *inversion* into polyhedra in one dimension higher. Below we sketch such construction, and refer the reader to Aurenhammer (1987) for further details.

We denote by $\hat{\mathbb{R}}^{d+1}$ the set $(\mathbb{R}^{d+1} \cup \{\infty\})$, and equip it with the topology which has as closed sets either compact subsets of $\mathbb{R}^{d+1}$ (with the Euclidean topology) or complements of such (including ∞). Note that this is the classical (in general topology) construction called the *one point compactification* which for $\mathbb{R}^{d+1}$ yields a topological space homeomorphic with S^{d+1}, the unit sphere in $\mathbb{R}^{d+2}$. Balls, halfspaces and hyperplanes in $\hat{\mathbb{R}}^{d+1}$ are, respectively, sets of the form

$$\{x \in \mathbb{R}^{d+1} : \|x - a\| \le r\} \qquad (r \ge 0),$$

$$\{x \in \mathbb{R}^{d+1} : \langle a, x\rangle \le r\} \cup \{\infty\} \qquad (a \ne 0),$$

$$\{x \in \mathbb{R}^{d+1} : \langle a, x\rangle = r\} \cup \{\infty\} \qquad (a \ne 0).$$

Moreover, we will say that a set is a polyhedron in $\hat{\mathbb{R}}^{d+1}$ if it can be expressed as a finite intersection of halfspaces in $\hat{\mathbb{R}}^{d+1}$.

Finally, $\mathbb{R}^d$ is embedded into $\hat{\mathbb{R}}^{d+1}$ by identifying $\mathbb{R}^d$ with the set $h_0 : x_{d+1} = 0$.

Take now an arbitrary point $c_0 \in \mathbb{R}^{d+1}\backslash h_0$, ($c_0$ is to play the role of the *center of inversion*) and consider the involutive inversion map $T : \hat{\mathbb{R}}^{d+1} \to \hat{\mathbb{R}}^{d+1}$ defined as

$$T(x) = \begin{cases} c_0 + \dfrac{1}{\|x - c_0\|^2}(x - c_0), & \text{if } x \in \mathbb{R}^{d+1}\backslash\{c_0\}, \\ \infty, & \text{if } x = c_0, \\ c_0, & \text{if } x = \infty. \end{cases}$$

In other words, $T(x)$ is the point on the ray of $\hat{\mathbb{R}}^{d+1}$ with origin at c_0 and containing x such that

$$\|T(x) - c_0\| = \frac{1}{\|x - c_0\|}.$$

It is well known and easy to check that T defines an involutive one-one correspondence between the following pairs of sets of objects in $\hat{\mathbb{R}}^{d+1}$:

- balls with c_0 on their boundary and closed halfspaces containing c_0 in their interior
- closed complements of balls with c_0 on their boundary and closed half-spaces not containing c_0
- spheres through c_0 and hyperplanes not passing through c_0

Hyperplanes in $\hat{\mathbb{R}}^{d+1}$ through c_0 are invariant under T. Finally, $T(h_0 \cup \{\infty\})$ is a sphere through c_0 which we will denote by B_0.

These properties enable us to embed lsb-sets (in particular S and each $C_{k:r}$) into objects in $\hat{\mathbb{R}}^{d+1}$ of similar shapes as follows.

- For any $s^1 \in \Delta_1$, let $E(s^1)$ be the ball in $\mathbb{R}^{d+1}$ whose boundary contains c_0 and such that $E(s^1) \cap h_0 = s^1$.
- For any $s^2 \in \Delta_2$ which is the closed complement of a nondegenerate ball $s^1 \in \Delta_1$, $s^2 = \mathbb{R}^d\backslash\text{int}(s^1)$, let $E(s^2) = (\mathbb{R}^{d+1}\backslash\text{int}(E(s^1))) \cup \{\infty\}$.
- For any $s^3 \in \Delta_3$, let $E(s^3)$ be the halfspace in $\hat{\mathbb{R}}^{d+1}$ whose boundary contains c_0 such that $E(s^3) \cap h_0 = s^3$.

We then evidently have

PROPERTY 11. *Let $s \in (\Delta_1 \cup \Delta_2 \cup \Delta_3)$. Then*

(1) *s can be recovered from $E(s)$ as its intersection with h_0:*

$$s = E(s) \cap h_0. \tag{4.27}$$

(2) *$T(E(s))$ is a closed halfspace in $\hat{\mathbb{R}}^{d+1}$.*

(3) $s = T(T(E(s)) \cap B_0)$.

Since any lsb-set is the intersection of a finite number of elements of $\Delta_1 \cup \Delta_2 \cup \Delta_3$ and T, being involutive, is bijective, we easily obtain the following

PROPERTY 12. *Let X be an lsb-set in $\mathbb{R}^d$. Then, there exists a polyhedron $\tilde{X}$ in $\mathbb{R}^{d+1}$ such that*

$$X = T(\tilde{X}) \cap h_0 = T(\tilde{X} \cap B_0).$$

PROOF. Since $X = \bigcap_{s\in\Delta} s$ for some $\Delta \subset \Delta_1 \cup \Delta_2 \cup \Delta_3$, it follows from Property 11 that $\tilde{X} := \bigcap_{s\in\Delta} T(E(s))$ is a polyhedron in $\mathbb{R}^{d+1}$ and

$$\begin{aligned} T(\tilde{X}) \cap h_0 &= \bigcap_{s\in\Delta} T(T(E(s))) \cap h_0 \\ &= \bigcap_{s\in\Delta} (E(s) \cap h_0) \\ &= \bigcap_{s\in\Delta} s \\ &= X. \quad \square \end{aligned}$$

By Property 9 each nondegenerate V_{kj} is an lsb-set, and S is such by assumption. Therefore each set $C_{k:r}(S)$ is an lsb-set as intersection of a finite number of lsb-sets, and, when nondegenerate, corresponds by Property 12 with a polyhedron $\tilde{C}_{k:r}(\tilde{S})$ in $\hat{\mathbb{R}}^{d+1}$. In addition we have

$$C_{k:r} \cap S = T(\tilde{C}_{k:r} \cap \tilde{S} \cap B_0).$$

For fixed r the family $\Gamma_r = \{\tilde{C}_{k:r}\}_{1\le k\le r}$ (excluding the degenerate cells) constitutes a polyhedral subdivision or cell-complex of $\hat{\mathbb{R}}^{d+1}$ and the family $\Gamma_r \cap \tilde{S} = \{\tilde{C}_{k:r} \cap \tilde{S}\}_{1\le k\le r}$ is a cell-complex with union $\tilde{S}$. The (generalized) extreme points of the sets $C_{k:r} \cap S$ to be checked for optimality at iteration r (see Theorem 8) may thus be obtained by intersecting the edges (1-dimensional faces) of the cell-complex $\Gamma_r \cap \tilde{S}$ with the sphere B_0 and inversion by T. To this end it is not necessary to have access to a full description of the facial structure of this cell-complex but only to its 1-skeleton (i.e., vertices and edges). One may note that each edge yields at most two intersection points with B_0.

At the next iteration a new cell-complex $\Gamma_{r+1} = \{\tilde{C}_{k:r+1}\}_{1\le k\le r+1}$ arises, and the same construction should be repeated. However, this Γ_{r+1} is the result of adding just one new cell to the cell-complex Γ_r, modifying some of its cells to make room for it.

In fact, using the results in Aurenhammer (1987)—Lemma 5—and evident properties of the weighted power diagram, each cell-complex Γ_r may be interpreted as a power diagram in $\mathbb{R}^{d+1}$, Γ_{r+1} arising from Γ_r by addition of a new sphere to the set of spheres defining Γ_r.

According to Aurenhammer (1991, p. 381) this may be viewed after the lifting process hinted at before, as adding a cutting hyperplane to a polyhedron in $\mathbb{R}^{d+2}$, which clearly shows the cutting ‘‘plane’’ nature of our method. At the same time, this also shows that the whole procedure may be implemented as an on line vertex enumeration scheme such as proposed in Chen, Hansen and Jaumard (1991), but in $\mathbb{R}^{d+2}$.

To sum up, in the general each (Q_r) can be solved by inspecting the finite set of points obtained as the images by T of the intersection points of the ball $T(h_0 \cup \{\infty\})$ with the 1-skeleton of a certain polyhedral subdivision; moreover, by means of standard techniques of Computational Geometry, this subdivision can be built by updating the one used in the previous step of the algorithm.

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