COMBINING MINSUM AND MINMAX: A GOAL PROGRAMMING APPROACH

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(Received March 1998; revision received February 1999; accepted July 1999)

A number of methods for multiple-objective optimization problems (MOP) give as solution to MOP the set of optimal solutions for some single-objective optimization problems associated with it. Well-known examples of these single-objective optimization problems are the minsum and the minmax. In this note, we propose a new parametric single-objective optimization problem associated with MOP by means of Goal Programming ideas. We show that the minsum and minmax are particular instances, so we are somehow combining minsum and minmax by means of a parameter. Moreover, such parameter has a clear meaning in the value space. Applications of this parametric problem to classical models in Locational Analysis are discussed.

1. INTRODUCTION

Let $S \subset \mathbb{R}^m$ be a nonempty compact set. For each $j = 1, \ldots, n$, let $f_j : S \to \mathbb{R}$ be continuous, and consider the vector optimization problem

$$\min_{x \in S} (f_1(x), \ldots, f_n(x)). \tag{1}$$

The two standard scalar problems associated with (1) are the *minsum*

$$\min_{x \in S} \sum_{j=1}^{n} \omega_j f_j(x), \tag{2}$$

where $\omega_1, \ldots, \omega_n$ are strictly positive weights assumed, without loss of generality, to sum unity, and the *minmax*

$$\min_{x \in S} \max_{j=1,\ldots,n} f_j(x), \tag{3}$$

which minimize respectively the average and the highest value of the functions f_j.

When both the average and the highest value are important, a compromise criterion must be constructed, the usual choice being a convex combination of the two criteria, i.e., for a fixed λ in [0, 1], one obtains

$$(1-\lambda)\left(\sum_{j=1}^{n} \omega_j f_j\right) + \lambda \max_{j=1,\ldots,n} f_j$$

yielding the scalar problem $Q(\lambda)$ defined as

$$\min_{x \in S} \left\{ (1-\lambda)\left(\sum_{j=1}^{n} \omega_j f_j(x)\right) + \lambda \max_{j=1,\ldots,n} f_j(x) \right\}. \qquad Q(\lambda)$$

We may observe that problem $Q(\lambda)$ has as particular cases (2) and (3) when λ is equal to zero and one, respectively. However, finding real meaning to the parameter λ to be chosen in $Q(\lambda)$ is not an easy task (e.g., Carrizosa et al. 1994, Saaty 1980).

In this note, we introduce another parametric problem which, as $Q(\lambda)$, includes as particular choices both (2) and (3). However, the parameter involved in our model has a clear interpretation as targets of a certain Goal Programming model (see e.g., Ignizio 1978, Romero 1991, Steuer 1986, Tamiz et al. 1998, Tamiz et al. 1995 and the references therein for further details on Goal Programming).

The remainder of this note is structured as follows. In §2 we introduce a new parametric problem $P(z)$ by using a Goal Programming model. We prove properties of this new problem. Moreover, we show that, in general, the parametric problem $Q(\lambda)$ and ours are not equivalent. The efficiency of the solutions for $P(z)$ is discussed in §3. We devote §4 to applications of our methodology to derive new models in Locational Analysis. Finally, §5 contains a short summary.

2. A NEW PARAMETRIC PROBLEM

Suppose we could give for all the functions f_j a common target $f_j(x) \leqslant z$ for some $z \in \mathbb{R}$. This is possible when all functions measure in the same units (see §4) or after normalizing in such a way that e.g.,

$$0 = \min_{x \in S} f_j(x),$$

$$1 = \max_{x \in S} f_j(x)$$

(see Tamiz et al. 1998). This yields the following Goal Programming formulation:

$$\min_{x \in S} \sum_{j=1}^{n} \omega_j \max(f_j(x) - z, 0). \tag{4}$$

Subject classifications: Decision analysis: multiple criteria theory. Facilities: Continuous location/discrete location.
Area of review: Decision Making.

Assuming ω_j to sum unity, the objective function of (4) can be rewritten as

$$\left(\sum_{j=1}^{n} \omega_j \max(f_j(x), z)\right) - z.$$

Define for each $z \in \mathbb{R}$, the scalar problem $P(z)$,

$$\min_{x\in S} \sum_{j=1}^{n} \omega_j \max(f_j(x), z), \qquad P(z)$$

which, by the previous discussion, is equivalent to minimizing the average deviation with respect to the common target $f_j(x) \leqslant z$.

We discuss in this section properties of $P(z)$. Denote by $z_{\min\min}$ and $z_{\min\max}$ respectively the optimal values of the minmin and minmax problems,

$$z_{\min\min} = \min_{x\in S} \min_{j=1,\dots,n} f_j(x),$$
$$z_{\min\max} = \min_{x\in S} \max_{j=1,\dots,n} f_j(x),$$

which, due to the compactness of S and the continuity of functions f_j, are finite and attained.

As $Q(\lambda)$, the parametric problem $P(z)$ includes as instances the *minsum* and *minmax* problems, in the sense that particular choices of z yield them.

Proposition 2.1. *The following statements hold:*

1. *For any $z \leqslant z_{\min\min}$, the problem $P(z)$ is equivalent to the minsum problem* (2).
2. *The problem $P(z_{\min\max})$ is equivalent to the minmax problem* (3).
3. *For $z \geqslant z_{\min\max}$, any optimal solution to $P(z_{\min\max})$ also solves $P(z)$.*

Proof. Since, for $z \leqslant z_{\min\min}$, $x \in S$ and j, $1 \leqslant j \leqslant n$, one has $f_j(x) \geqslant z$, it then follows that

$$\sum_{j=1}^{n} \omega_j \max(f_j(x), z) = \sum_{j=1}^{n} \omega_j f_j(x), \quad \text{for each } x \in S, \qquad (5)$$

thus part 1 follows.

Let $x_{\min\max}$ be an optimal solution to (3). In particular,

$$f_j(x_{\min\max}) \leqslant z_{\min\max}, \quad \text{for each } j = 1, \dots, n. \qquad (6)$$

Thus, for any $x \in S$, we have by (6)

$$\sum_{j=1}^{n} \omega_j \max(f_j(x), z_{\min\max}) \geqslant \sum_{j=1}^{n} \omega_j z_{\min\max} = \sum_{j=1}^{n} \omega_j \max(f_j(x_{\min\max}), z_{\min\max}),$$

thus $x_{\min\max}$ solves $P(z_{\min\max})$.

Conversely, let x^* be optimal for $P(z_{\min\max})$ and we will show that x^* also solves (3). Indeed, by the optimality of x^* for $P(z_{\min\max})$ and (6), we have that

$$\begin{aligned}\sum_{j=1}^{n} \omega_j z_{\min\max} &\leqslant \sum_{j=1}^{n} \omega_j \max(f_j(x^*), z_{\min\max}) \\ &\leqslant \sum_{j=1}^{n} \omega_j \max(f_j(x_{\min\max}), z_{\min\max}) \\ &= \sum_{j=1}^{n} \omega_j z_{\min\max}.\end{aligned}$$

Hence, the inequalities above are equalities and thus it follows for each j that

$$\max(f_j(x^*), z_{\min\max}) \leqslant z_{\min\max},$$

thus

$$\max_{j=1,\dots,n} f_j(x^*) \leqslant z_{\min\max},$$

showing that x^* also solves (3), and this shows part 2.

For part 3, observe first that any optimal solution x^* to $P(z_{\min\max})$ also solves (3), thus,

$$\max(f_j(x^*), z_{\min\max}) = z_{\min\max}, \quad \text{for each } j = 1, 2, \dots, n,$$

and then, for any $z \geqslant z_{\min\max}$,

$$\max(f_j(x^*), z) = z, \quad \text{for each } j = 1, 2, \dots, n,$$

thus for any $x \in S$,

$$\begin{aligned}\sum_{j=1}^{n} \omega_j \max(f_j(x^*), z) &= \sum_{j=1}^{n} \omega_j z \\ &\leqslant \sum_{j=1}^{n} \omega_j \max(f_j(x), z),\end{aligned}$$

and the result follows. □

We have two parametric problems $Q(\lambda)$ and $P(z)$ associated with the vector optimization problem (1). It could happen that these two parametric problems were equivalent, i.e., the solution set obtained by $Q(\lambda)$ when λ varies on [0, 1] is the same as the solution set obtained by $P(z)$ when z varies $[z_{\min\min}, z_{\min\max}]$. However, an easy example shows that these parametric problems are not equivalent.

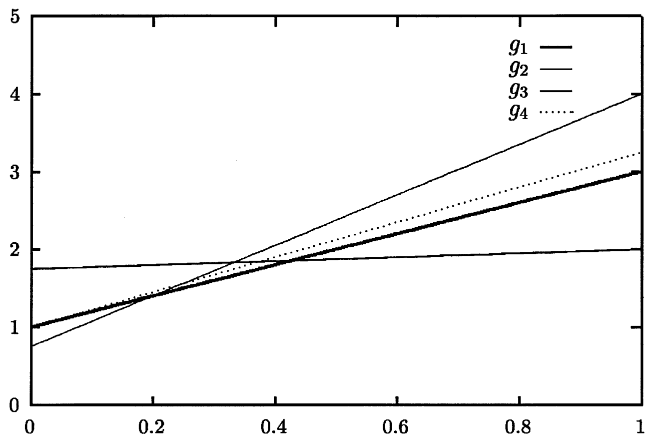
Example 2.2. *Consider the discrete multiobjective linear problem in* $\mathbb{R}^2$

$$\min_{x\in S}(f_1(x), f_2(x), f_3(x)),$$

where

$$f_1(x_1, x_2) = -3x_1 + 4x_2,$$
$$f_2(x_1, x_2) = 3x_1 - x_2,$$
$$f_3(x_1, x_2) = x_1 + x_2,$$
$$S = \{(1, 0), (0, 1), (1, 1), (0.25, 1)\},$$

Figure 1. Trajectory of $Q(\lambda)$ for each point in S.



and strictly positive weights $\omega_1 = 1/4$, $\omega_2 = 1/2$, and $\omega_3 = 1/4$. Figure 1 plots functions g_1, g_2, g_3, and g_4 which correspond to the function on $\lambda \in [0, 1]$,

$$g_{(x_1,x_2)}(\lambda) = (1-\lambda)\sum_{j=1}^{3}\omega_j f_j(x_1,x_2) + \lambda \max_{j=1,2,3} f_j(x_1,x_2),$$

where (x_1, x_2) is equal to $(1, 0)$, $(0, 1)$, $(1, 1)$, and $(0.25, 1)$ respectively. It is straightforward to see that the solution set obtained by $Q(\lambda)$ is equal to

$$\begin{cases}(0,1) & \text{if } 0 \leqslant \lambda \leqslant 0.2,\\ (1,0) & \text{if } 0.2 \leqslant \lambda \leqslant 3/7,\\ (1,1) & \text{if } 3/7 \leqslant \lambda \leqslant 1.\end{cases}$$

Figure 2 plots functions h_1, h_2, h_3, and h_4 which correspond to the function on $z \in [z_{\min\min} = -3, z_{\min\max} = 2]$,

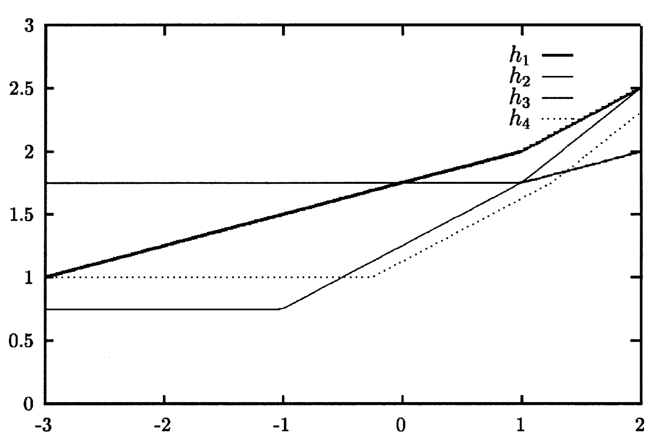
$$h_{(x_1,x_2)}(z) = \sum_{j=1}^{3}\omega_j \max(f_j(x_1,x_2), z),$$

where (x_1, x_2) is equal to $(1, 0), (0, 1), (1, 1)$, and $(0.25, 1)$ respectively. We may observe that the solution set obtained by this criterion is equal to

$$\begin{cases}(0,1) & \text{if } -3 \leqslant z \leqslant -0.5,\\ (0.25,1) & \text{if } -0.5 \leqslant z \leqslant 1.375,\\ (1,1) & \text{if } 1.375 \leqslant z \leqslant 2.\end{cases}$$

Observe that $Q(0)$ and $P(-3)$ are equivalent, as well as for $Q(1)$ and $P(2)$, as stated in Proposition 2.1. □

Figure 2. Trajectory of $P(z)$ for each point in S.



3. EFFICIENCY

One desirable property when finding a solution for (1) is efficiency. It is well known that $Q(\lambda)$ gives efficient solutions when $\lambda \in [0, 1)$. As usual in Goal Programming, there could exist optimal solutions for $P(z)$ which are not efficient. However, we can show that at least one of them is efficient by using standard restoration techniques (Romero 1991, Tamiz et al. 1998).

PROPOSITION 3.1. *Let $x(z)$ be an arbitrary optimal solution for $P(z)$. Then, any optimal solution for*

$$\begin{aligned} \min \ & \sum_{j=1}^{n}\omega_j f_j(y)\\ \text{s.t.}\ & y \in S \qquad\qquad (7)\\ & f_j(y) \leqslant f_j(x(z)) \quad \text{for each } j = 1,\dots,n \end{aligned}$$

is optimal for $P(z)$ and efficient for (1).

PROOF. Since S is assumed to be compact and each f_j is continuous on S, for each $z \in \mathbb{R}$, an optimal solution $x(z)$ for $P(z)$ exists. The feasible region of (7), $S(x(z))$, is a nonempty compact set. It is evident that any point in $S(x(z))$ also solves $P(z)$; moreover, any optimal solution to (7) is efficient for (1). □

We cannot ensure efficiency for all the optimal solutions for $P(z)$. Nevertheless, the next proposition proves weak efficiency for all of them.

PROPOSITION 3.2. *For each $z \leqslant z_{\min\max}$, any optimal solution for $P(z)$ is a weakly efficient solution for problem (1).*

PROOF. By contradiction, we will assume that the result is not true. Then, there exists some optimal solution $x(z)$ for $P(z)$ and some $y \in S$ such that

$$f_j(y) < f_j(x(z)), \quad \text{for each } j = 1,\dots,n. \qquad (8)$$

Moreover, there exists at least an index j_0 such that

$$f_{j_0}(y) \geqslant z. \qquad (9)$$

Indeed, otherwise, for each j, $f_j(y) < z$, thus

$$\max_{j=1,\dots,n} f_j(y) < z \leqslant z_{\min\max},$$

but this is a contradiction with the definition of $z_{\min\max}$. So, from (8),

$$\max(f_j(y), z) \leqslant \max(f_j(x(z)), z), \quad \text{for each } j = 1,\dots,n,$$

and, by multiplying by ω_j, we have

$$\omega_j \max(f_j(y), z) \leqslant \omega_j \max(f_j(x(z)), z), \quad \text{for each } j = 1, \dots, n.$$

Moreover, by (8) and (9), the last inequality is strict for $j = j_0$. Then,

$$\sum_{j=1}^{n} \omega_j \max(f_j(y), z) < \sum_{j=1}^{n} \omega_j \max(f_j(x(z)), z),$$

but this is a contradiction with the fact that $x(z)$ is optimal for $P(z)$. $\square$

4. APPLICATIONS

4.1. *p*-Facility Location Problem

A set of n customers, with demand $\omega_1, \dots, \omega_n$, must be served from a set of p plants, to be chosen from m candidate sites. For each $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$, d_{ij} denotes the distance from customer j to candidate plant i.

The two basic p-facility location problems are the p-median and the p-center (e.g., Daskin 1995, Haudler 1990, Mirchandani 1990),

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} \omega_j \min_{i:\, y_i=1} d_{ij} \\ \text{s.t.} \quad & \sum_{i=1}^{m} y_i = p \\ & y_i \in \{0, 1\} \forall i \end{aligned}$$

and

$$\begin{aligned} \min \quad & \max_{j=1,\dots,n} \min_{i:\, y_i=1} d_{ij} \\ \text{s.t.} \quad & \sum_{i=1}^{m} y_i = p \\ & y_i \in \{0, 1\} \forall i, \end{aligned}$$

where, for each $i = 1, \dots, n$, y_i is the binary variable

$$y_i = \begin{cases} 1 & \text{if plant } i \text{ is open} \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

It is well known that both problems can be readily rewritten as Linear Integer problems; indeed, by defining for each $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$ the allocation variable x_{ij},

$$x_{ij} = \begin{cases} 1 & \text{if customer } j \text{ is served by plant } i, \\ 0 & \text{otherwise,} \end{cases}$$

and the region S,

$$S = \left\{ (x, y) \colon x_{ij} \leqslant y_i \text{ for each } i, j, \quad \sum_{i=1}^{m} y_i = p, x_{ij}, y_i \in \{0, 1\} \text{ for each } i, j \right\},$$

the p-median problem can then be written as

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} \omega_j \sum_{i=1}^{m} d_{ij} x_{ij} \\ \text{s.t.} \quad & (x, y) \in S \end{aligned}$$

and the p-center becomes

$$\begin{aligned} \min \quad & W \\ \text{s.t.} \quad & \sum_{i=1}^{m} d_{ij} x_{ij} \leqslant W \forall j \\ & (x, y) \in S. \end{aligned}$$

Define $f_j(x) = \sum_{i=1}^{m} d_{ij} x_{ij}$. In this case, the parametric problem $P(z)$ can be written as follows

$$\begin{aligned} & \min_{(x, y) \in S} \sum_{j=1}^{n} \omega_j \max(f_j(x), z) \\ & = \min_{(x, y) \in S} \sum_{j=1}^{n} \omega_j \max\left(\sum_{i=1}^{m} d_{ij} x_{ij}, z \right). \end{aligned}$$

The next result will show that the parametric problem $P(z)$ is equivalent to a p-median problem.

Lemma 4.1. *For any $z \in \mathbb{R}$, problem $P(z)$ is equivalent to the p-median problem with distances $\hat{d}_{ij}(z) := \max(d_{ij}, z)$.*

Proof. Let $P'(z)$ denote the p-median problem with distances $\hat{d}_{ij}(z)$, i.e.,

$$\min_{(x, y) \in S} \sum_{j=1}^{n} \omega_j \sum_{i=1}^{m} \max(d_{ij}, z) x_{ij}. \tag{$P'(z)$}$$

The feasible regions of both problems are the same. Then, we only need to show that the objective functions coincide in both problems at any feasible point. Let (x, y) be a feasible solution for $P(z)$ and $P'(z)$. The value of the objective function associated with index j in $P(z)$ is equal to

$$\omega_j \max\left(\sum_{i=1}^{m} d_{ij} x_{ij}, z \right) = \omega_j \max\left(\sum_{i=1}^{m} d_{ij} x_{ij}, \sum_{i=1}^{m} z x_{ij} \right) \tag{11}$$

$$= \omega_j \sum_{i=1}^{m} \max(d_{ij}, z) x_{ij}, \tag{12}$$

where equality (11) follows from $\sum_{i=1}^{m} x_{ij} = 1$, and (12) since exactly one variable x_{ij} is equal to 1 and the rest are zeroes. We have just shown that, for each j, the objective function associated with j in $P(z)$ and $P'(z)$ are equal, and then, the result follows. $\square$

4.2. Locating an Obnoxious Service

The models described in the previous section are suitable when the facilities to be located are desirable, in the sense that customers want them near, and the minimization of transportation costs is the main concern. When, due to the nature of the facilities, the main concern is, on the contrary, the environmental impact (individuals want the facilities far), one faces completely different location models, as reviewed in Erkut and Neuman (1989) and Plastria (1996).

Suppose that a single obnoxious facility is to be located within a region $S \subset \mathbb{R}^2$ with a polygonal boundary. Such location affects a set A of population areas each modeled as a point a in $\mathbb{R}^2$, with population ω_a.

The two basic models are the euclidean maxsum and maxmin, maximizing the average and minimum euclidean distance (Erkut and Neuman 1989 and Plastria 1996). Such models correspond to cost minimization models if one assumes that locating the facility at x yields for $a \in A$ a per-habitant (environmental) cost $K - d_a(x)$, where $d_a(x)$ is the euclidean distance from $a \in A$ to x and K is a positive constant, such as

$$K \geqslant \max_{x \in S} \max_{a \in A} \omega_a d_a(x).$$

Thus, we obtain the following equivalent formulations for the maxsum,

$$\min_{x \in S} \sum_{a \in A} \omega_a (K - d_a(x)) \tag{13}$$

and the maxmin,

$$\min_{x \in S} \max_{a \in A} (K - \omega_a d_a(x)). \tag{14}$$

The corresponding parametric problem $P(z)$ becomes now

$$\min_{x \in S} \sum_{a \in A} \omega_a \max(K - d_a(x), z), \tag{15}$$

which was suggested in Erkut and Neuman (1989) as a more realistic version of (13) (it models environmental impact as linearly decaying until reaching a threshold value, after which it remains constant/negligible), but no solution procedure has been proposed since then.

The extreme cases of $P(z)$, (13) and (14) have *finite dominating sets*, since an optimal solution to (13) always exists within a finite set of points (Erkut and Neuman 1989 and Plastria 1996). This property is enjoyed also by the remaining instances of $P(z)$, as shown in the following proposition.

Denote by $bd(c, r)$ the boundary of the ball centered at c with radius r. One then has

PROPOSITION 4.2. *Given $z \leqslant z_{\min\max}$, let D_z be the finite set consisting of the points in*

- *the set of vertices of S,*
- *the intersection of some edge of S with some $bd(a, K - z)$, with $a \in A$,*
- *the intersection of two sets of the form $bd(a, K - z)$, $bd(c, K - z)$, with $a, c \in A$.*

Then, D_z is a finite dominating set for $P(z)$.

PROOF. Let $\bar{x} \in S \backslash D_z$ be optimal to $P(z)$. We will show that another optimal solution belongs to D_z. Define the partition of A

$$A^{\geqslant}(\bar{x}) = \{a \in A\colon d_a(\bar{x}) \geqslant K - z\},$$

$$A^{<}(\bar{x}) = \{a \in A\colon d_a(\bar{x}) < K - z\}$$

and the feasible subset $S(\bar{x})$,

$$S(\bar{x}) = \{x \in S\colon \; d_a(x) \geqslant K - z \quad \text{for each } a \in A^{\geqslant}(\bar{x}) \\ d_a(x) \leqslant K - z \quad \text{for each } a \in A^{<}(\bar{x})\},$$

which is nonempty since $\bar{x} \in S(\bar{x})$. Then, one has

$$F(x) := \sum_{a \in A} \omega_a \max(K - d_a(x), z) = \sum_{a \in A^{\geqslant}(\bar{x})} \omega_a z + \sum_{a \in A^{<}(\bar{x})} \omega_a (K - d_a(x)) \quad \text{for each } x \in S(\bar{x}). \tag{16}$$

Hence, the objective function F is concave on the (nonconvex) set $S(\bar{x})$.

Since we assume $\bar{x} \notin D_z$, two cases may arise:

1. $\bar{x}$ is in the boundary of S, or
2. $\bar{x}$ is interior to S.

In the first case, the intersection of the boundary of S with $S(\bar{x})$ is a set of closed intervals, one of which (say I) contains $\bar{x}$ in its relative interior. By construction, both endpoints of I are in D_z, and, by (16), F is concave on I, thus attains its minimum at one of its endpoints, say x^*; since $\bar{x}$ was supposed to be absolute minimum, x^* also enjoys this property and the result holds.

We will show now, by contradiction, that $\bar{x}$ cannot be interior to S. Indeed, suppose that $\bar{x}$ is in the interior of S. For $z = z_{\min\max}$, it follows from Proposition 2.1 that $\bar{x}$ also solves the maxmin problem, thus (e.g., Erkut and Neuman 1989), at least three points of A are equidistant from $\bar{x}$, showing that $\bar{x}$ is in D_z, which is a contradiction with the definition of $z_{\min\max}$. Hence,

$$z < z_{\min\max}. \tag{17}$$

In particular,

$$A^{<}(\bar{x}) \neq \emptyset. \tag{18}$$

Indeed, else, for all $a \in A$, $d_a(\bar{x}) \geqslant K - z$, thus $\max_{a \in A}(K - d_a(\bar{x})) \leqslant z < z_{\min\max}$, which is a contradiction. Hence (18) holds.

Since $\bar{x} \notin D_z$, one can construct a triangle T whose vertices are in $S(\bar{x})$ and that contains $\bar{x}$ in its interior. Then (recall that F is concave on $S(\bar{x})$ and $\bar{x}$ is a minimizer) F is constant on the nonempty interior set T, thus F is also constant on $S(\bar{x})$. Hence, by (16) and (18), the function $x \to \sum_{a \in A^{<}(\bar{x})} \omega_a d_a(x)$ is constant on the set with nonempty interior $S(\bar{x})$, which is a contradiction.

EXAMPLE 4.3. To illustrate Proposition 4.2, consider the feasible region S and the set of population areas $A = \{a_1, a_2, a_3\} = \{(1, 2), (3, 1), (4, 3)\}$ depicted in Figure 3.

Suppose the population areas are equally weighted, i.e., $\omega_a = 1/3$ for each $a \in A$. We set $K = 6$, which satisfies

Figure 3. Feasible region S and population areas A.

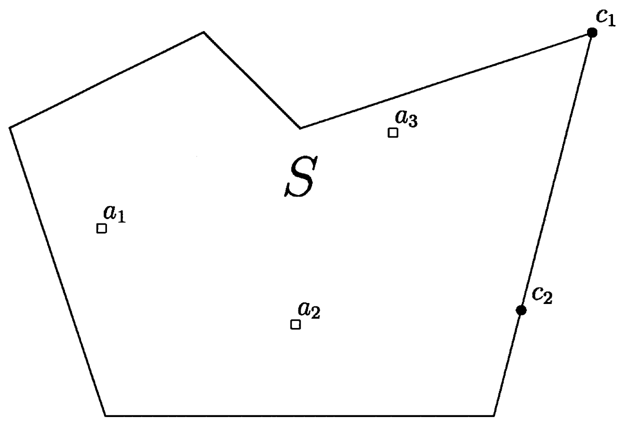
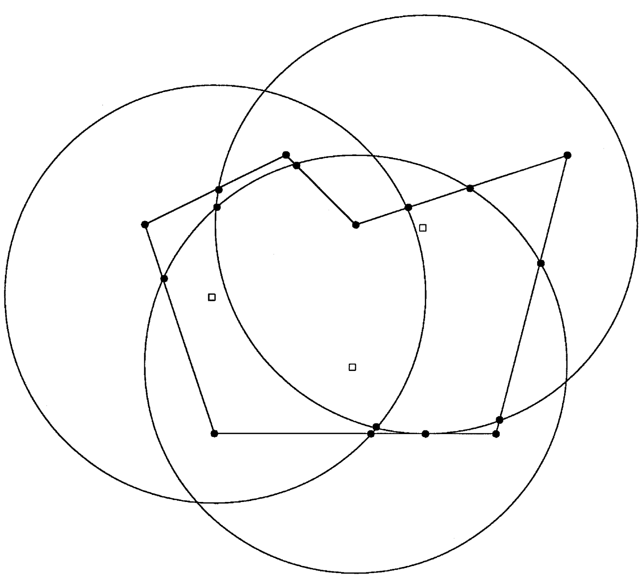


Figure 4. Dominating set for $z = 3$.



$K \geqslant \max_{x\in S} \max_{a\in A} \omega_a d_a(x)$ $(= \sqrt{29}/3)$, as checked after inspecting the convex vertices of the feasible region S.

By inspecting the convex vertices of S, we obtain as optimal solution for (13) the point $c_1 = (6, 4)$. However, the optimal solution of (14) is obtained (after constructing the Voronoi diagram of A) at $c_2 = (95/18, 10/9)$. Hence, the optimal solutions for (13) and (14) do not coincide.

When both criteria are combined through problems $P(z)$, a finite dominating set can be constructed.

For instance, for $z = 3$ (for which $z \leqslant z_{\min\max} = 6 - \sqrt{(4-95/18)^2 + (3-10/9)^2/3}$), Figure 4 represents the dominating set.

5. CONCLUSIONS

The two basic scalarizations of a vector optimization problem are the minsum (minimization of the average) and minmax (minimization of the maximum) problems. When both the average and the maximum have to be considered, the usual way of aggregating them has consisted of taking a convex combination of these two criteria. However, the weight used in this convex combination has no clear meaning, thus it could be difficult to obtain. In this note, we propose a new parametric problem based on a Goal Programming formulation, in which the parameter used is defined as the target fixed for each objective. We show that both minsum and minmax problems can be obtained as particular cases. Hence, our approach can be seen as a new way of combining minsum and minmax, which, in general, turns out to be not equivalent to the convex combination approach. Two applications in Locational Analysis are presented as an alternative way to the standard convex combination criterion.

ACKNOWLEDGMENT

The research of this author is partially supported by Grant PB96-1416-C02-02 of Ministerio de Educación y Cultura, Spain.

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