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Location and shape of a rectangular facility in $\mathbb{R}^n$. Convexity properties [1]

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Abstract

In this paper we address a generalization of the Weber problem, in which we seek for the center and the shape of a rectangle (the facility) minimizing the average distance to a given set (the demand-set) which is not assumed to be finite. Some theoretical properties of the average distance are studied, and an expression for its gradient, involving solely expected distances to rectangles, is obtained. This enables the resolution of the problem by standard optimization techniques. © 1998 The Mathematical Programming Society, Inc. Published by Elsevier Science B.V.

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1. The model

In the classical Weber Problem there exists a finite set of demand points that request some kind of service, and the problem is to locate a new point, called facility, minimizing the weighted sum of distances to the demand points, see [1].

If the demand set is not discrete or the facility has non-void area, then the problem is the *Regional Weber Problem* [2,3]. The problem with regional demand has been addressed in [4,5], while the regional server case is studied in [6]; other papers such as [7,8] are devoted to the evaluation of the expected distances to some regions. The problem of locating regional facilities has been previously addressed, but a new insight can be added to it if the shape of the facility is considered to be a decision variable. This case appears, for example, when one wants to locate an industrial park in a city or a chip in an integrated circuit. These examples show that, as a consequence of technological or infrastructure constraints, modelling the facility to be located as a region may be more accurate than modelling it as a single point.

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The classical approach to solve the location of regional facilities consists of replacing each region by its centroid. Nevertheless, Aly [9] and Vaughan [8] have shown that this methodology is not satisfactory, since it strongly depends on the norm in use, (for example, usually good for the Euclidean norm and not so good for the l_1 norm), and the shape of the demand region. In addition what is even worse is that through this methodology there is no way to incorporate the shape of the region as a decision variable into the problem.

In this paper we address the problem of determining the location and shape of an oriented rectangle, that is, a rectangle parallel to the axes of the coordinate system. The goal is to minimize the expected distance to the demand, distributed over a region $A \in \mathbb{R}^n$ according to a probability measure p, when a lower bound $k \geqslant 0$ is given for the volume of the rectangle. If the location of the server is uniformly distributed over a set S, then the expected distance between the demand and the server is given by

$$\overline{d}_A(S) = \frac{1}{\mu(S)} \int_A \int_S \gamma(s-a)\,\mathrm{d}s\,\mathrm{d}p(a),$$

where γ is a gauge, see [10], that measures the distance between two points and μ is the Lebesgue measure in $\mathbb{R}^n$. The use of gauges instead of norms as a distance measure is discussed in detail in [11].

To formulate the problem we need to introduce some notation. Given $c \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}^n_+$ consider the rectangle

$$R(c,\alpha) = \prod_{i=1}^{n} [c_i - \alpha_i, c_i + \alpha_i].$$

Under the above assumptions, the problem can be formulated as

$$\begin{aligned} \min_{(c,\alpha)} \quad & \overline{R}_A(c,\alpha) \\ \text{s.t.} \quad & 2^n \prod_{i=1}^{n} \alpha_i \geqslant k, \\ & \alpha \geqslant \underline{0}, \quad cx \in \mathbb{R}^n, \end{aligned} \tag{1}$$

where $\overline{R}_A(c,\alpha)$ is the average distance between A and the rectangle $R(c,\alpha)$, i.e.,

$$\overline{R}_A(c,\alpha) = \frac{1}{2^n \prod_{i=1}^{n} \alpha_i} \int_A \int_{R(c,\alpha)} \gamma(x-a)\,\mathrm{d}x\,\mathrm{d}p(a).$$

2. Properties of the average distance to a rectangle

2.1. General properties

First of all, we must note that Problem 1 is well defined if and only if $E(A)$ exists and is finite, see [12], $E(A)$ being the mathematical expectation of the random vector A. Thus, hereafter, we impose this condition.

For a given set S consider

$$S(c,\alpha) = \{(c_1 + \alpha_1 s_1, \ldots, c_n + \alpha_n s_n) \in \mathbb{R}^n\colon\ (s_1, \ldots, s_n) \in S\}.$$

In particular, if S is a sphere centered at the origin, then the family obtained, defines all the ellipsoids with symmetrical edges parallel to the coordinate axis, while taking S as the unit cube centered at the origin, gives all the rectangles with sides parallel to the coordinate axis.

The following result holds.

Theorem 1. *Given a compact set S containing the origin in its interior, the function*

$$(c,\alpha) \in \mathbb{R}^n \times \mathbb{R}^n_+ \mapsto \frac{1}{\mu(S(c,\alpha))} \int_A \int_{S(c,\alpha)} \gamma(x-a)\,\mathrm{d}x\,\mathrm{d}p(a)$$

is convex.

Proof. The function $\gamma\colon x \in \mathbb{R}^n \mapsto \gamma(x)$ is a gauge, thus convex. For a, u fixed denote $u \cdot \alpha = (u_1\alpha_1, \ldots, u_n\alpha_n)$. The function $(c,\alpha) \mapsto c + u \cdot \alpha - a$ is linear. Thus $(c,\alpha) \mapsto \gamma(c + u \cdot \alpha - a)$ is convex.

Integrating, one concludes that

$$(c,\alpha) \mapsto \frac{1}{\mu(S)} \int_S \gamma(c + u \cdot \alpha - a)\,\mathrm{d}u \tag{2}$$

is convex. Now

$$\begin{aligned}\frac{1}{\mu(S)} \int_S \gamma(c + u \cdot \alpha - a)\,\mathrm{d}u &= \frac{1}{\mu(S)\prod_{i=1}^n \alpha_i} \int_{S(c,\alpha)} \gamma(x-a)\,\mathrm{d}x \\ &= \frac{1}{\mu(S(c,\alpha))} \int_{S(c,\alpha)} \gamma(x-a)\,\mathrm{d}x\end{aligned}$$

and integrating again, one concludes that the function

$$(c,\alpha) \mapsto \frac{1}{\mu(S(c,\alpha))} \int_A \int_{S(c,\alpha)} \gamma(x-a)\,\mathrm{d}x\,\mathrm{d}p(a)$$

is convex.

Taking S as $\prod_{i=1}^n [-1,1]$ in Theorem 1, one has the following

Theorem 2. *The function $\overline{R}_A$ is convex.*

Corollary 3. *Problem 1 is a convex program.*

Proof. An equivalent formulation of Problem 1 is

$$\begin{aligned}&\min_{(c,\alpha)} \quad \overline{R}_A(c,\alpha) \\ &\text{s.t.} \quad \frac{k}{2^n \prod_{i=1}^n \alpha_i} - 1 \leqslant 0, \quad \alpha \geqslant \underline{0}, \quad c \in \mathbb{R}^n;\end{aligned}$$

in this formulation both the objective function and the constraints are convex. Thus the problem is a convex program.

Theorem 2 shows that Problem 1 is a convex program, thus solvable by a variety of algorithms available for convex programs in the literature, see [13], as soon as one can evaluate the average distance to a rectangle, see [12,7].

Remark 4. The convexity properties of the objective function enable the resolution of the problem also under convex constraints. An interesting special case of this problem is the following: *the location of a rectangle with area at least of k units fully inside a convex set*. This can be easily written as convex constraints. Since we have already shown that the objective function is convex this is also a convex program.

2.2. As a function of the shape

The functional expression of the expected distance to a rectangle is only known for some particular gauges, see [14,7,8]. For this reason, the use of an explicit form of the objective function is a technique unavailable in a general approach to this problem. This fact makes the study of the problem a hard task. This section is devoted to develop further properties of the average distance function to a rectangle in order to ease the practical resolution of the problem.

Observe that $\overline{R}_x(c,\alpha)$ equals $\overline{R}_{\underline{0}}(c-x,\alpha)$, where $\underline{0}$ is the null vector in $\mathbb{R}^n$. If the demand A is a degenerate random vector (in short A is degenerated) at the point x, this means A is equal to x with probability 1, the expected distance reduces to

$$\begin{aligned}\overline{R}_x(c,\alpha) &= \frac{1}{2^n\prod_{i=1}^n \alpha_i}\int_{R(c,\alpha)} \gamma(r-x)\,\mathrm{d}r\\ &= \frac{1}{2^n\prod_{i=1}^n \alpha_i}\int_{R(\underline{0},\alpha)} \gamma(c+r-x)\,\mathrm{d}r\\ &= \frac{1}{2^n\prod_{i=1}^n \alpha_i}\int_{-\alpha_1}^{\alpha_1}\cdots\int_{-\alpha_n}^{\alpha_n}\gamma(c+r-x)\,\mathrm{d}r_n\ldots\mathrm{d}r_1,\end{aligned}$$

where $r=(r_1,\ldots,r_n)$.

If A is not degenerate then the expected distance between R and A is given by

$$\overline{R}_A(c,\alpha) = \int_A \overline{R}_a(c,\alpha)\,\mathrm{d}p(a).$$

For simplicity in the expressions the following notation is introduced. Let

$$\begin{aligned}L_i^-(c,\alpha) &= c+[-\alpha_1,\alpha_1]\times\cdots\times\{-\alpha_i\}\times\cdots\times[-\alpha_n,\alpha_n],\\ L_i^+(c,\alpha) &= c+[-\alpha_1,\alpha_1]\times\cdots\times\{\alpha_i\}\times\cdots\times[-\alpha_n,\alpha_n]\end{aligned}$$

be the facets of $R(c,\alpha)$, for $i=1,\ldots,n$. Or equivalently,

$$\begin{aligned}L_i^-(c,\alpha) &= R(c,\alpha)\cap\{x\in\mathbb{R}^n\colon\ x_i=c_i-\alpha_i\},\\ L_i^+(c,\alpha) &= R(c,\alpha)\cap\{x\in\mathbb{R}^n\colon\ x_i=c_i+\alpha_i\}.\end{aligned}$$

First of all, we study some properties of the objective function with respect to the variable α.

Theorem 5. *The function $\overline{R}_x$ is non-decreasing and differentiable with respect to each α_i, the partial derivatives being given by*

$$\frac{\partial}{\partial \alpha_i}\overline{R}_x(c,\alpha) = \frac{1}{\alpha_i}\left(\frac{\overline{d}_x(L_i^+(c,\alpha)) + \overline{d}_x(L_i^-(c,\alpha))}{2} - \overline{R}_x(c,\alpha)\right).$$

Proof. The proof will be made for α_1, for the remaining indices it runs analogously.

First of all, the partial derivatives are calculated. By definition one has

$$\begin{aligned}\overline{R}_x(c,\alpha) &= \frac{1}{2^n\prod_{i=1}^n \alpha_i}\int_{R(c,\alpha)} \gamma(u-x)\,\mathrm{d}u \\ &= \frac{1}{2^n\prod_{i=1}^n \alpha_i}\int_{-\alpha_1}^{\alpha_1}\cdots\int_{-\alpha_n}^{\alpha_n}\gamma(c+u-x)\,\mathrm{d}u_n\ldots\mathrm{d}u_1.\end{aligned}$$

Taking partial derivatives

$$\frac{\partial}{\partial \alpha_1}\overline{R}_x(c,\alpha) = \frac{\partial}{\partial \alpha_1}\left\{\frac{1}{2^n\prod_{i=1}^n \alpha_i}\int_{-\alpha_1}^{\alpha_1}\cdots\int_{-\alpha_n}^{\alpha_n}\gamma(c+u-x)\,\mathrm{d}u_n\ldots\mathrm{d}u_1\right\}$$

and taking into account that γ is continuous, one has

$$\begin{aligned}\frac{\partial}{\partial \alpha_1}\overline{R}_x(c,\alpha) &= \frac{1}{2^n\prod_{i=1}^n \alpha_i}\left(\int_{L_1^+(c,\alpha)}\gamma(c+u-x)\,\mathrm{d}u_n\ldots\mathrm{d}u_2\right. \\ &\quad\left. +\int_{L_1^-(c,\alpha)}\gamma(c+u-x)\,\mathrm{d}u_n\ldots\mathrm{d}u_2\right) - \frac{1}{\alpha_1 2^n\prod_{i=1}^n\alpha_i}\int_{R(c,\alpha)}\gamma(u-x)\,\mathrm{d}u \\ &= \frac{1}{\alpha_1}\left(\frac{\overline{d}_x(L_1^+(c,\alpha)) + \overline{d}_x(L_1^-(c,\alpha))}{2} - \overline{R}_x(c,\alpha)\right).\end{aligned}$$

Showing that $\overline{R}_x$ is non-decreasing with respect to α reduces to show that the partial derivatives, with respect to each α_i, are non-negative.

Indeed, for $i = 2,\ldots,n$, let u_i be in $[-\alpha_i,\alpha_i]$, λ be in $[0,1]$ and take $\lambda' = 1-\lambda$. Consider the convex combination

$$\lambda'(c+(-\alpha_1,u_2,\ldots,u_n)-x) + \lambda(c+(\alpha_1,u_2,\ldots,u_n)-x).$$

Since γ is convex then one has

$$\begin{aligned}&\gamma\Big(\lambda'(c+(-\alpha_1,u_2,\ldots,u_n)-x) + \lambda(c+(\alpha_1,u_2,\ldots,u_n)-x)\Big) \\ &\quad\leqslant \lambda'\gamma(c+(-\alpha_1,u_2,\ldots,u_n)-x) + \lambda\gamma(c+(\alpha_1,u_2,\ldots,u_n)-x)\end{aligned}$$

and integrating with respect to λ

$$\int_0^1 \gamma\Big(\lambda'(c+(-\alpha_1,u_2,\dots,u_n)-x)+\lambda(c+(\alpha_1,u_2,\dots,u_n)-x)\Big)\,\mathrm{d}\lambda$$

$$\leqslant \int_0^1 \Big(\lambda'\gamma(c+(-\alpha_1,u_2,\dots,u_n)-x)+\lambda\gamma(c+(\alpha_1,u_2,\dots,u_n)-x)\Big)\,\mathrm{d}\lambda,$$

then making the change of variable $u_1 = -\alpha_1 + 2\lambda\alpha_1$ and integrating

$$\frac{1}{2\alpha_1}\int_{-\alpha_1}^{\alpha_1} \gamma(c+u-x)\,\mathrm{d}u_1 \leqslant \tfrac{1}{2}\{\gamma(c+(-\alpha_1,u_2,\dots,u_n)-x)$$

$$+\gamma(c+(\alpha_1,u_2,\dots,u_n)-x)\}.$$

Now integrating with respect to the remaining u_i and dividing by $2^n\prod_{i=1}^n \alpha_i$

$$\frac{1}{2\alpha_1 2^n\prod\limits_{i=1}^{n}\alpha_i}\int_{-\alpha_1}^{\alpha_1}\cdots\int_{-\alpha_n}^{\alpha_n}\gamma(c+u-x)\,\mathrm{d}u_n\dots\mathrm{d}u_1$$

$$\leqslant \frac{1}{2\;2^n\prod_{i=1}^n\alpha_i}\int_{-\alpha_2}^{\alpha_2}\cdots\int_{-\alpha_n}^{\alpha_n}\gamma(c+(-\alpha_1,u_2,\dots,u_n)-x)\,\mathrm{d}u_n\dots\mathrm{d}u_2$$

$$+\frac{1}{2\;2^n\prod_{i=1}^n\alpha_i}\int_{-\alpha_2}^{\alpha_2}\cdots\int_{-\alpha_n}^{\alpha_n}\gamma(c+(\alpha_1,u_2,\dots,u_n)-x)\,\mathrm{d}u_n\dots\mathrm{d}u_2$$

or equivalently,

$$\frac{1}{2\alpha_1}\overline{R}_x(c,\alpha)\leqslant\frac{1}{4\alpha_1}(\overline{d}_x(L_1^-(c,\alpha))+\overline{d}_x(L_1^+(c,\alpha))),$$

that is to say

$$\frac{\partial}{\partial\alpha_1}\overline{R}_x(c,\alpha)\geqslant 0.$$

Thus $\overline{R}_x$ is non-decreasing with respect to α_1.

The above theorem can be generalized to the case in which the demand is regional.

Corollary 6. *The function $\overline{R}_A$ is non-decreasing and differentiable with respect to each α_i, the partial derivatives being given by*

$$\frac{\partial}{\partial\alpha_i}\overline{R}_A(c,\alpha)=\frac{1}{\alpha_i}\left(\frac{\overline{d}_A(L_i^+(c,\alpha))+\overline{d}_A(L_i^-(c,\alpha))}{2}-\overline{d}_A(c,\alpha)\right).$$

Proof. The function $\overline{R}_A$ is non-decreasing, since the integral operator is monotone and $\overline{R}_x$ is non-decreasing with respect to α.

Theorem 1, Section 2, of [15] allows to exchange the integral and differential operators, since $\overline{R}_x$ is convex, so that

$$\begin{aligned}\frac{\partial}{\partial\alpha_i}\overline{R}_A(c,\alpha) &= \int_A \frac{\partial}{\partial\alpha_i}\overline{R}_a(c,\alpha)\,\mathrm{d}p(a)\\ &= \frac{1}{\alpha_i}\int_A\left\{\frac{\overline{d}_a(L_i^+(c,\alpha))+\overline{d}_a(L_i^-(c,\alpha))}{2}-\overline{d}_a(c,\alpha)\right\}\mathrm{d}p(a)\\ &= \frac{1}{\alpha_i}\left(\frac{\overline{d}_A(L_i^+(c,\alpha))+\overline{d}_A(L_i^-(c,\alpha))}{2}-\overline{d}_A(c,\alpha)\right)\end{aligned}$$

and the equality is proved.

As a consequence, one has the following

Corollary 7. *The optimal solution for Problem 1 for $k = 0$ is some rectangle degenerate to a point.*

Thus Problem 1 with $k = 0$ becomes the standard Regional Weber Problem [2,12]. This result has been previously shown in [12,16] by a different procedure. Hereafter, we suppose that k is strictly positive.

2.3. As a function of the location

Now let us study the differentiability with respect to the location variable c.

Theorem 8. *The function $\overline{R}_x$ is differentiable with respect to each c_i, the partial derivatives being given by*

$$\frac{\partial}{\partial c_i}\overline{R}_x(c,\alpha) = \frac{\overline{d}_x(L_i^+(c,\alpha))-\overline{d}_x(L_i^-(c,\alpha))}{2\alpha_i}.$$

Proof. By definition

$$\begin{aligned}\overline{R}_x(c,\alpha) &= \frac{1}{2^n\prod_{i=1}^n\alpha_i}\int_{-\alpha_1}^{\alpha_1}\cdots\int_{-\alpha_n}^{\alpha_n}\gamma(c+u-x)\,\mathrm{d}u_n\ldots\mathrm{d}u_1\\ &= \frac{1}{2^n\prod_{i=1}^n\alpha_i}\int_{c_1-\alpha_1-x_1}^{c_1+\alpha_1-x_1}\cdots\int_{c_n-\alpha_n-x_n}^{c_n+\alpha_n-x_n}\gamma(u)\,\mathrm{d}u_n\ldots\mathrm{d}u_1.\end{aligned}$$

The proof is analogous for each index. Then we take $i = 1$; differentiation in the previous expression yields

$$\begin{aligned}
&\frac{\partial}{\partial c_1}\overline{R}_x(c,\alpha) \\
&= \frac{\partial}{\partial c_1}\left(\frac{1}{2^n\prod_{i=1}^n \alpha_i}\int\limits_{c_1-\alpha_1-x_1}^{c_1+\alpha_1-x_1}\cdots\int\limits_{c_n-\alpha_n-x_n}^{c_n+\alpha_n-x_n}\gamma(u)\,\mathrm{d}u_n\ldots\mathrm{d}u_1\right) \\
&= \frac{1}{2^n\prod_{i=1}^n \alpha_i}\left(\int\limits_{L_1^+(c,\alpha)-x}\gamma(u)\,\mathrm{d}u_n\ldots\mathrm{d}u_2 - \int\limits_{L_1^-(c,\alpha)-x}\gamma(u)\,\mathrm{d}u_n\ldots\mathrm{d}u_2\right) \\
&= \frac{\overline{d}_x(L_1^+(c,\alpha)) - \overline{d}_x(L_1^-(c,\alpha))}{2\alpha_1}.
\end{aligned}$$

Again, in the case of regional demand a similar result is obtained.

Corollary 9. *The function $\overline{R}_A$ is differentiable with respect to each c_i, the partial derivatives being given by*

$$\frac{\partial}{\partial c_i}\overline{R}_A(c,\alpha) = \frac{\overline{d}_A(L_i^+(c,\alpha)) - \overline{d}_A(L_i^-(c,\alpha))}{2\alpha_i}.$$

Proof. Using the equality

$$\overline{R}_A(c,\alpha) = \int_A \overline{R}_a(c,\alpha)\,\mathrm{d}p(a)$$

and taking derivatives, one has

$$\frac{\partial}{\partial c_i}\overline{R}_A(c,\alpha) = \frac{\partial}{\partial c_i}\int_A \overline{R}_a(c,\alpha)\,\mathrm{d}p(a),$$

since $\overline{R}_x$ is convex, the above-mentioned theorem of [15], enables us to exchange the integral and differential operators

$$\begin{aligned}
&= \int\limits_A \frac{\partial}{\partial c_i}\overline{R}_a(c,\alpha)\,\mathrm{d}p(a) \\
&= \int\limits_A \frac{\overline{d}_a(L_i^+(c,\alpha)) - \overline{d}_a(L_i^-(c,\alpha))}{2\alpha_i}\,\mathrm{d}p(a) \\
&= \frac{\overline{d}_A(L_i^+(c,\alpha)) - \overline{d}_A(L_i^-(c,\alpha))}{2\alpha_i},
\end{aligned}$$

which proves the corollary.

Remark 10. The condition of optimality with respect to c is, from the above theorem,

$$\frac{\partial}{\partial c_i}\overline{R}_A(c,\alpha) = 0 \quad i = 1,\ldots,n.$$

This expression admits a remarkable interpretation: in the optimal rectangle the expected distance to every pair of opposite facets must be equal.

The convexity property enables us to extend the previous theorems.

Theorem 11. *The functions $\overline{R}_x$ and $\overline{R}_A$ are differentiable.*

Proof. By Theorem 25.2 of [17] a convex function is differentiable if its partial derivatives exist and are finite. The partial derivatives of $\overline{R}_A$ are given by Corollaries 6 and 9, and are finite as soon as $E(A)$ exists and is finite. Since this condition was taken as assumption, the result holds.

2.4. Uniqueness properties

Some conditions must be imposed in order to guarantee the uniqueness of the solution.

Following [18], we recall that γ is said to be a strict gauge or round gauge if its unit ball does not contain segments not reduced to a point. Recall that a segment is called degenerate if it reduces to a point.

Theorem 12. *If γ is a strict gauge, then the function $\overline{R}_A$ is increasing in α.*

Proof. The proof is analogous to Theorem 5 and Corollary 6. One only needs to prove that $\overline{R}_x(c,\alpha)$ is increasing in α and then take integral. Given α there exist non-collinear x, $(-\alpha_1, u_2, \ldots, u_n)$ and $(\alpha_1, u_2, \ldots, u_n)$. Then $\forall \lambda \in (0,1), \lambda' = 1-\lambda$:

$$\begin{aligned}&\gamma(\lambda'((-\alpha_1, u_2, \ldots, u_n) - x) + \lambda((\alpha_1, u_2, \ldots, u_n) - x)) \\ &\quad < \lambda'\gamma((-\alpha_1, u_2, \ldots, u_n) - x) + \lambda\gamma((\alpha_1, u_2, \ldots, u_n) - x).\end{aligned}$$

Thus the first inequality of the proof of Theorem 5 is strict. Since γ is continuous the inequality holds strictly in a neighbourhood of such point. This enables us to assure that the strict inequality is kept after integration. Consequently $\frac{\partial}{\partial \alpha_i}\overline{R}_x(c,\alpha) > 0$. Thus $\overline{R}_x(c,\alpha)$ and $\overline{R}_A(c,\alpha)$ are increasing with respect to α.

Theorem 13. *If γ is a strict gauge, then the function $\overline{R}_A$ is strictly convex in the interior of its domain.*

Proof. Given x and α, there exists u such that x and $(u_1\alpha_1, \ldots, u_n\alpha_n)$ are not collinear with $\underline{0}$, then $(x,u) \mapsto \gamma((u_1\alpha_1, \ldots, u_n\alpha_n) - x)$ is strictly convex. By continuity, there is also a neighbourhood of u where the non-collinearity holds. This guarantees the strict inequality of γ regarding any strictly convex combination in a set of non-zero Lebesgue measure and this implies the strict convexity of the expected distance function.

Theorems 12 and 13, together with the property of inf-compactness, enable us to obtain sufficient conditions for the uniqueness of the solutions.

Theorem 14. *If γ is a strict gauge, then Problem 1 has a unique optimal solution.*

3. Determining an optimal solution

Using the monotonicity property of the objective function, in order to search for solutions of Problem 1 only the boundary of the feasible set has to be considered.

Theorem 15. *There exists an optimal solution to Problem 1 such that*

$$2^n \prod_{i=1}^{n} \alpha_i = k.$$

According to this theorem an optimal solution of Problem 1 can be obtained by solving the following problem:

$$\begin{aligned} \min_{(c,\alpha)} \quad & \overline{R}_A(c,\alpha) \\ \text{s.t.} \quad & 2^n \prod_{i=1}^{n} \alpha_i = k, \quad c \in \mathbb{R}^n, \ \alpha > \underline{0}. \end{aligned} \tag{3}$$

This problem has a non-linear equality constraint. This converts it into a non-convex problem, increasing the resolution difficulty from a practical point of view. Hence, it would be convenient to eliminate the constraint $2^n \prod_{i=1}^{n} \alpha_i = k$, as will be made below through its accommodation into the objective function. For this purpose consider the function $\overline{R}'_A$ defined by

$$\overline{R}'_A : (c,\alpha') \in \mathbb{R}^n \times \mathbb{R}_+^{n-1} \mapsto \overline{R}_A\left(c, \left(\alpha'_1, \ldots, \alpha'_{n-1}, \frac{k}{\prod_{i=1}^{n-1} \alpha'_i} \right) \right).$$

It is straightforward to show the following.

Lemma 16. *Let $f: \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ be a convex function, such that $f(x,.)$ is non-decreasing and let $g: \mathbb{R}^k \to \mathbb{R}$ be convex. Then the function $h: (x,y) \mapsto f(x,g(y))$ is convex.*

Furthermore, if $f(x,.)$ is increasing and f and g are strictly convex, then h is strictly convex.

Theorem 17. *The function $\overline{R}'_A$ is convex and differentiable.*

Proof. The function

$$(c,\alpha') \in \mathbb{R}^n \times \mathbb{R}_+^{n-1} \mapsto \left(c, \left(\alpha'_1, \ldots, \alpha'_{n-1}, \frac{k}{\prod_{i=1}^{n-1} \alpha'_i} \right) \right)$$

is convex and differentiable. Furthermore, the function $(c,\alpha) \in \mathbb{R}^n \times \mathbb{R}^n_+ \mapsto \overline{R}_A(c,\alpha)$ is convex and increasing in α. The composition of both is, by Lemma 16, convex and differentiable, as we wish to prove.

Using these results, Problem 1 turns out to be equivalent to

$$\begin{aligned} &\min_{(c,\alpha')} \quad \overline{R}'_A(c,\alpha') \\ &\text{s.t.} \quad c \in \mathbb{R}^n, \ \ \alpha' > \underline{0}. \end{aligned} \tag{4}$$

Corollary 18. *The partial derivatives of $\overline{R}'_A$ with respect to each α'_i, are given by*

$$\begin{aligned} \frac{\partial}{\partial \alpha'_i}\overline{R}'_A(c,\alpha') = \frac{1}{2\alpha'_i}\big(&\overline{d}_A(L_i^+(c,\alpha)) + \overline{d}_A(L_i^-(c,\alpha)) - \overline{d}_A(L_n^+(c,\alpha)) \\ &-\overline{d}_A(L_n^-(c,\alpha))\big). \end{aligned}$$

Proof. The function $\overline{R}'$ is differentiable. Thus

$$\begin{aligned} \frac{\partial}{\partial \alpha'_i}\overline{R}'_A(c,\alpha') &= \frac{\partial}{\partial \alpha_i}\overline{R}_A(c,\alpha) + \frac{\partial \alpha_n}{\partial \alpha_i}\frac{\partial}{\partial \alpha_n}\overline{R}_A(c,\alpha) \\ &= \frac{\partial}{\partial \alpha_i}\overline{R}_A(c,\alpha) - \frac{1}{\alpha_i}\frac{k}{2^n\prod_{j=1}^{n-1}\alpha_j}\frac{\partial}{\partial \alpha_n}\overline{R}_A(c,\alpha). \end{aligned}$$

By Theorem 6, one has

$$\begin{aligned} \frac{\partial}{\partial \alpha'_i}\overline{R}'_A(c,\alpha') &= \frac{1}{\alpha_i}\left(\frac{\overline{d}_A(L_i^+(c,\alpha)) + \overline{d}_A(L_i^-(c,\alpha))}{2} - \overline{d}_A(c,\alpha)\right) \\ &\quad - \frac{1}{\alpha_i}\frac{k}{2^n\prod_{j=1}^{n-1}\alpha_j}\frac{2^n\prod_{j=1}^{n-1}\alpha_j}{k}\left(\frac{\overline{d}_A(L_n^+(c,\alpha)) + \overline{d}_A(L_n^-(c,\alpha))}{2} - \overline{d}_A(c,\alpha)\right) \\ &= \frac{1}{2\alpha_i}\big(\overline{d}_A(L_i^+(c,\alpha)) + \overline{d}_A(L_i^-(c,\alpha)) - \overline{d}_A(L_n^+(c,\alpha)) - \overline{d}_A(L_n^-(c,\alpha))\big). \end{aligned}$$

Remark 19. Observe that the partial derivatives of $\overline{R}'$ can be evaluated, with respect all its variables, evaluating only the expected distances to the sides of the rectangle. This means that we can solve the problem in a n-dimensional space evaluating only the expected distances to $(n-1)$-dimensional rectangles. From a practical point of view, this reduces drastically the total amount of computation required to solve the problem.

To solve Problem 4 one only needs to solve the equations

$$\begin{aligned} &\frac{\partial}{\partial c_i}\overline{R}'_A(c,\alpha) = 0, \quad i = 1,\dots,n, \\ &\frac{\partial}{\partial \alpha'_i}\overline{R}'_A(c,\alpha') = 0, \quad i = 1,\dots,n-1. \end{aligned}$$

Or equivalently

$$\overline{d}_A(L_i^+) - \overline{d}_A(L_i^-) = 0, \quad i = 1, \dots, n,$$
$$\overline{d}_A(L_i^+) + \overline{d}_A(L_i^-) - \overline{d}_A(L_n^+) - \overline{d}_A(L_n^-) = 0, \quad i = 1, \dots, n-1.$$

Furthermore,

$$\overline{d}_A(L_i^+(c,\alpha)) - \overline{d}_A(L_i^-(c,\alpha)) = 0, \quad i = 1, \dots, n,$$
$$\overline{d}_A(L_i^+(c,\alpha)) - \overline{d}_A(L_n^+(c,\alpha)) = 0, \quad i = 1, \dots, n-1.$$

Theorem 20. *If γ is a strict gauge, then the function $\overline{R}'_A$ is strictly convex.*

Proof. The proof is analogous to that of Theorem 17, taking into account the increasing monotonicity and the strict convexity of $\overline{R}_A$.

Theorem 21. *If γ is a strict gauge, Problem 4 has only one optimal solution.*

4. An example

In the previous sections we have developed some properties of the considered location problem. The purpose of this section is to show, from a practical point of view, that these properties really ease the resolution of the problem.

Given a finite set of point $A = \{a_1, \dots, a_m\} \subset \mathbb{R}^2$, we look for the position of a rectangle, with sides parallel to the axes of the coordinate system, and an area at least of k, minimizing the average distance to A. This problem can be formulated as:

$$\min_{(c,\alpha)} \quad \overline{d}_A(R(c,\alpha)) = \frac{1}{m}\sum_{i=1}^{m} \overline{d}_{a_i}(R(c,\alpha))$$
$$\text{s.t.} \quad 4\alpha_1\alpha_2 \geqslant k, \quad \alpha > \underline{0}, \quad c \in \mathbb{R}^2.$$

Making the transformation suggested before Lemma 16 and taking $\alpha_l = (l, 4k/l)$, by Theorems 17 and 18 an optimal solution can be found solving the equations

$$\frac{\partial}{\partial c_1}\overline{d}_A(R(c,\alpha_l)) = 0,$$
$$\frac{\partial}{\partial c_2}\overline{d}_A(R(c,\alpha_l)) = 0,$$
$$\frac{\partial}{\partial l}\overline{d}_A(R(c,\alpha_l)) = 0.$$

These equations reduce to

$$\overline{d}_A(L_1^+(c,\alpha_l)) - \overline{d}_A(L_1^-(c,\alpha_l)) = 0,$$
$$\overline{d}_A(L_2^+(c,\alpha_l)) - \overline{d}_A(L_2^-(c,\alpha_l)) = 0,$$
$$\overline{d}_A(L_1^+(c,\alpha_l)) - \overline{d}_A(L_2^+(c,\alpha_l)) = 0.$$

Table 1
Optimal rectangle

Norm	Center	Semi-edges	Expected distance
l_1	(0.4367,0)	(0.8165,1.2247)	4.2532
l_2	(0.2172,0)	(1.0389,0.9625)	3.7765
l_∞	(0,0)	(1,1)	3.6944

Notice that these expressions only involve the expected distance to segments. The expected distances from a point to a segment can be evaluated by explicit formula, when known, or by numerical integration.

Given $A = \{(1,0),(-1,0),(0,5),(0,-5)\}$ and $k = 4$. The center of the optimal rectangle is the point (0,0) for all l_p norms. The optimal lengths of the semiedges are (1,1), (1.1326, 0.8829) and (1,1), respectively for l_1, l_2 and l_∞ norms, and the respective expected distances are 3.5, 3.1060, and 3.0417.

Finally, for $A = \{(1,0),(-1,0),(0,5),(0,-5),(1,5),(1,-5)\}$, the optimal rectangle for several l_p norms are shown in Table 1.

5. Conclusions

In this paper we consider an extension of the Regional Weber Problem, in which both the location and the shape of the facility are sought. It is shown that this problem is convex, and solvable as soon as one can evaluate the objective function, namely, the expected distance.

We have shown that the objective function is differentiable and we have obtained easily valuable expressions for the gradient of the objective function, easing the resolution process.

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