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Theory and Methodology

Multi-criteria analysis with partial information about the weighting coefficients

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Abstract

In this paper we address the problem of ranking a set of alternatives with partial information about the weighting coefficients. We introduce a family of quasiorders that are easily interpretable and manageable, which includes among others, the natural quasiorder in $\mathbb{R}^n$ and other well known preference structures in the literature. The enrichment of the preference structure with respect to the natural quasiorder is measured by means of an absolute measure we introduce.

Keywords: Multiple criteria decision making; Partial information; Weights

1. Introduction

Nearly all the real world decision problems involve more than one objective and can be formulated in a natural way using the multi-criteria approach:

$$\text{'Max' } f(x) \doteq (f_1(x), \ldots, f_n(x)) \tag{1}$$

where $f_i : X \to \mathbb{R}$ is the evaluation of the i-th objective $(i = 1, \ldots, n)$ and X is the set of alternatives to be ranked.

In this formulation, and without additional information about the objectives, one can obtain a partial order R_I in X: Given two alternatives x, $y \in X$, we say that x is as preferred as y following R_I $(xR_I y)$ iff

$$f_i(x) \geq f_i(y) \quad \forall i = 1, \ldots, n,$$

or equivalently,

$$xR_I y \quad \text{iff} \quad \sum_{1 \leq i \leq n} w_i f_i(x) \geq \sum_{1 \leq i \leq n} w_i f_i(y) \quad \forall w \geq 0,\ w \in \mathbb{R}^n.$$

As the partial order R_I is typically too vague (and the set of nondominated alternatives is too large) a number of procedures has been proposed in order to enrich the preference structure above (Promethee [5], Electre [19], interactive methods [11], the utility approach [9], etc.). The interested reader is referred to the seminal paper of Roy [20] for a synthesis of the main approaches to this problem which have been studied.

In the utility approach, one assumes the existence of a function $U : \mathbb{R}^n \to \mathbb{R}$, in such a way that an alternative x is considered as preferred to y iff

$$U(f_i(x), \ldots, f_n(x)) \geq U(f_1(y), \ldots, f_n(y)).$$

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Furthermore, under mild regularity conditions [12], U is a linear function, i.e.: one assumes the existence of a vector $w^* = (w_1^*, \ldots, w_n^*) \geq 0$ such that the Decision Maker (D-M) preferences are given by the order R:

$$xRy \quad \text{iff} \quad \sum_{1 \leq i \leq n} w_i^* f_i(x) \geq \sum_{1 \leq i \leq n} w_i^* f_i(y).$$

However, one of the main drawbacks of this approach is that the D-M would have to estimate numerically the weight (the vector w^*) that each criterion brings to the final score of an alternative and he is not always willing to do so [6]. This choice is a critical step, and, although some methodologies have been developed to attain this goal, such as the Entropy method [11], the Saaty method [21], the Solymosi and Dombi technique [23,15], etc., they require, in our opinion, too much specialized information from the D-M.

Instead of providing the exact w^*, the D-M might have some knowledge about w^* and this information can be used to obtain a (partial) ranking in X which enriches the original preference structure R_I.

In this paper we show that exact numerical weights are not always necessary. Instead of this exact estimation, the D-M gives only certain linear relations which express partial information about the marginal substitution rates between the criteria. This approach is not new and some work in this field can be found in the literature; Kirwood and Sarin [13], Hazen [10], or Eiselt and Laporte [8], derive conditions to determine ordinal rankings between alternatives using partial information about weighting constants.

For instance, in the qualitative approach [17] the D-M is requested to estimate only the rank order of the criteria. If the criteria are arranged in decreasing order of preference, with $w_1 \geq \cdots \geq w_n$, then

$$xRy \quad \text{iff} \quad w(f(x) - f(y)) \geq 0 \quad \forall w \geq 0,$$
$$w_1 \geq \cdots \geq w_n$$

or, equivalently (see [17])

$$xRy \quad \text{iff} \quad \sum_{j \leq i} f_j(x) \geq \sum_{j \leq i} f_j(y) \quad \forall i = 1, \ldots, n.$$

In a more general setting, the D-M is requested to estimate a linear operator which mixes the weights, i.e. the D-M asserts that the vector w^* belongs to a certain polyhedron. Thus, given a linear operator A, we define the preference between the alternatives by means of the binary relation R_A as

$$xR_A y \quad \text{iff} \quad w(f(x) - f(y)) \geq 0 \quad \forall w \geq 0, \quad Aw \geq 0. \tag{2}$$

All these binary relations are quasi-orders (i.e. reflexive and transitive) [26] and can be used to partially rank the alternatives, although their use requires the solution of a linear problem to compare every pair of alternatives.

However, if the extreme points $w^1, \ldots, w^m$ of the polytype $\{w : Aw \geq 0,\ w \geq 0,\ \sum_{i=1}^n w_i = 1\}$ are known, the relation R_A is

$$xR_A y \quad \text{iff} \quad w^k(f(x) - f(y)) \geq 0 \quad \forall k = 1, \ldots, m, \tag{3}$$

which simplifies its use.

The following example illustrates the comments.

Example 1.1. The staff manager of a consulting firm must rank four different executives of a bank according to the budget they estimate for next year based on three financial criteria. These criteria are: loans given to clients (C_1), clients' savings deposits (C_2) and redemptions achieved (C_3).

Each manager's budget must adhere to the financial policy of the bank. This policy is given in form of three constraints:

1. The bank wishes that the credits are greater than the sum of the redemptions plus 1.5 times the savings.
2. The redemptions must be less than 0.1 times the savings deposits.
3. The redemptions must be non-negative.

Consider the following matrix where the row i represents the action proposed by the manager i, $i = 1, \ldots, 4$:

	C_1	C_2	C_3
a_1	11	12.2	0
a_2	5	4	5
a_3	11	11	13
a_4	11	12	2.3

Let F denote the above matrix. The constraints imposed by the bank can be written as

$$w_1 \geq 1.5w_2 + w_3, \qquad w_2 \geq 10w_3, \qquad w_3 \geq 0.$$

Thus, the linear operator A and the extreme points matrix E, of the polytope $\{w : Aw \geq 0, w \geq 0, \sum_{i=1}^{n} w_i = 1\}$ are respectively

$$A = \begin{pmatrix} 1 & -1.5 & -1 \\ 0 & 1 & -10 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } E = \begin{pmatrix} 1 & \frac{3}{5} & \frac{16}{27} \\ 0 & \frac{2}{5} & \frac{10}{27} \\ 0 & 0 & \frac{1}{27} \end{pmatrix},$$

and the quasiorder given by the above constraints in the action's set is obtained from the matrix $F \times E$ by means of the natural quasiorder. Hence,

$$F \times E = \begin{bmatrix} 11 & \frac{287}{25} & \frac{298}{27} \\ 5 & \frac{23}{5} & \frac{125}{27} \\ 11 & 11 & \frac{299}{27} \\ 11 & \frac{285}{25} & \frac{298.3}{27} \end{bmatrix}$$

generates a relation which can be represented by the graph of Fig. 1.

However, with a different set of constraints imposed by the bank given by the operator A_1, and whose matrix representation and extremes are

$$A_1 = \begin{pmatrix} 1 & -2.5 & -2.5 \\ 0 & 1 & -13 \\ 0 & 0 & 1 \end{pmatrix},$$

$$E_1 = \begin{pmatrix} 1 & \frac{5}{7} & \frac{35}{49} \\ 0 & \frac{2}{7} & \frac{13}{49} \\ 0 & 0 & \frac{1}{49} \end{pmatrix},$$

it is easy to see that the quasiorder generated is actually an order whose graph is given in Fig. 2. Thus, it is possible in many cases to identify the preferred alternatives without knowing the numerical estimates of the criterion weights.

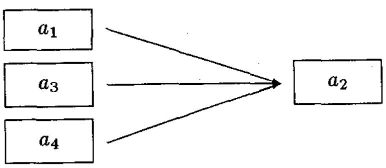

Fig. 1. The graph of R_A.

Fig. 2. The graph of R_{A_1}.

However, the preference structure determination in its general formulation (the determination of the set of non-dominated alternatives) is equivalent to the vertex polyhedron enumeration, which is not a polynomially solvable problem in the size of the inequality system defining the polyhedron [7,22].

The paper is structured as follows: In Section 2, we introduce a class of matrices (*Q-operators*) and study some properties of the quasiorders they induce. In Section 3, we characterize some Q-operators that are easily understandable and manageable. In Section 4, we extend the results obtained in previous sections to a broader class. The paper finishes with some conclusions and possible extensions.

2. The class of Q-operators

Every linear operator A belonging to the set of real matrices of dimension $k \times n$, defines a quasi-order R_A on the set X of alternatives: Given a matrix $A \in \mathbb{R}^{k \times n}$ we define the polytopes C_A^+ and C_A:

$$C_A^+ = \left\{ w : Aw \geq 0, w \geq 0, \sum_{i=1}^{n} w_i = 1 \right\} \tag{4}$$

and

$$C_A = \left\{ w : Aw \geq 0, \sum_{i=1}^{n} w_i = 1 \right\}, \tag{5}$$

and the quasiorder R_A on X:

$$xR_A y \quad \text{iff} \quad \sum_{i=1}^{n} w_i f_i(x) \geq \sum_{i=1}^{n} w_i f_i(y) \quad \forall w \in C_A^+. \tag{6}$$

In this section we introduce a class of quasiorders induced by linear operators (*Q-operators*) that have very interesting properties: the extreme points of the corresponding C_A^+ can be very easily obtained.

Definition 2.1. A linear operator $A \in \mathbb{R}^{n \times n}$ is said to be a *Q-operator* if $\det(A) \neq 0$ and $A^{-1} \geq 0$ (componentwise).

For a Q-operator A, the inverse operator A^{-1} exists; we denote its elements by α_{ij}, and by μ the vector of sums of the columns of A^{-1}:

$$A^{-1} = (\alpha_{ij}) \text{ and } \mu_j = \sum_{1 \leq i \leq n} \alpha_{ij}. \tag{7}$$

Theorem 2.1. *If $A \in \mathbb{R}^{n \times n}$ is a Q-operator, then C_A^+ is the convex hull of the columns of A^{-1}, each one normalized in order to add* 1.

Proof. As $A^{-1} \geq 0$, it follows that $\mu_j > 0$, $\forall j = 1, \dots, n$.

Let D be the diagonal matrix such that $d_{ii} = 1/\mu_i$, $\forall i$, and let $e \in \mathbb{R}^n$ denote the vector of ones. Given $w \in \mathbb{R}^n$, one has

$$w \in C_A^+ \quad \text{iff } \exists z \text{ such that } w = A^{-1}Dz, \quad Dz \geq 0,$$

$$A^{-1}Dz \geq 0, \quad eA^{-1}Dz = 1$$

i.e. (recall that $D \geq 0$, $A^{-1} \geq 0$, $eA^{-1}D = e$):

$$w \in C_A^+ \quad \text{iff } \exists z \text{ such that } w = A^{-1}Dz, \quad z \geq 0,$$

$$ez = 1.$$

In other words, $w \in C_A^+$ iff w can be represented as a convex combination of the columns of $A^{-1}D$, as asserted. □

Remark 2.1. The well-known Paelinck theorem [17] proven, among others, in [17,2,6,13], reduces to the calculus of the extreme points of the polyhedron $\{w \in \mathbb{R}^n : w_1 \geq w_2 \geq \cdots \geq w_n \geq 0, \sum_{i=1}^n w_i = 1\}$, which is of the form C_A^+ for the following matrix A:

$$\begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

The matrix above is easily shown to be a Q-operator, thus Paelinck's theorem appears as a direct consequence of our Theorem 2.1. Indeed, A^{-1} can be readily obtained, and the extreme points of C_A^+ are the columns of the following matrix:

$$\begin{pmatrix} 1 & \frac{1}{2} & \dots & 1/n \\ 0 & \frac{1}{2} & \dots & 1/n \\ 0 & 0 & \dots & 1/n \\ \vdots & \vdots & \ddots & 1/n \\ 0 & 0 & \dots & 1/n \end{pmatrix}.$$

Remark 2.2. In the *centroid method* of Solymosi and Dombi [23,15], given the polyhedron C_A^+, one proposes as w^* the average vector of the extreme points of C_A^+. By the theorem above, this w^* is easy to obtain when A is a Q-operator: $w^* = (1/n)A^{-1}e$.

Remark 2.3. It should be noted that these kinds of quasi-orders do not necessarily need n linear relations. If the D-M is only able to supply $k \leq n$, these operators can be transformed without incorporating any additional information. This is possible by adding the natural relations $w_i \geq 0$ and removing the redundant ones. As an illustration, consider a problem with three objectives, where the D-M states that $w_1 \geq w_2$ but is unable to provide more information about the weights. Then he will have the following operator A and its inverse A^{-1}:

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad A^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and the quasi-order would be

$$xR_A y \quad \text{iff} \begin{cases} f_1(x) \geq f_1(y), \\ f_1(x) + f_2(x) \geq f_1(y) + f_2(y), \\ f_3(x) \geq f_3(y). \end{cases}$$

Remark 2.4. Another important property of Q-operators is the fact that they induce interval weights, which are easy to obtain, allowing a certain degree of sensitivity analysis in the numerical estimation of weights. The exact interval of $w_i \in C_A^+$, $\forall i = 1, \dots, n$, is given by

$$w_i \in \left[\min_j (\alpha_{ij}/\mu_j), \max_j (\alpha_{ij}/\mu_j) \right], \quad i = 1, \dots, n,$$

where α_{ij} and μ_j were defined in (7).

Indeed, for $i = 1, \dots, n$, let z_i be the optimal value of the linear program $\max\{w_i : w \in C_A^+\}$. This z_i is attained at an extreme point of C_A^+ and, thus, by Theorem 2.1, $z_i = \max_j(\alpha_{ij}/\mu_j)$. Similarly, one concludes with the minimum.

For instance, for the quasiorder R_A described in Remark 2.1, it is easily seen that

$$w_1 \in [1/n, 1], \quad w_2 \in [0, \tfrac{1}{2}], \dots, \quad w_n \in [0, 1/n].$$

As a final consequence, observe that one can also obtain the maximum and minimum value associated with each alternative $x \in X$ when the weight w varies in C_A^+, which is the basis of some decision-making methods (see, e.g. [3]). Indeed, as finding the maximum (respect. the minimum) value of $wf(x)$ when w varies in C_A^+ reduces to solving the linear program $\max\{wf(x) : w \in C_A^+\}$ (respectively $\min\{wf(x) : w \in C_A^+\}$), Theorem 2.1 implies that

$$\min_k \sum_{i=1}^{n} \frac{\alpha_{ik}}{\mu_k} f_i(x) \le wf(x) \le \max_k \sum_{i=1}^{n} \frac{\alpha_{ik}}{\mu_k} f_i(x)$$

$$\forall w \in C_A^+, \quad x \in X.$$

The family of operators proposed in the previous theorem is maximal in the sense that the unique set of weights in $\mathbb{R}^n$ with n extreme points whereby all the weights generated are non-negative are those with $A^{-1} \ge 0$. This is stated in the following theorem:

Theorem 2.2. *Let* $A \in \mathbb{R}^{n \times n}$ *be a linear operator such that* $\det(A) \ne 0$ *and* $C_A \ne \emptyset$. *Then* $C_A^+ = C_A$ *iff* $A^{-1} \ge 0$.

Proof. It is evident that, if $A^{-1} \ge 0$ then $C_A^+ = C_A$. We now show the converse. Let A be an $n \times n$ matrix with $\det(A) \ne 0$ such that $C_A^+ = C_A$. Define the sets D_A and D_A^+:

$$D_A = \{w \in \mathbb{R}^n : Aw \ge 0\},$$

$$D_A^+ = \{w \in \mathbb{R}^n : Aw \ge 0, w \ge 0\}.$$

First, we show that $D_A = D_A^+$. Indeed, it is evident that $D_A^+ \subset D_A$. Now, let $w \in D_A$, and we will show that $w \in D_A^+$. Obviously, if $w = 0$, then $w \in D_A^+$, so we can assume that $w \ne 0$; in other words, we only have to consider the cases $(ew > 0)$ and $(ew \le 0, w \ne 0)$.

Case 1. $ew > 0$. Let $w^1 = (1/(ew))w \in C_A = C_A^+$. Hence, $w^1 \ge 0$, thus $w \ge 0$, which (recall that $w \in D_A$) implies that $w \in D_A^+$.

Case 2. $ew \le 0$, $w \ne 0$. As by assumption $C_A \ne \emptyset$, there exists $w^0 \in C_A = C_A^+$. Define w^1 as follows:

$$w^1 = \frac{1}{1 - ew} w + \frac{-ew}{1 - ew} w^0.$$

Such w^1 verifies that $w^1 \in D_A$ and $ew^1 = 0$. Furthermore, at least one component of w^1 is negative. Indeed, if $w^1 \ge 0$, then, as $ew^1 = 0$, it would follow that $w^1 = 0$, thus $(ew)w^0 = w$; as $w \ne 0$, $ew \le 0$, one would obtain $ew < 0$, thus $0 \le Aw = (ew)Aw^0$; as $0 \le Aw^0$ and $ew < 0$, this would imply that $Aw^0 = 0$, i.e.: (recall that A^{-1} exists) $w^0 = 0$, which is a contradiction.

Hence, w^1 has at least a negative component, thus there exists some λ, $0 < \lambda < 1$ such that the vector $w^2 = \lambda w^1 + (1 - \lambda)w^0$ verifies that $Aw^2 \ge 0$, $ew^2 > 0$, w^2 has at least a negative component. By case 1, $w^2 \in D_A^+$, which is a contradiction. Hence, $D_A = D_A^+$, as we claimed.

With this, we are in position to show that $A^{-1} \ge 0$. Indeed, let v be a column of A^{-1}; as AA^{-1} gives the identity matrix, it follows that $Av \ge 0$, i.e., $v \in D_A$, thus $v \ge 0$. Then, we have shown that all the columns v of A^{-1} verify that $v \ge 0$, thus $A^{-1} \ge 0$, as asserted. □

The process of supplying information to the initial multi-criteria problem transforms the preference scheme. Thus in the beginning, i.e. when no information is available, one alternative $x \in X$ is preferred to another $y \in X$ iff $f(x) \ge f(y)$ (component-wise). That is, with no information, the preference scheme coincides with the Pareto quasi-order. So, it seems natural that in the process of supplying information, the more precise information the D-M gives, the more accurate quasi-order will be generated. It is evident that every quasi-order R_A with $A^{-1} \geq 0$, improves the no-information-relation because it reduces the set of weights. But given two relations R_A and R_B it is not clear how to determine which of them is the most accurate.

However, this is very important because it allows us to know the degree of knowledge shown by the D-M about his own problem. The more accurate the quasi-order, the better the knowledge of the problem. Moreover, it seems natural that the accuracy of a relation is inversely proportional to the magnitude of its set of weights as is proposed by Rios [18], so we define the accuracy of R_A in the following way.

Definition 2.2. Given a Q-operator A, the *accuracy* of R_A, $\mathrm{AC}(R_A)$, is defined as $\mu_{n-1}(C_I^+)/\mu_{n-1}(C_A^+)$, where μ_{n-1} represents the Lebesgue measure in $\mathbb{R}^{n-1}$.

The limit values for the accuracy are given in the following proposition.

Proposition 2.1. *If A is a Q-operator, then* $1 \leq \mathrm{AC}(R_A) < +\infty$.

Proof. First of all, $\mu_{n-1}(C_A^+) \leq \mu_{n-1}(C_I^+)$, $\forall A \in \mathbb{R}^{n\times n}$. Second, as $A^{-1} \geq 0$, C_A^+ is a simplex in the hyperplane $\Sigma_{1\leq i\leq n} w_i = 1$ and $\mu_{n-1}(C_A^+) > 0$. Then $1 \leq \mathrm{AC}(R_A) < +\infty$. □

The value 1 corresponds to the first no-information case and, hence, we can see the accuracy as a reduction measure of the set of weights. More precisely every operator that generates an *order* on $\mathbb{R}^n$ has an accuracy $+\infty$. An order relation of this kind could be seen as the limit case of a convergent sequence of quasi-orders with increasing and more precise information.

Finally we shall give the expression of the accuracy:

Theorem 2.3. *Let A be a Q-operator. Then,* $\mathrm{AC}(R_A) = |\det(A)| \Pi_{1\leq j\leq n} \mu_j$, *where μ_k was defined in* (7).

Proof. First, recall that the volume of the simplex generated by the points $x^1, \dots, x^n$ *in* $\mathbb{R}^{n-1}$ is [1]

$$\frac{1}{(n-1)!}\left|\det\begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1^1 & x_2^2 & \dots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-1}^1 & x_{n-1}^2 & \dots & x_{n-1}^n \end{pmatrix}\right|.$$

Let $e^i \in \mathbb{R}^n$ be the vector whose i-th component is one and zero the rest. Consider the system of reference $\mathscr{R}$ in the hyperplane $H = \{w : ew = 1\}$ that has e^1 as origin and the vectors $e^i - e^1$, $i = 2, \dots, n$, as generators. Then, any point $w = (w_1, \cdots, w_n) \in H$ has coordinates $(w_2, \cdots, w_n)$ in $\mathscr{R}$.

By Theorem 2.1, C_A^+ is generated by the vectors

$$\begin{pmatrix} \alpha_{1i}/\mu_i \\ \alpha_{2i}/\mu_i \\ \vdots \\ \alpha_{ni}/\mu_i \end{pmatrix}, \quad i = 1, \dots, n.$$

Hence,

$$\mathrm{AC}(R_A) = \left|\det\begin{pmatrix} 1 & 1 & \dots & 1 \\ \alpha_{21}/\mu_1 & \alpha_{22}/\mu_2 & \dots & \alpha_{2n}/\mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1}/\mu_1 & \alpha_{n2}/\mu_2 & \dots & \alpha_{nn}/\mu_n \end{pmatrix}\right|^{-1}.$$

Thus (recall that $\mu_i = \Sigma_{j=1}^n \alpha_{ij}$, $\forall i$),

$$\mathrm{AC}(R_A) = \left|\det\begin{pmatrix} \alpha_{11}/\mu_1 & \alpha_{12}/\mu_2 & \dots & \alpha_{1n}/\mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1}/\mu_1 & \alpha_{n2}/\mu_2 & \dots & \alpha_{nn}/\mu_n \end{pmatrix}\right|^{-1}.$$

In other words,

$$\mathrm{AC}(R_A) = \frac{1}{|\det(A^{-1})| \Pi_{j=1}^n (1/\mu_j)} = |\det(A)| \prod_{j=1}^n \mu_j,$$

as asserted. □

Example 2.1. For the quasiorder R_A described in Remark 2.1, one has that $\det(A) = 1$ and $\mu_j = j$, $\forall j$, thus $\mathrm{AC}(R_A) = n!$

For the quasiorders R_A and R_{A_1} introduced in Example 1.1, the improvement in the order relation from R_A to R_{A_1} can be measured by means of the accuracy: $\mathrm{AC}(R_A) = 67.5$ and $\mathrm{AC}(R_{A_1}) = 171.5$.

3. Some families of Q-operators

The widest class of operators we can deal with is characterized in Theorem 2.2. However, the condition shown in the previous paragraph is difficult to check beforehand. So in order to enable the D-M to apply these relations, we propose two sub-classes belonging to the original one with three important properties:

1. to know beforehand that they belong to the broad class;
2. to be easy for the D-M to understand and accept;
3. to be sure that the set of weights it generates is not empty.

A procedure to obtain a class of these operators consists in offering the D-M the comparison of each criterion $f_i(.)$ with at most a coalition of the remaining criteria.

Usually, the exact determination of the weights is made by the trade-off between a criterion and the remainders. But this methodology cannot be used if the D-M does not give its preferences so precisely.

Alternatively, our approach proposes to replace this equivalence by inequalities which are quite acceptable for the D-M. In this process, the more precise information the D-M supplies, the more accurate the quasi-order it generates, and hence it is closer to the cardinal utility.

Therefore, the information required from the D-M about the criterion $f_i(.)$ which he is willing to give, will have the following form:

$$w_i \geq \sum_{j \neq i} \tilde{a}_{ij} w_j, \quad \tilde{a}_{ij} \geq 0, \quad \sum_{j \neq i} \tilde{a}_{ij} \leq 1,$$

where a_{ij} represents the minimum marginal substitution rate of f_i for f_j. We should notice that when no information is available, this kind of relation will be $w_i \geq 0$. But, even by supplying small values of a_{ij} one improves the accuracy of the quasi-order given by the D-M and avoids the problem of the exact estimation of the weights (cardinal utility).

Example 3.1. We shall deal with the following example in which the D-M has three objectives, a set X of feasible alternatives, and he is able to give the required information in the form of inequalities.

$$\max_{x \in X} (f_1(x), f_2(x), f_3(x))$$

$$w_1 \geq 0.5 w_2,$$

$$w_2 \geq 0.5 w_1 + 0.5 w_3,$$

$$w_3 \geq 0.5 w_2.$$

Then the operator is defined by the matrix A, whose extreme points are given by the columns of the second matrix and the interval weights are

$$A = \begin{pmatrix} 1 & -0.5 & 0 \\ -0.5 & 1 & -0.5 \\ 0 & -0.5 & 1 \end{pmatrix},$$

$$E = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \quad \begin{matrix} w_1 \in [\frac{1}{6}, \frac{1}{2}] \\ w_2 \in [\frac{1}{3}, \frac{1}{2}]. \\ w_3 \in [\frac{1}{6}, \frac{1}{2}] \end{matrix}$$

Hence its accuracy $\mathrm{AC}(R_A) = 18$.

This example suggests the possibility that these kinds of operators have the property of inverse positive. In order to clarify the terms used in the following results we introduce two classical concepts [16].

Definition 3.1. A linear operator $A \in \mathbb{R}^{n \times n}$ is diagonally dominant if

$$\sum_{j \neq i} |a_{ij}| \leq |a_{ii}|, \quad i = 1, \ldots, n,$$

and strictly diagonally dominant if strict inequalities hold for all $i = 1, \ldots, n$.

The result suggested by the example above is stated in the following theorem.

Theorem 3.1. *Let $A \in \mathbb{R}^{n \times n}$ be a diagonally dominant linear operator such that $a_{ij} \leq 0$, $i \neq j$, $a_{ii} > 0$, $\forall i = 1, \ldots, n$, and $\det(A) \neq 0$. Then $A^{-1} \geq 0$.*

Proof. For simplicity and without loss of generality, we consider $a_{ii}=1$, $\forall i$. As A is diagonally dominant, the inequalities $a_{ii} \geq -\sum_{j \neq i} a_{ij}$, $\forall i = 1,\dots,n$, hold. Then every cofactor A_{ij} of the matrix A is nonnegative [4].

Let A^{-1} be the inverse of A, whose elements are $\alpha_{ij} = A_{ji}/\det(A)$. Then the sign of α_{ij}, $\forall i, j$, coincides with the sign of $\det(A)$ and

$$\det(A) = \det\begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 0 & 1-a_{12}a_{21} & \cdots & a_{2n}-a_{1n}a_{21} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2}-a_{12}a_{n1} & \cdots & 1-a_{1n}a_{n1} \end{pmatrix} = \det\left(\begin{array}{c|c} 1 & a_{12} \cdots a_{1n} \\ \hline \Theta & A_1 \end{array}\right) = \det(A_1).$$

Besides, the matrix A_1 is diagonally dominant because for all $i=2,\dots,n$ their elements verify $1-a_{1i}a_{i1}>0$ and

$$1 + \sum_{j \neq i, j \neq 1} a_{ij} + a_{i1}\Big(-\sum_{j \neq 1} a_{1j}\Big) \geq 1 + \sum_{j \neq i} a_{ij} \geq 0.$$

Thus, we can repeat this reasoning n times and we obtain that $\det(A) > 0$. □

Corollary 3.1. *Let $A \in \mathbb{R}^{n \times n}$ be a strictly diagonally dominant linear operator such that $a_{ij} \leq 0$, $i \neq j$, $a_{ii} > 0$, $\forall i = 1,\dots,n$. Then $A^{-1} \geq 0$.*

This class of operators is closely related to a well known family of linear operators called M-operators [16,25], but the first one exhibits in its favor the easy interpretability and manipulation because the M-operators require properties of irreducibility or strict diagonal dominance, not needed in the proposed class.

It must also be noted that the well known ordinal relation mentioned in Remark 2.1 is an example of such an operator.

Another important sub-class of operators belonging to the class characterized by Theorem 2.2 are the triangular and nonpositive off-diagonal elements. This type of operator corresponds to the situation in which the D-M is able to do a certain rank order in the criteria and in addition, he can compare the importance of several criteria between them. So the expression obtained for the criterion f_i ranked in i-th position is

$$w_i \geq \sum_{j>i} a_{ij} w_j, \quad a_{ij} \geq 0,$$

where a_{ij} represents the minimum marginal substitution rate of f_i for f_j.

In this sub-class the three properties enumerated at the beginning of Section 3 also hold and they belong to the class characterized by Theorem 2.2.

Theorem 3.2. *Let $A \in \mathbb{R}^{n \times n}$ be a diagonal positive triangular linear operator such that $a_{ij} \leq 0$, $\forall j > 1$. Then $A^{-1} \geq 0$.*

Proof. In this situation $\det(A) = \prod_{1 \leq i \leq n} a_{ii} > 0$ and the cofactors are

$$A_{ij} = \prod_{k \neq i, k \neq j} a_{kk} > 0 \quad \forall i < j,$$

$$A_{ii} = \prod_{k \neq i} a_{kk} > 0, \quad i = 1,\dots,n,$$

$$A_{ij} = 0 \quad \forall i > j.$$

Hence, reasons analogous to the ones we used in Theorem 3.1 prove this theorem. □

4. Non-homogeneous Q-operators

On many occasions, the D-M is willing to obtain at least certain levels in his weights [14], so the quasi-order is given by a linear operator A and a level vector $\lambda \geq 0$. In this case, the set of feasible weights is given by the polytope

$$C^{+}_{(A,\lambda)} = \Big\{ w \in \mathbb{R}^n : w \geq 0,\ Aw \geq \lambda, \sum_{i=1,\dots,n} w_i = 1 \Big\}. \tag{8}$$

Hence the pair (A, λ) induces the relation $R_{(A,\lambda)}$ given by

$$xR_{(A,\lambda)}y \quad \text{iff} \quad w(f(x)-f(y)) \geq 0 \quad \forall w \in C^{+}_{(A,\lambda)}. \tag{9}$$

An interesting type of these relations are those defined by a matrix A with non-negative inverse (*componentwise*) and $eA^{-1}\lambda < 1$. In what follows we say these relations are induced by a *non-homogeneous Q-operator* (A, λ). To this kind of relations we can extend all the previous results with minimum effort as can be seen from the following theorem.

Theorem 4.1. *Let* (A, λ) *be a non-homogeneous Q-operator. Then*

$$w \in C^+_{(A,\lambda)} \quad \text{iff} \quad \frac{w - A^{-1}\lambda}{1 - eA^{-1}\lambda} \in C^+_A.$$

Proof. $w \in C^+_{(A,\lambda)}$ iff $\exists t \geq 0 \mid Aw = \lambda + t,\ w \geq 0,$ $ew = 1$; i.e. (recall that $A^{-1} \geq 0$):

$$w \in C^+_{(A,\lambda)} \quad \text{iff} \quad \exists t \geq 0 \mid w - A^{-1}\lambda = A^{-1}t,$$

$ew = 1$.

As $e(w - A^{-1}\lambda) = 1 - eA^{-1}\lambda > 0$, it follows that

$$w \in C^+_{(A,\lambda)} \quad \text{iff} \quad \exists t \geq 0 \mid \frac{1}{1 - eA^{-1}\lambda}(w - A^{-1}\lambda)$$

$$= A^{-1}\frac{t}{1 - eA^{-1}\lambda} \quad \text{iff} \quad \exists \hat{t} \geq 0 \mid \hat{w} = A^{-1}\hat{t},$$

$e\hat{w} = 1$,

$$\text{where } \hat{w} = \frac{w - A^{-1}\lambda}{1 - eA^{-1}\lambda} \quad \text{iff} \quad \hat{w} \in C^+_A.$$

Which concludes the proof. □

Corollary 4.1. *If* (A, λ) *is a non-homogeneous Q-operator, then the set* $C^+_{(A,\lambda)}$ *is the convex hull of the columns of the following matrix*:

$$A^{-1}\begin{pmatrix} 1 - \mu_1^{-1}(1 - \Sigma_{k \neq 1}\mu_k\lambda_k) & \lambda_1 & \cdots & \lambda_1 \\ \lambda_2 & 1 - \mu_2^{-1}(1 - \Sigma_{k \neq 2}\mu_k\lambda_k) & \cdots & \lambda_2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_n & \lambda_n & \cdots & 1 - \mu_n^{-1}(1 - \Sigma_{k \neq n}\mu_k\lambda_k) \end{pmatrix}$$

where μ_j, $j = 1, \ldots, n$, *were defined in* (7).

Although all the properties holding in the homogeneous case can be extended to the non-homogeneous case, we wish to pay some attention to the following problem. We know (recall Remark 2.3) that there exist many quasi-orders that produce the same intervals for their weights; but, are there any of them which must be emphasized?

The answer is affirmative and among these operators one is particularly interesting; it is known in the literature as E-cone [24] which is one of our non-homogeneous Q-operators. This kind of operator (the E-cone) is based on relations of the form $w_i \geq k_i$, $i = 1, \ldots, n$ (one per objective). Their two main properties are that the set of weights it defines includes the interval weights considered beforehand and also is minimal (in the inclusion sense among the operators of this kind). We state and prove this in the following theorem.

Let $k = (k_1, \ldots, k_n) \geq 0$. Consider the set

$$C^+_{(I,k)} = \left\{ w \in \mathbb{R}^n : w_i \geq k_i;\ \sum_{1 \leq i \leq n} w_i = 1 \right\}$$

and for each α, $\beta \in \mathbb{R}^n$ such that $0 \leq \alpha_i \leq \beta_i \leq 1$, $\forall i$, and $\Sigma_{i=1}^n \alpha_i < 1 < \Sigma_{i=1}^n \beta_i$, the set (see [2])

$$\chi_{(\alpha,\beta)} = \left\{ w \in \mathbb{R}^n : \alpha \leq w \leq \beta;\ \sum_{1 \leq i \leq n} w_i = 1 \right\}.$$

Theorem 4.2. *The vector* $k = (k_1, \ldots, k_n)$ *with* $k_i = \max(\alpha_i, 1 - \Sigma_{j \neq i}\beta_j)$, $i = 1, \ldots, n$, *determines the minimum set* $C^+_{(I,k)}$ *containing the set* $\chi_{(\alpha,\beta)}$.

Proof. For all $(k_1, \ldots, k_n) \geq 0$, $\Sigma_{1 \leq i \leq n} k_i \leq 1$, one can obtain, using Corollary 4.1, that $C^+_{(I,k)}$ is the convex hull of the columns of the following matrix:

$$\begin{pmatrix} 1-\sum_{i\neq 1}k_i & k_1 & \cdots & k_1 \\ k_2 & 1-\sum_{i\neq 2}k_i & \cdots & k_2 \\ \vdots & \vdots & \ddots & \vdots \\ k_n & k_n & \cdots & 1-\sum_{i\neq n}k_i \end{pmatrix}.$$

The lower limit for w_i, $i=1,\dots,n$, is given by $k_i = \min(k_i, 1-\sum_{j\neq i}k_j)$, because $\sum_{1\leq i\leq n}k_i \leq 1$.

To obtain the value of k_i, we solve the linear program

$$\min\left\{w_i:\ \sum_{1\leq j\leq n} w_j = 1;\ \alpha_j \leq w_j \leq \beta_j,\ j=1,\dots,n\right\}$$

which by assumption verifies $0 \leq \alpha_j \leq w_j \leq \beta_j \leq 1$, $\forall j$. Thus, it follows that

$$k_i = \max\left(\alpha_i, 1-\sum_{j\neq i}\beta_j\right), \quad i=1,\dots,n.$$

Besides, it is easy to see that if in one j, $k_j' < k_j$ is taken, the set $C^+_{(I,k)} \subset C^+_{(I,(k_1,\dots,k_j',\dots,k_n))}$. In addition, if in one j, $k_j' > k_j$, at least one interval is not included in $C^+_{(I,(k_1,\dots,k_j',\dots,k_n))}$:

1. If $\alpha_j = \max(\alpha_j, 1-\sum_{k\neq j}\beta_k)$, then $k_j' > \alpha_j = k_j$ and $[\alpha_j, \beta_j]$ is not included.
2. Otherwise, consider $\hat{w}$, whose components are: $\hat{w}_l = \beta_l$, $\forall l \neq j$, and $\hat{w}_j = 1-\sum_{l\neq j}\hat{w}_l$. Then $\hat{w} \in C^+_{(I,k)} \cap \chi_{(\alpha,\beta)}$, but $\hat{w} \notin C^+_{(I,(k_1,\dots,k_j',\dots,k_n))}$. Hence, some of the intervals $[\alpha_l, \beta_l]$, $l\neq j$, are not included.

Thus, we conclude the theorem. □

As a direct consequence of this theorem we can obtain a sufficient condition on $\chi_{(\alpha,\beta)}$ in order for it to be a set of the form $C^+_{(I,k)}$.

Corollary 4.2. *If* $\min_{j=1,\dots,n}(\beta_j) > \max_{j=1,\dots,n}(1-\sum_{k\neq j}\alpha_k)$, *then the set* χ *defines a set* $C^+_{(I,k)}$.

Proof. Because of $\min_{j=1,\dots,n}(\beta_j) > 1-\sum_{k\neq j}\alpha_k$, $\forall j=1,\dots,n$, the points of the form $(\alpha_1, \alpha_2,\dots,\alpha_{j-1}, 1-\sum_{k\neq j}\alpha_k, \alpha_{j+1},\dots,\alpha_n)$ are the unique extreme points of the set χ. Then, the set

$$C^+_{(I,(\alpha_1,\dots,\alpha_n))}$$

is the one sought. □

5. Conclusions and extensions

In this paper we address the multicriteria decision problem under the linear utility approach with partial information about the weighting coefficients.

We introduce a family of polytopes of weights, generated by homogeneous linear relations whose induced quasi-orders are easily manageable.

We study some particular cases that D-M might understand and accept.

Finally, we broaden these relations to the non-homogeneous case, showing that all the previous properties can be extended.

Some extensions of this work can be considered. Concretely, it is possible to extend these results for more relations than the number of criteria. This extension follows under certain conditions of the generalized inverse of the matrix A. In addition, the results obtained in this work can be used as a basis for the development of an interactive procedure based on Q-operators.

However, these extensions are not trivial, and will be discussed in depth in a forthcoming paper.

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