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Dominators for Multiple-objective Quasiconvex Maximization Problems

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Abstract. In this paper we address the problem of finding a dominator for a multiple-objective maximization problem with quasiconvex functions. The one-dimensional case is discussed in some detail, showing how a Branch-and-Bound procedure leads to a dominator with certain minimality properties. Then, the well-known result stating that the set of vertices of a polytope S contains an optimal solution for single-objective quasiconvex maximization problems is extended to multiple-objective problems, showing that, under upper-semicontinuity assumptions, the set of $(k-1)$-dimensional faces is a dominator for k-objective problems. In particular, for biobjective quasiconvex problems on a polytope S, the edges of S constitute a dominator, from which a dominator with minimality properties can be extracted by Branch-and Bound methods.

Key words: Multiple-objective problems; Quasiconvex maximization; Dominators

1. Introduction

Given a nonempty closed subset S of $\mathbb{R}^n$ and a function $F : S \subset \mathbb{R}^n \to \mathbb{R}^k$, define the *multiple-objective* problem $(P[F; S])$,

$$\max_{x \in S} F(x) , \qquad (P[F; S])$$

which seeks those alternatives maximizing simultaneously the components $F_1, F_2, \dots, F_k$ of F, [7, 28, 31].

Although the term simultaneous maximization is not uniquely defined, it customarily means finding the set $\mathscr{E}[F; S]$ of *efficient* or *Pareto-optimal* solutions to $(P[F; S])$,

$$\mathscr{E}[F; S] = \{x \in S : \text{no } y \in S \text{ verifies } F_i(y) \geqslant F_i(x)\ \forall i = 1, 2, \dots, k \\ \text{with at least one inequality strict}\}$$

In general $\mathscr{E}[F; S]$ lacks many desirable properties such as being connected or closed, and this seems to be quite often the case and not only in pathological

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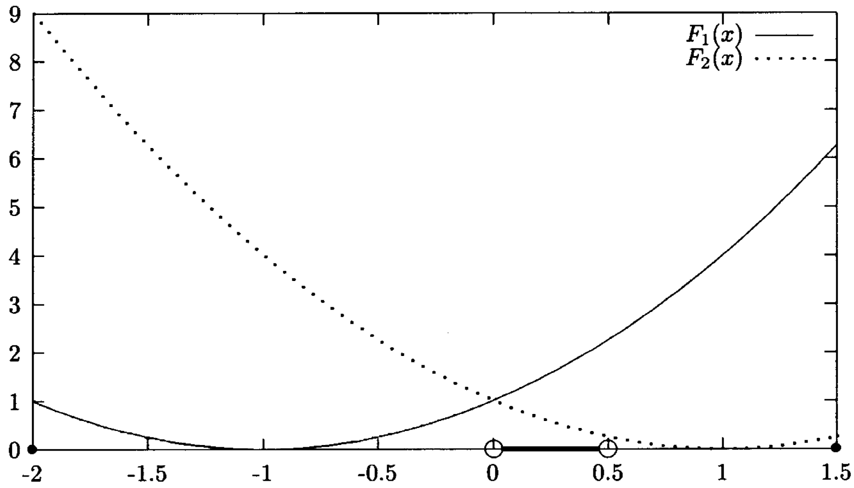

Figure 1. Biobjective convex maximization.

examples: take, for instance, the biobjective convex maximization problem in one variable ($n = 1$, $k = 2$) with $F(x) = ((x + 1)^2, (x - 1)^2)$ and $S = [-2, 1.5]$, plotted in Figure 1.

Since $F(-2) \geqslant F(x)\ \forall x \in]-2, 0]$, with at least one inequality strict, and $F(1.5) \geqslant F(x)\ \forall x \in [0.5, 1.5[$, with at least one ineuality strict too, it follows that the set of Pareto-optimal points must be contained in $\{-2\} \cup]0, 0.5[\cup \{1.5\}$. In fact, it is readily seen from the plot that

$$\mathscr{E}[F; S] = \{-2\} \cup]0, 0.5[\cup \{1.5\},$$

which is a disconnected non-closed set. See following sections and also e.g. [3] for other instances.

Moreover, although there exist procedures to check whether a given point is efficient or not, e.g. [7, 31], an algorithm to construct $\mathscr{E}[F; S]$ is only available for a few classes of problems, such as multiple-objective linear problems, [28].

This drawback has been overcome in the literature by means of two strategies: either $\mathscr{E}[F; S]$ is sought, but, due to the unability for obtaining it, an approximation (sometimes with unknown degree of precision) is provided, e.g. [8, 18], or else the concept of efficiency is relaxed and replaced by a manageable surrogate of it.

In this paper we follow the second approach by using the concept of *dominator*, [5, 16, 21, 30], also called *weak kernel*, e.g. in [31] which is defined as any subset $S_0 \subset S$ such that, for any feasible $x \not\in S_0$, S_0 contains a feasible alternative at least as good as x with respect to all objectives. See Section 2 for a formal definition.

It should be remarked that this concept is not only useful as a surrogate of the idea of Pareto-efficiency, but also as a tool in the resolution of some single-objective problems. Indeed, some of the most popular optimization methods for single-objective problems of the form

$$\max_{x \in S} \Psi(x) \tag{1.1}$$

require the feasible region S to be bounded. Such is the case, among others, of the Branch and Bound methods for global optimization, e.g. [15], which, in their simplest version, require, as pre-processing, the construction of a bounded polyhedron P (usually a hyper-rectangle, or a simplex) including either the whole feasible region, or, at least a bounded subset $S_0 \subset S$ known to contain an optimal solution. Moreover, the speed of convergence of the procedure is known to deteriorate with the volume of P, so P should be as small as possible in order to obtain reasonable computation times.

How to construct P will depend, of course, on the specific properties of the problem at hand. In particular, if (1.1) has the form

$$\max_{x \in S} \Phi(F(x)), \tag{1.2}$$

for some $\Phi : F(S) \to \mathbb{R}$ componentwise non-decreasing, then it is well known that, if (1.1) has optimal solutions, then any dominator for the multiple-objective problem $\max_{x \in S} F(x)$ also contains optimal solutions for (1.1), [21]. In other words, we can take as S_0 any bounded dominator for the multiple-objective problem, and as P any superset of S_0 with the required geometry.

This property has been successfully exploited, among others, in [5, 21, 22, 30] for problems of Linear Regression and Continuous Location, in which the globalizing function Φ is an arbitrary non-decreasing function and the function F is componentwise concave. Our aim here is to address the (harder) problem in which the function F is componentwise (quasi)-convex, showing as main result (Proposition 19) that, under upper-semicontinuity assumptions, the search of a dominator can be restricted to the $(k-1)$-dimensional faces of S.

The rest of this paper is structured as follows. In Section 2 we formally introduce the concept of dominators and discuss some general properties. These properties are used in Section 3 to address the one-dimensional case, for which dominators with certain minimality properties can be obtained.

Section 4 is devoted to show that, for multiple-objective multi-dimensional problems, one can construct dominators contained in low dimensional faces of the polytope S.

The paper ends with an application of these results to the construction of a dominator for a biobjective problem in Continuous Location. The reader is referred also to [25] for another successful application of the technique developed in this paper.

2. Dominators

Defining for each $x \in S$ the upper level set at x of F on S, $\mathscr{S}^{\geqslant}(x)$ as

$$\mathscr{S}^{\geqslant}(x) = \{y \in S : F_i(y) \geqslant F_i(x) \text{ for all } i = 1, 2, \ldots, k\},$$

the set $\mathscr{E}[F;S]$ of efficient solutions may be defined by

$$\begin{aligned}\mathscr{E}[F;S] &= \{x \in S : \text{If } y \in \mathscr{S}^{\geqslant}(x) \text{ then } x \in \mathscr{S}^{\geqslant}(y)\} \\ &= \{x \in S : \text{If } y \in \mathscr{S}^{\geqslant}(x) \text{ then } F(x) = F(y)\}\end{aligned}$$

DEFINITION 1. *A set $S^* \subset S$ is said to be a dominator for $(P[F;S])$ iff for each $x \in S$ there exists some $x^* \in S^*$ which has, componentwise, a value not smaller than x. In other words, S^* is a dominator iff*

$$(\forall x \in S)\ \exists x^* \in \mathscr{S}^{\geqslant}(x) \cap S^*$$

Hereafter, the class of dominators for $(P[F;S])$ will be denoted by $\mathscr{D}[F;S]$.

A direct consequence of the definition is the following:

PROPOSITION 2. *One has*

1. *$S \in \mathscr{D}[F;S]$. In particular, $\mathscr{D}[F;S]$ is nonempty.*
2. *If $D \in \mathscr{D}[F;S]$ and D^* satisfies $D \subset D^* \subset S$, then $D^* \in \mathscr{D}[F;S]$.*
3. *For any class $\{S_j : j \in J\}$ of nonempty sets in $\mathbb{R}^n$,*

   $$\text{If } S_j^* \in \mathscr{D}[F;S_j](\forall j \in J) \text{ then } \bigcup_{j \in J} S_j^* \in \mathscr{D}\left[F;\bigcup_{j \in J} S_j\right]$$

4. *For any class $\{S_j : j \in J\}$ of nonempty sets in $\mathbb{R}^n$,*

   $$\bigcap_{j \in J} \mathscr{D}[F;S_j] \subset \mathscr{D}\left[F;\bigcup_{j \in J} S_j\right]$$

5. *If $D \in \mathscr{D}[F;S]$, then*

   $$\mathscr{D}[F;D] \subset \mathscr{D}[F;S]\,.$$

By Proposition 2, the class $\mathscr{D}[F;S]$ is nonempty since the whole feasible set S is one of its elements. However S does not seem to be the most appropriate dominator since it possibly contains (too) many dominated alternatives, being too far from the ideal aim of a smallest possible dominator.

PROPOSITION 3. *Suppose each F_j is upper-semicontinuous on S, then any class of compact nested dominators is closed under intersections. In other words: if $(I, \preceq)$ is a totally ordered set, and $\{D_i\}_{i \in I}$ is a class of compact dominators with $D_i \subset D_j$, $j \in I$, $i \preceq j$, then*

$$\bigcap_{i \in I} D_i \in \mathscr{D}[F;S]\,.$$

Proof. Take any $x \in S$. By the upper-semicontinuity of the functions F_j, all upper level sets $\{y \in S : F_j(y) \geqslant F_j(x)\}$ are closed, so their intersection $\mathscr{S} \geqslant (x)$ is also closed. By the definition of dominators and their compactness, it follows for each $i \in I$ that $\mathscr{S}^{\geqslant}(x) \cap D_i$ is a nonempty compact set, thus $\{\mathscr{S}^{\geqslant}(x) \cap D_i\}_{i \in I}$ constitutes a class of nested compact sets. By compactness their intersection (i.e., $\mathscr{S}^{\geqslant}(x) \cap \bigcap_{i \in I} D_i$) is nonempty.

However, it is evident that the whole class $\mathscr{D}[F; S]$ is not closed under intersections (take constant functions $F_1, \dots, F_k$, then any singletons $\{x\}, \{y\} \subset S$ are dominators, with empty intersection). Hence, a unique smallest dominator is unlikely to exist. We then relax the idea of smallest dominator by introducing the concept of (weak) minimal dominators. First define for each $x \in S$ the strict upper level set of F on S, $\mathscr{S}^{>}(x)$ as

$$\mathscr{S}^{>}(x) = \{y \in S : F_i(y) > F_i(x) \text{ for all } i = 1, 2, \dots, k\} .$$

DEFINITION 4. *A dominator S^* is said to be* minimal *for* $(P[F; S])$ *iff no proper subset of S^* belongs to $\mathscr{D}[F; S]$. In other words, $S^* \subset S$ is minimal iff*

$$(x, y \in S^*, x \neq y) \Rightarrow x \notin \mathscr{S}^{\geqslant}(y)$$

A dominator $S^* \subset S$ is said to be weak minimal *for* $(P[F; S])$ *iff*

$$(x, y \in S^*) \Rightarrow x \notin \mathscr{S}^{>}(y)$$

The class of minimal (respectively weak minimal) dominators for problem $(P[F; S])$ *will be denoted by* $\mathscr{D}_M[F; S]$ *(respectively* $\mathscr{D}_{WM}[F; S]$*).*

As a simple illustration of the concepts, consider the 2-dimensional 2-objective optimization problem $\max_{x \in S} F(x)$, depicted in Figure 2, where the feasible region S is the polyhedron in $\mathbb{R}^2$ with vertices $a = (0, -3)$, $b = (4, -1)$, $c = (4, 0)$, $d = (0, 3)$, and F is given by

$$F_1(x_1, x_2) = x_1$$
$$F_2(x_1, x_2) = |x_2|$$

Then, the Pareto optimal set is given by

$$\mathscr{E}[F; S] = \{d\} \cup [a, b] ,$$

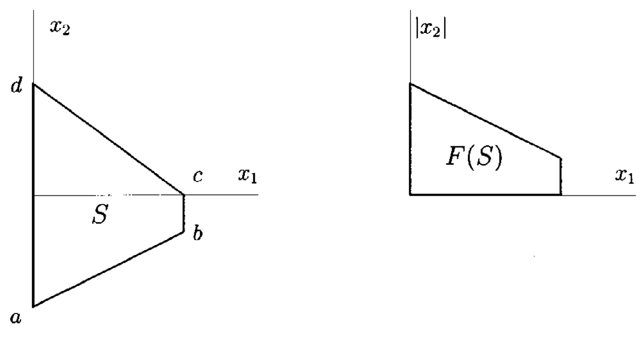

Figure 2. S and $F(S)$.

only two minimal dominators exist, namely

$$S_1 = [a, b]$$
$$S_2 =]a, b] \cup \{d\},$$

whereas the polygonal S_3,

$$S_3 = \{d\} \cup [a, b] \cup [b, c]$$

is also weak minimal.

We observe in this example that the two minimal dominators are proper subsets of $\mathscr{E}[F; S]$. This result is more general, as stated in the following:

PROPOSITION 5. *Suppose that S is compact and each F_i is upper semicontinuous on S. Then*

1. *$\mathscr{E}[F; S]$ is a weak minimal dominator.*
2. *Minimal dominators exist.*
3. *$\mathscr{E}[F; S] = \cup_{S^* \in \mathscr{D}_M[F;S]} S^*$.*

Proof. By the upper-semicontinuity assumption, for each $x \in S$ the set $\mathscr{S}^{\geqslant}(x)$ is compact. Hence, by Theorem 6 of Chapter 2 of [31] $\mathscr{E}[F; S]$ is a dominator, which, by construction, is also weak minimal. Hence 1 holds.

To show 2, define on $\mathscr{E}[F, S]$ the equivalence relation

$$\rho = \{(x, y) \in \mathscr{E}[F; S] \times \mathscr{E}[F; S] : F(x) = F(y)\}.$$

Taking exactly one element in every equivalence class, we obtain a set S^* which is, by construction, a minimal dominator. Indeed, it is a dominator because $\mathscr{E}[F; S]$ is a dominator, as shown in Part 1. Moreover it is minimal: if there exists some dominator $M \subset S^*$, $M \neq S^*$, for any $x \in S^* \setminus M$ there would exist some $y \in M$ with $F(y) \geqslant F(x)$. But by construction of S^* we would have $F(y) \neq F(x)$ contradicting the fact that x is efficient. Hence, minimal dominators exist.

For Part 3, we first show that every efficient point is in some minimal dominator: let $x^* \in \mathscr{E}[F; S]$, and construct a subset S^* of $\mathscr{E}[F; S]$ taking exactly one element of every equivalence class (with respect to the equivalence relation ρ above), x^* being the element chosen from its equivalence class. Using the reasoning above, it is seen that S^* is a minimal dominator, and $x^* \in S^*$.

Finally to show that any minimal dominator is included in the efficient set, take $x^* \in S^*$, for some $S^* \in \mathscr{D}_M[F; S]$, and assume $x^* \not\in \mathscr{E}[F; S]$. Then, there exists some $y \in S$ with $F(y) \geqslant F(x)$, and at least one inequality strict. Since $S^* \in \mathscr{D}_M[F; S]$, there must exist some $y^* \in S^*$ with $F(y^*) \geqslant F(y) \geqslant F(x^*)$, thus the set $S^* \setminus \{x^*\}$ will also be a dominator, contradicting the minimality of S^*. Hence, $x^* \in \mathscr{E}[F; S]$. □

REMARK 6. The upper-semicontinuity assumption is needed in order to guarantee the nonvoidness of $\mathscr{D}_{WM}[F; S]$, as the following counterexample shows: Let $S \subset \mathbb{R}^2$ be the triangle whose endpoints are $(-1, 0)$, $(1, 0)$, $(0, 1)$, and let $F_1 : S \to \mathbb{R}$ be defined as $1/(1-x_2)$ on the relative interior of the two top-edges, and zero elsewhere. Since

$$\lim_{\substack{(x_1,x_2)\to(0,\,1),\\ (x_1,x_2)\in bd(S)}} F_1(x_1,x_2) = +\infty\,,$$

the maximum of F_1 on S is not attained, thus any $D \in \mathscr{D}[F_1; S]$ must contain a sequence of boundary points converging to $(0, 1)$, implying that D contains points x, y with $F_1(x) > F_1(y)$. Hence, no weak minimal dominator exists. □

3. Multiple-objective one-dimensional problems

In this section we address the multiple-objective problem $(P[F; S])$ when S is given as a finite union of compact intervals in $\mathbb{R}$, and each F_i is quasiconvex on each interval. We first discuss some properties of one-dimensional single-objective quasiconvex minimization problems, which are then used to tackle $(P[F; S])$, first when S reduces to a single compact interval and then in the general case. For the basic properties of quasiconvex functions we refer the reader to [1].

3.1. SINGLE-OBJECTIVE QUASICONVEX MINIMIZATION PROBLEMS ON AN INTERVAL

Let $I \subset \mathbb{R}$ be a nonempty compact interval, and let $g : I \to \mathbb{R}$ be quasiconvex. We will denote by $\mathrm{cl}_\mathrm{I}\, g$ the closure of g relative to I, namely

$$\begin{aligned}\mathrm{cl}_\mathrm{I}\, g(x) &= \inf\{t : \exists\{x_r\}_r \subset I, \text{ such that } x_r \to x,\ g(x_r) \to t\}\\ &= \liminf_{x_r \to x}\, g(x_r)\end{aligned} \tag{3.3}$$

LEMMA 7. *One has:*
1. $g(x) \geqslant \mathrm{cl}_\mathrm{I}\, g(x)$ *for all* $x \in I$.
2. $\inf_{x\in I} g(x) = \inf_{x\in I} \mathrm{cl}_\mathrm{I}\, g(x)$.
3. $\mathrm{cl}_\mathrm{I}\, g$ *is quasiconvex and lower-semicontinuous*.
4. *The set* $\arg\min_{x\in I} \mathrm{cl}_\mathrm{I}\, g(x)$ *of optimal solutions to* $\min_{x\in I} \mathrm{cl}_\mathrm{I}\, g(x)$ *is a nonempty compact subinterval of* I.

Proof. 1 to 3 immediately follow from the definition of quasiconvexity and (3.3).
By the lower semicontinuity of $\mathrm{cl}_\mathrm{I}\, g$, the set $\arg\min_{x\in I} g(x)$ is compact and nonempty; since $\mathrm{cl}_\mathrm{I}\, g$ is also quasiconvex, it follows that $\arg\min_{x\in I} \mathrm{cl}_\mathrm{I}\, g(x)$ is also convex, thus it is a compact interval, and Part 4 follows. □

We recall that a function g is said to be *semistrictly quasiconvex*, [1], if it satisfies the following:

$$\left.\begin{array}{l} g(a) < g(b)\\ c \in\,]a, b[\end{array}\right\} \Rightarrow g(c) < g(b)$$

The next lemma shows that, due to the quasiconvexity of g, the behavior of g and

$\mathrm{cl}_I\, g$ are closely related, the relationship being stronger for semistrictly quasiconvex g:

LEMMA 8. *Let* $x^* \in \arg\min_{x\in I} \mathrm{cl}_I\, g(x)$, *and let* $z_1, z_2 \in I$ *such that* $z_1 \in\,]x^*, z_2[$. *One has:*

1. $g(z_1) \leqslant g(z_2)$.
2. *If* g *is also semistrictly quasiconvex and* $g(z_1) = g(z_2)$, *then* $]x^*, z_2[\, \subset \arg\min_{x\in I} g(x)$.

Proof. By definition of $\mathrm{cl}_I\, g$ and Part 2 of Lemma 7, one can take a sequence $\{x_r\}$ in I converging to x^* such that $\inf_r g(x_r) = \inf_{x\in I} g(x) = \mathrm{cl}_I\, g(x^*)$.

Since $z_1 > x^*$, there exists r_0 such that $x_r < z_1$ for all $r \geqslant r_0$, thus

$$z_1 \in\,]x_r, z_2[\text{ for all } r \geqslant r_0$$

Given $r \geqslant r_0$, if it were the case that $g(z_2) < g(z_1)$, then

$$\begin{aligned} g(z_2) &< g(z_1) \\ &\leqslant \max\{g(z_2), g(x_r)\} \end{aligned}$$

Hence, $g(z_1) \leqslant g(x_r)$ for each $r \geqslant r_0$ thus one would have

$$\begin{aligned} g(z_2) &< g(z_1) \\ &\leqslant \inf_r g(x_r) \\ &= \inf_{x\in I} g(x)\,, \end{aligned}$$

which is a contradiction. Hence, $g(z_2) \geqslant g(z_1)$, which shows 1.

To show 2, by the quasiconvexity of g it is enough to show that, if $g(z_1) = g(z_2)$, then $\{z_1, z_2\} \subset \arg\min_{x\in I} g(x)$. Suppose that, on the contrary, $g(z_1) = g(z_2) > \inf_{x\in I} g(x)$. Then, by Lemma 7,

$$\begin{aligned} g(z_1) &= g(z_2) \\ &> \mathrm{cl}_I\, g(x^*)\,, \end{aligned}$$

and we could take a sequence $\{x_r\}$ converging to x^* with $g(x_r)$ converging to $\mathrm{cl}_I\, g(x^*)$ and $g(x_r) < g(z_2)$ for each r. Since $z_1 \in\,]x^*, z_2[$, it would follow that $z_1 \in\,]x_r, z_2[$ for some r, thus, by the strict quasiconvexity of g, $g(z_1) < g(z_2)$, which would be a contradiction. Hence $g(z_1) = g(z_2) = \mathrm{cl}_I\, g(x^*)$, showing that

$$[z_1, z_2] \subset \arg\min_{x\in I} g(x)\,.$$

By the quasiconvexity of both g and $\mathrm{cl}_I\, g$, and the optimality of x^* and $[z_1, z_2]$ for $\min_{x\in I} \mathrm{cl}_I\, g(x)$, it then follows that

$$[x^*, z_1] \subset \arg\min_{x\in I} g(x)\,,$$

and the result holds. □

Another interesting property, which will be exploited in the sequel, states that, once problem $\min_{x\in I} \mathrm{cl}_I\, g(x)$ has been solved, any problem $\inf_{x\in J} g(x)$ with nested feasible interval $J\subset I$ is immediately solved. Indeed, denoting by $\mathrm{i}(J)$ the interior of J, one has:

PROPOSITION 9. *Let $J := [a, b] \subset I$ be two compact intervals in $\mathbb{R}$. One has:*

1. $\mathrm{cl}_I\, g \leqslant \mathrm{cl}_J\, g$ *on* J, *and*

$$\mathrm{cl}_I\, g(x) = \mathrm{cl}_J\, g(x) \quad \text{for all } x \in \mathrm{i}(J) \tag{3.4}$$

2. If $(\arg\min_{x\in I} \mathrm{cl}_I\, g(x)) \cap \mathrm{i}(J) \neq \emptyset$, *then*

$$\inf_{x\in J} g(x) = \min_{x\in I} \mathrm{cl}_I\, g(x) \tag{3.5}$$

3. If $(\arg\min_{x\in I} \mathrm{cl}_I\, g(x)) \cap \mathrm{i}(J) = \emptyset$, *then*

$$\inf_{x\in J} g(x) = \min\{g(a), g(b)\} \tag{3.6}$$

Proof. Part 1 is a direct consequence of the definition of the closure of g and Lemma 7.

For Part 2, let $x^* \in \arg\min_{x\in I} \mathrm{cl}_I\, g(x) \cap \mathrm{i}(J)$; then, by Parts 1, 2 of Lemma 7 and Part 1 of this proposition,

$$\begin{aligned}\min_{x\in I} \mathrm{cl}_I\, g(x) &= \mathrm{cl}_I\, g(x^*)\\ &= \mathrm{cl}_J\, g(x^*)\\ &= \min_{x\in J} \mathrm{cl}_J\, g(x)\\ &= \inf_{x\in J} g(x)\\ &\geqslant \inf_{x\in I} g(x)\\ &= \min_{x\in I} \mathrm{cl}_I\, g(x)\end{aligned}$$

Part 3 immediately follows from Lemma 8 if $\arg\min_{x\in I} \mathrm{cl}_I\, g(x)$ contains points in $I\setminus J$. In the remaining case, $\arg\min_{x\in I} \mathrm{cl}_I\, g(x)$ consists of just one endpoint of J, say a. If a sequence $\{x_i\}\subset J$ exists converging to a with $g(x_i)$ converging to $\min_{x\in I} \mathrm{cl}_I\, g(x) = \mathrm{cl}_I\, g(a)$, then the result follows from the definition of $\mathrm{cl}_I\, g$. Otherwise there exists $x^* < a$ with $g(x^*) < g(a)$ and then the quasiconvexity of g implies that, for any $x \in J$,

$$\begin{aligned}g(x^*) &< g(a)\\ &\leqslant \max\{g(a), g(x)\},\end{aligned}$$

thus $g(x) \geqslant g(a)$, showing (3.6). □

3.2. MULTIPLE-OBJECTIVE QUASICONVEX MAXIMIZATION PROBLEMS ON AN INTERVAL

In this subsection we show how to find a (weak) minimal dominator for the problem $(P[F; I])$ when $I = [a, b]$ is a compact interval of $\mathbb{R}$.

By Lemma 7, for each $i = 1, 2, \dots, k$, the set $\arg\min_{x\in I} \mathrm{cl}_I F_i(x)$ is a nonempty closed subinterval of I, thus it has the form $[\alpha^i, \beta^i]$.

LEMMA 10. *Let* $x \in]a, \min_{1\leqslant i\leqslant k} \beta^i[$ (*respectively* $x \in]\max_{1\leqslant i\leqslant k} \alpha^i, b[$). *Then,* $a \in \mathscr{S}^{\geqslant}(x)$ (*respectively* $b \in \mathscr{S}^{\geqslant}(x)$).

Proof. Given $x \in]a, \min_{1\leqslant i\leqslant k} \beta^i[$ and $j \in \{1, 2, \dots, k\}$, it follows that $x < \beta^j$, thus, by the definition of β^j there exists $y^j \in \arg\min_{y\in I} \mathrm{cl}_I F_j(y)$ such that $x \in]a, y^j[$. Hence, by Lemma 8, $F_j(x) \leqslant F_j(a)$ for all j, showing that $a \in \mathscr{S}^{\geqslant}(x)$. The other case is similar.

PROPOSITION 11. *Define* D_I^0 *as*

$$D_I^0 = \{a, b\} \cup [\min_{1\leqslant i\leqslant k} \beta^i, \max_{1\leqslant i\leqslant k} \alpha^i],$$

where it is understood that $[\min_{1\leqslant i\leqslant k} \beta^i, \max_{1\leqslant i\leqslant k} \alpha^i] = \emptyset$ *if* $\min_{1\leqslant i\leqslant k} \beta^i > \max_{1\leqslant i\leqslant k} \alpha^i$. *Define also*

$$D_I = \begin{cases} \{a\}, \text{ if } F(a) \geqslant F(b) \\ \{b\}, \text{ if } F(b) \geqslant F(a),\ F(b) \neq F(a) \\ D_I^0 \setminus (\{x \neq a : a \in \mathscr{S}^{\geqslant}(x)\} \cup \{x \neq b : b \in \mathscr{S}^{\geqslant}(x)\}), \text{ otherwise} \end{cases}$$

One then has

1. $D_I^0 \in \mathscr{D}[F; I]$.
2. $D_I \in \mathscr{D}_{WM}[F; I]$.
3. *If* $a \in \mathscr{S}^{\geqslant}(b)$, $b \in \mathscr{S}^{\geqslant}(a)$, *or each* F_i *is semistrictly quasiconvex on* $[a, b]$, *then* $D_I \in \mathscr{D}_M[F; I]$.

Proof. Part 1 follows from Lemma 10. To show 2, if $a \in \mathscr{S}^{\geqslant}(b)$ one would have for each $x \in I$ and $i \in \{1, 2, \dots, k\}$ that

$$\begin{aligned} F_i(x) &\leqslant \max\{F_i(a), F_i(b)\} \\ &= F_i(a), \end{aligned}$$

thus $a \in \mathscr{S}^{\geqslant}(x)$; hence $D_I = \{a\} \in \mathscr{D}[F; I]$, which is (weak) minimal being a singleton. A similar result is obtained when $b \in \mathscr{S}^{\geqslant}(a)$, thus to finish the proof of 2, we assume that $a \not\in \mathscr{S}^{\geqslant}(b)$ and $b \not\in \mathscr{S}^{\geqslant}(a)$. In particular, $\{a, b\} \subset D_I$. Given $x \in [a, b]$, it follows from Part 1 that there exists some $y \in \mathscr{S}^{\geqslant}(x) \cap D_I^0$; if $y \not\in D_I$, then $y \not\in \{a, b\}$ and either $a \in \mathscr{S}^{\geqslant}(y) \subset \mathscr{S}^{\geqslant}(x)$ or $b \in \mathscr{S}^{\geqslant}(y) \subset \mathscr{S}^{\geqslant}(x)$, hence $\emptyset \neq \{a, b\} \cap \mathscr{S}^{\geqslant}(x) \subset D_I \cap \mathscr{S}^{\geqslant}(x)$; if $y \in D_I$ then $D_I \cap \mathscr{S}^{\geqslant}(x) \neq \emptyset$. Thus $D_I \in \mathscr{D}[F; S]$.

To show that $D_I \in \mathscr{D}_{WM}[F; I]$, suppose that, by contradiction, $x, y \in D_I$ exist such that $x \in \mathscr{S}^{>}(y)$. Since either $x \in [a, y[$ or $x \in]y, b]$, we can assume w.l.o.g. that $x \in [a, y[$. Then, for each i

$$\begin{aligned} F_i(y) &< F_i(x) \\ &\leqslant \max\{F_i(a), F_i(y)\} \end{aligned}$$

thus $F_i(y) < F_i(a)$ for each i. Hence $a \in \mathscr{S}^{>}(y)$, thus $y \not\in D_I$, which is a contradiction. Hence $D_I \in \mathscr{D}_{WM}[F; I]$, and this shows 2.

The minimality property of Part 3 was shown above for $a \in \mathscr{S}^{\geqslant}(b)$ or $b \in \mathscr{S}^{\geqslant}(a)$, so we show now the case of semistrictly quasiconvex functions F_i. Suppose that, on the contrary, $x, y \in D_I$ exist such that $x \in \mathscr{S}^{\geqslant}(y) \setminus \{y\}$. Since $y \in D_I$, one gets that $x \not\in \{a, b\}$; then, $x \in]a, y[\cup]y, b[$, thus w.l.o.g. we assume $x \in]a, y[$. Since $x \in D_I \setminus \{a\}$, $a \not\in \mathscr{S}(x)$, thus there exists some i with $F_i(a) < F_i(x)$, thus

$$\begin{aligned} F_i(a) &< F_i(x) \\ &\leqslant \max\{F_i(a), F_i(y)\}\,, \end{aligned}$$

thus $F_i(x) \leqslant F_i(y)$, and, since $x \in \mathscr{S}^{\geqslant}(y)$, $F_i(x) = F_i(y)$, which contradicts the semistrict quasiconvexity of F_i. Hence, $D_I \in \mathscr{D}_M[F; I]$.

REMARK 12. In Part 1 of Proposition 11, a dominator has been constructed, consisting of at most three intervals, two of which are reduced to a point. Moreover, such a dominator is easily derived once all the single-objective one-dimensional problems $\min_{x \in I} \mathrm{cl}_{\mathrm{I}}\, F_i(x)$, $i = 1, 2, \ldots, k$ have been solved.

On the other hand, it follows from the quasiconvexity of the functions F_i that the set $\{x \in I : a \in \mathscr{S}^{\geqslant}(x)\}$ (respectively $\{x \in I : b \in \mathscr{S}^{\geqslant}(x)\}$) is an interval having a (respectively b) as one of its endpoints. This implies that the set D_I, shown in Part 2 of Proposition 11 to be weak minimal, also consists of at most three intervals, two of which are reduced to the endpoints of I.

In the case of continuous F_i, finding the set $\{x : a \in \mathscr{S}^{\geqslant}(x)\}$ is reduced to finding, for each $i = 1, \ldots, k$, the highest root of the nonlinear equation $F_i(x) = F_i(a)$, which, due to the quasiconvexity of F_i can be solved with any prespecified accuracy by e.g. binary search.

REMARK 13. For the biobjective case ($k = 2$), the interval $[\min_k \beta^k, \max_k \alpha^k]$ is, by construction, such that, within it, both F_1 and F_2 are monotonic: one nondecreasing and the other nonincreasing. Hence, for the biobjective case, there is no loss of generality in assuming that functions F_i are not only quasiconvex but also quasiconcave on the intervals $[\min_k \beta^k, \max_k \alpha^k]$.

EXAMPLE 1. Let $I = [0, 4]$, and consider the three quasiconvex functions F_1, F_2, F_3 defined as

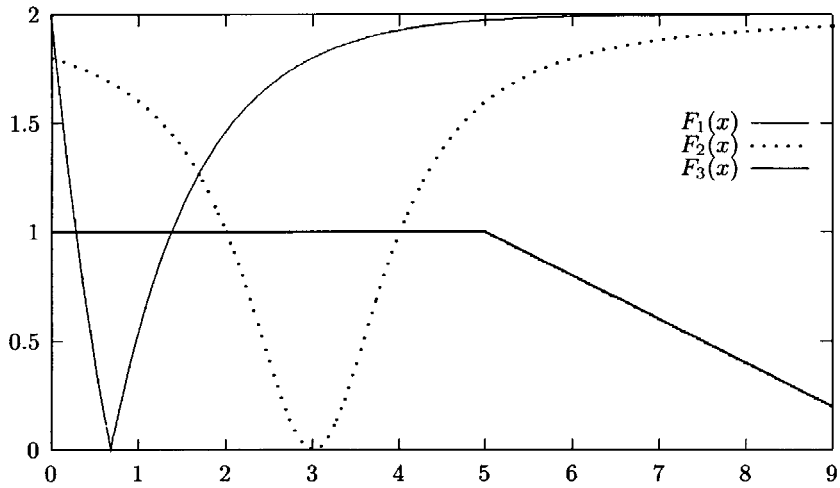

Figure 3. Functions of Example 1.

$$F_1(x) = |4e^{-x} - 2|$$

$$F_2(x) = \frac{2(x-3)^2}{1+(x-3)^2}$$

$$F_3(x) = \min\left\{1, 2 - \frac{x}{5}\right\},$$

depicted in Figure 3.

In order to construct the dominator(s) described in Proposition 11, we must determine first the set $[\alpha_i, \beta_i]$ of minima on I for each F_i. These are respectively $\{\ln 2\} = \{0.6931\}$, $\{3\}$ and $[0, 4]$. This yields

i	α_i	β_i
1	0.6931	0.6931
2	3	3
3	0	4

For this we obtain the dominator

$$D_I^0 = \{0, 4\} \cup [0.6931, 3]$$

Moreover, by comparing the endpoints, we get

$$F(0) = (2, 1.8, 1)$$
$$F(4) = (1.9267, 1, 1),$$

thus $F(0) \geqslant F(4)$. Hence, by Proposition 11, the set $D_I = \{0\}$ is not only a weak minimal dominator but also a minimal dominator.

Suppose now that the feasible region is the interval $I = [5, 9]$. In this case we obtain

i	α_i	β_i
1	5	5
2	5	5
3	9	9

From this it is easily seen that

$$D_I^0 = D_I = [5, 9] .$$

Since all the functions are semistrictly quasiconvex in I, it follows that I is a minimal dominator for $\max_{x \in I} F(x)$.

Finally, for $I = [4, 9]$ we similarly obtain $D_I^0 = D_I = [4, 9]$, but in this case D_I^0 is not a minimal dominator, since $[5, 9]$ is a strictly included dominator (which may be seen to be minimal).

3.3. MULTI-OBJECTIVE QUASICONVEX MAXIMIZATION PROBLEMS ON A SET OF INTERVALS

As a natural extension of the model presented in Section 3.2, we address here the problem

$$\max_{x \in X} F(x) ,$$

where

- $X = \bigcup_{1 \leqslant i \leqslant t} I_i$, with $\{I_i\}_{1 \leqslant i \leqslant t}$ being a family of compact (possibly degenerate) intervals of the real line, not necessarily disjoint,
- $F_1, \ldots, F_k$ are quasiconvex on each I_i, $i = 1, \ldots, t$. (Note that this is a weaker assumption than each component of F to be quasiconvex in the convex hull of $\bigcup_{1 \leqslant i \leqslant t} I_i$).

By Proposition 2, if one finds, for each $i = 1, 2, \ldots, t$ some dominator $D_i \in \mathscr{D}[F; I_i]$, then any $D \in \mathscr{D}[F; \bigcup_{1 \leqslant i \leqslant t} D_i]$ would serve as a dominator for $(P[F; \bigcup_{1 \leqslant i \leqslant t} I_i])$. Moreover, if a (weak) minimal dominator is sought, redundant alternatives should be purged, either in the construction of the sets D_i (by imposing e.g. $D_i \in \mathscr{D}_{\mathscr{WM}}[F; I_i]$) or when they are merged to produce a (small) final dominator.

To approximate this goal one can use a Branch-and-Bound scheme, similar to the one described in [14]: we start with a list $\mathscr{L}$ of compact intervals, the union of which is known to be a dominator for $(P[F; \bigcup_{1 \leqslant i \leqslant t} I_i])$, and then refine iteratively the elements in $\mathscr{L}$, by making pairwise comparisons, in such a way that, at any stage, one has

$$\bigcup_{I \in \mathscr{L}} I \in \mathscr{D}[F; \cup_{1 \leqslant i \leqslant t} I_i]$$

To perform comparisons among elements in $\mathscr{L}$ we introduce, for each interval $I := [a, b]$ contained in some I_i, the vectors $M(I)$, $UB(I) \in \mathbb{R}^k$ of evaluations at the midpoint of I and a componentwise upper bound of F, respectively:

$$M(I)_j = F_j\left(\frac{a+b}{2}\right)$$
$$UB(I)_j \geqslant \max_{x \in I} F_j(x)$$

REMARK 14. By the quasiconvexity of F_j on $I \subset I_i$, it follows that one may choose

$$UB(I)_j = \max\{F_j(a), F_j(b)\} \quad j = 1, 2, \ldots, k$$

Note also that for $I = \{a\}$ we have M(I) = UB(I).

From the definitions of the vectors M and UB one immediately obtains the following way to check whether some interval J can be discarded from further consideration in the Branch-and-Bound scheme.

PROPOSITION 15. *Given nonempty compact intervals I, J, suppose F is continuous on I and on J. Then the following statements are equivalent:*

1. $I \in \mathscr{D}[F; J]$, *i.e. for any $y \in J$ there exists $x \in I$ with $F(x) \geqslant F(y)$*
2. $0 \leqslant \min_{y \in J} \max_{x \in I} \min_{1 \leqslant j \leqslant k} (F_j(x) - F_j(y))$.

This is implied by both

$$\bigcap_{1 \leqslant j \leqslant k} \{x \in I : F_j(x) \geqslant UB(J)_j\} \neq \emptyset \tag{3.7}$$

and

$$0 \leqslant \min_{1 \leqslant j \leqslant k} (M(I)_j - UB(J)_j)\,, \tag{3.8}$$

while (3.8) always implies (3.7).

Proof. The equivalence between 1 and 2 is evident. Since (3.7) is equivalent to the existence of some $x \in I$ with

$$F(x) \geqslant F(y)\ \forall y \in J\,, \tag{3.9}$$

it clearly implies 1. On the other hand, (3.8) is equivalent to (3.9) for x fixed to the midpoint of I. Hence, (3.8) implies (3.7) and the result follows. □

Although condition (3.8) is easier to implement, the stronger test (3.7) is also of practical interest since this intersection set, if nonempty, has a simple structure due to the quasiconvexity of F, as indicated by the following simple result:

PROPOSITION 16. *One has for any values* c_j

1. *Each set* $\{x \in I : F_j(x) \geqslant c_j\}$ *consists of at most two intervals, each with an endpoint of* I *as one of its endpoints.*
2. *For* $k_0 = 1, 2, \ldots, k$, *the set* $\bigcap_{1 \leqslant j \leqslant k_0} \{x \in I : F_j(x) \geqslant c_j\}$ *is a collection of* $n(k_0)$ *intervals, with*

$$n(1) \leqslant 2$$

$$n(k_0) \leqslant n(k_0 - 1) + 1 , \quad k_0 = 2, 3, \ldots, k$$

The basic steps of the Branch-and-Bound procedure are described below:

Algorithm 1

Initialization:

Set $\mathscr{L} := \{cl(D_{I_j}), \ j = 1, \ldots, t\}$

Set $r := 1$

Iteration $r = 1, 2, \ldots,$:

for all $I \in \mathscr{L}$ do

  If, for some $J \in \mathscr{L}$, $J \neq I$, (3.8) or (3.7) hold, then

    delete I from $\mathscr{L}$;

  Else, if I is non-degenerate do

    split I into I_1 and I_2 at the midpoint of I;

    replace I by I_1 and I_2 in $\mathscr{L}$;

GoTo Iteration $r + 1$

Before discussing the output of the algorithm in the limit ($r = \infty$), let us present an illustrative example.

EXAMPLE 1 (Cont.)

Let F be the three-objective one-dimensional function described in the first part of the Example, and assume now that the feasible region X consists of the two compact segments $I_1 = [0, 4]$, and $I_2 = [5, 9]$.

In the Initialization phase, we must construct the sets $cl(D_{I_j})$, $j = 1, 2$. This was already done in the first part of the Example, thus we start with the list

$$\mathscr{L} = \{\{0\}, [5, 9]\} .$$

Then, we go to Iteration 1. For each interval I (degenerate or not) in $\mathscr{L}$, the vectors $M(I), UB(I)$ must be constructed. (Observe that this task becomes trivial using Remark 14 above.) Evaluations at the endpoints, 0, 5, 9 and the midpoint 7 yield

$F(0) = (2, 1.8000, 1)$

$F(5) = (1.9730, 1.6000, 1)$

$F(7) = (1.9964, 1.8824, 0.6000)$

$F(9) = (1.9995, 1.9459, 0.2000)$

We then obtain

I	$M(I)$	$UB(I)$
{0}	(2, 1.8000, 1)	(2, 1.8000, 1)
[5, 9]	(1.9964, 1.8824, 0.6000)	(1.9995, 1.9459, 1)

We will only use the simplest test, namely, (3.8) in the algorithm.

Since no pair of intervals in $\mathscr{L}$ satisfies condition (3.8), we go to Iteration 2 with the list of intervals

$$\mathscr{L} = \{\{0\}, [5, 7], [7, 9]\} .$$

Two new midpoints appear, namely, 6 and 8, with objective values

$F(6) = (1.9901, 1.8000, 0.8000)$

$F(8) = (1.9987, 1.9231, 0.4000)$.

This enables us to update the table of vectors M, UB yielding

I	$M(I)$	$UB(I)$
{0}	(2, 1.8000, 1)	(2, 1.8000, 1)
[5, 7]	(1.9901, 1.8000, 0.8000)	(1.9964, 1.8824, 1)
[7, 9]	(1.9987, 1.9231, 0.4000)	(1.9995, 1.9459, 0.6000)

As in the previous iteration, no pair of intervals satisfies condition (3.8), and we go to Iteration 3 with the updated list of intervals

$$\mathscr{L} = \{\{0\}, [5, 6], [6, 7], [7, 8], [8, 9]\}$$

The new midpoints give objective values

$F(5.5) = (1.9837, 1.7241, 0.9000)$

$F(6.5) = (1.9940, 1.8491, 0.7000)$

$F(7.5) = (1.9978, 1.9059, 0.5000)$

$F(8.5) = (1.9992, 1.9360, 0.3000)$

With this, our new table of vectors M, UB is given by

I	$M(I)$	$UB(I)$
{0}	(2, 1.8000, 1)	(2, 1.8000, 1)
[5, 6]	(1.9837, 1.7241, 0.9000)	(1.9901, 1.8000, 1)
[6, 7]	(1.9940, 1.8491, 0.7000)	(1.9964, 1.8824, 0.8000)
[7, 8]	(1.9978, 1.9059, 0.5000)	(1.9987, 1.9231, 0.6000)
[8, 9]	(1.9992, 1.9360, 0.3000)	(1.9995, 1.9459, 0.4000)

In this case, the sufficient condition for dominance is satisfied for the pair of intervals {0} and [5, 6], so the interval [5, 6] can be excluded for further considerations.

We would then obtain a reduced list

$$\mathscr{L} = \{\{0\}, [6, 7], [7, 8], [8, 9]\}$$

to start Iteration 4, if desired. □

The following theorem shows that the successive steps of the algorithm above provide a sequence of nested compact dominators, converging to a dominator which, under mild further assumptions on the functions F_i, enjoys minimality properties:

PROPOSITION 17. *Denote by D_r the union of all intervals of $\mathscr{L}$ at the end of iteration r, and by D^* the compact set*

$$D^* = \bigcap_{r=1}^{\infty} D_r$$

1. $D_1 = X = \bigcup_{1 \leqslant i \leqslant t} I_i$ *and* $D_{r+1} \subset D_r$ *for all* r.
2. *If F is upper-semicontinuous, then*

$$D^* \in \mathscr{D}[F; X] \tag{3.10}$$

3. *Moreover, if F is continuous, then*

$$D^* \in \mathscr{D}_{\mathscr{W}\mathscr{M}}[F; X]\,. \tag{3.11}$$

Proof. The first property is evident from the algorithm.

By construction, each D_r is compact, thus their intersection is also compact. Moreover, $D_r \in \mathscr{D}[F; X]$, thus, by Proposition 3, (3.10) follows.

To show (3.11), suppose, on the contrary, that there exist $x_1, x_2 \in D^*$ with $x_1 \in \mathscr{S}^{>}(x_2)$. If, for each $i = 1, 2$ and $r = 1, 2, \ldots$, we denote by $\mathscr{I}_i^r$ the class of intervals I_i^r in the list at stage r with $x_i \in I_i^r$, it will follow from the splitting process that there exists some r_0 such that, for each $r \geqslant r_0$, and each $I_i^r \in \mathscr{I}_i^r$

$$x_1 \not\in I_2^r\,, \quad \text{and} \quad x_2 \not\in I_1^r$$

Since the functions F_i are continuous, thus uniformly continuous on X, there would exist some r such that for each $I_i^r \in \mathscr{I}_i^r$

$$F_j(x) > F_j(y) \quad \text{for all } x \in I_1^r \text{ and } y \in I_2^r\,, \quad j = 1, 2, \ldots, k$$

Hence $UB(I_2^r) < M(I_1^r)$, implying that I_2^r (thus x_2) would have been deleted prior to stage r by (3.8), thus $x_2 \not\in D^*$, which is a contradiction. □

4. Multiple-objective multi-dimensional problems

For the single-objective case (i.e., if $k = 1$ in $(P[F_1; S])$), it is a well-known result of Global Optimization that, if S is a polytope and F_1 is quasiconvex on S, then the set of vertices of S is a dominator for $(P[F_1; S])$, [15].

In other words, if, for $j = 0, 1, \ldots, n$, $\mathscr{F}^j$ denotes the set of points of a polytope S contained in some j-dimensional face of S, then

$$\mathscr{F}^0 \in \mathscr{D}[F_1; S] \tag{4.12}$$

The next proposition extends assertion (4.12) to multiple-objective quasiconvex problems. To show it, we will use the following

LEMMA 18. *Let P be a polyhedron in $\mathbb{R}^n$, and let $H_1, H_2, \ldots, H_t$ be closed halfspaces in $\mathbb{R}^n$. If x^* is an extreme point of $P \cap \bigcap_{1 \leq i \leq t} H_i$, then x^* belongs to some face of P with dimension not greater than t.*

Proof. Let P be represented as

$$P = \{x \in \mathbb{R}^n : a_r' x \leq b_r \text{ for all } r \in R\}$$

for some finite index set R, and let each H_i be given as

$$\{x \in \mathbb{R}^n : c_i' x \leq d_i\}$$

Define the sets of active indices $R(x^*)$ and $T(x^*)$ as

$$R(x^*) = \{r \in R : a_r' x^* = b_r\}$$
$$T(x^*) = \{i, 1 \leq i \leq t : c_i' x^* = d_i\}$$

Then x^* belongs to the face F of P,

$$F = P \cap \{x \in \mathbb{R}^n : a_r' x = b_r \ \forall r \in R(x^*)\}$$

We will show that F has dimension not greater than t. Indeed, since x^* is, by assumption, an extreme point of $P \cap \bigcap_{1 \leq i \leq t} H_i$, then the set of vectors $\{a_r\}_{r \in R(x^*)} \cup \{c_i\}_{i \in T(x^*)}$ has rank

$$\operatorname{rank}(\{a_r\}_{r \in R(x^*)} \cup \{c_i\}_{i \in T(x^*)}) = n$$

Hence, denoting by $|T(x^*)|$ the cardinality of $T(x^*)$, one obtains

$$\begin{aligned}\operatorname{rank}(\{a_r\}_{r\in R(x^*)}) &\geqslant n - |T(x^*)| \\ &\geqslant n - t\,,\end{aligned}$$

thus the dimension of F cannot be greater than t. □

PROPOSITION 19. *Let S be a polytope in $\mathbb{R}^n$, let $k \leqslant n+1$, and let $F_1, \ldots, F_k$ be quasiconvex functions on S, all but possibly one of which are upper-semicontinuous. Then*

$$\mathscr{F}^{k-1} \in \mathscr{D}[F; S] \tag{4.13}$$

Proof. Without loss of generality we assume that $F_1, F_2, \ldots, F_{k-1}$ are upper-semicontinuous on S. We will show that, for any $x \in S$,

$$\mathscr{S}^{\geqslant}(x) \cap \mathscr{F}^{k-1} \neq \emptyset \tag{4.14}$$

Let $x \in S$, and denote by $\mathscr{A}(x)$ the index set

$$\mathscr{A}(x) = \{i,\ 1 \leqslant i \leqslant k-1,\ F_i(y) < F_i(x) \text{ for some } y \in S\}\,.$$

If $\mathscr{A}(x)$ is empty, we would have

$$F(y) \geqslant F(x)\ \forall y \in S\,,$$

thus any vertex y^* of S satisfies $y^* \in \mathscr{S}^{\geqslant}(x)$. Hence

$$\emptyset \neq \mathscr{S}^{\geqslant}(x) \cap \mathscr{F}^0 \subset \mathscr{S}^{\geqslant}(x) \cap \mathscr{F}^{k-1}\,,$$

showing (4.14).

We consider now the case $\mathscr{A}(x) \neq \emptyset$. For each $i \in \mathscr{A}(x)$, the convex set $\{y \in S : F_i(y) < F_i(x)\}$ is open in S (its complement is closed due to the upper-semicontinuity of F_i) and does not contain x. Hence, there exists some nonzero vector u^i such that

$$\langle u^i, y - x\rangle > 0 \quad \text{for all } y \in S \text{ with } F_i(y) < F_i(x)\,, \tag{4.15}$$

where $\langle\cdot,\cdot\rangle$ stands for the usual scalar product in $\mathbb{R}^n$.

Consider the polyhedron $S(x)$,

$$S(x) = S \cap \{y \in \mathbb{R}^n : \langle u^i, y - x\rangle \leqslant 0, \quad \forall i \in \mathscr{A}(x)\}\,,$$

which is nonempty because $x \in S(x)$. Consider the optimization problem

$$\max_{y\in S(x)} F_k(y) \tag{4.16}$$

Since F_k is quasiconvex on the nonempty polyhedron $S(x)$, (4.16) has an optimal solution at some vertex y^* of $S(x)$. We will show that

$$y^* \in \mathscr{F}^{k-1} \cap \mathscr{S}^{\geqslant}(x) \tag{4.17}$$

Since y^* is a vertex of $S(x)$, Lemma 18 implies that

$$y^* \in \mathscr{F}^{|\mathscr{A}(x)|} \subset \mathscr{F}^{k-1} \tag{4.18}$$

Since $x \in S(x)$ and y^* is optimal for (4.16),

$$F_k(y^*) \geqslant F_k(x) \tag{4.19}$$

By definition of $\mathscr{A}(x)$,

$$F_i(y^*) \geqslant F_i(x) \; \forall i \in \{1, 2, \ldots, k-1\} \setminus \mathscr{A}(x) \tag{4.20}$$

and by (4.15) and the fact that $y^* \in S(x)$,

$$F_i(y^*) \geqslant F_i(x) \; \forall i \in \mathscr{A}(x) \tag{4.21}$$

Joining (4.18–4.21), (4.17) holds, thus $\mathscr{S}^{\geqslant}(x) \cap \mathscr{F}^{k-1} \neq \emptyset$, as asserted. □

REMARK 20. The assumption of upper-semicontinuity of at least $k-1$ functions is not superfluous, as the following example shows: let $k=2$, $n=2$, $S=[0,1]\times[0,1]$, and the functions F_1, F_2 defined as

$$F_1(x) = \begin{cases} 0, & \text{if } x_2 > \frac{1}{2} \text{ or } x = (0, \frac{1}{2}) \\ 1, & \text{otherwise} \end{cases} \qquad F_2(x) = \begin{cases} 0, & \text{if } x_2 > \frac{1}{2} \text{ or } x = (1, \frac{1}{2}) \\ 1, & \text{otherwise} \end{cases}$$

Both functions are quasiconvex but are not upper-semicontinuous; let $x^* = (\frac{1}{2}, \frac{1}{2})$. It is easily seen that

$$\mathscr{S}^{\geqslant}(x^*) = \{(\lambda, \tfrac{1}{2}) : 0 < \lambda < 1\}$$

thus $\mathscr{S}^{\geqslant}(x^*) \cap \mathscr{F}^1 = \emptyset$, showing that $\mathscr{F}^1$ is not a dominator. □

As a consequence of Propositions 19 and 2, one obtains

COROLLARY 21. *Let S be the union of t polytopes $S_1, \ldots, S_t$ in $\mathbb{R}^n$. Let $F_1, \ldots, F_k$ be $k \leqslant n+1$ real-valued functions on S. On each S_j, let all F_i be quasiconvex and all but possibly one F_i be lower-semicontinuous. Then the union of all $k-1$-faces of all S_j is a dominator for $P[F; S]$.*

Proposition 19 also enables us to derive localization results for single-objective problems.

COROLLARY 22. *Let S be the union of t polytopes $S_1, \ldots, S_t$ in $\mathbb{R}^n$. Let $F_1, \ldots, F_k$ be $k \leqslant n+1$ real-valued functions on S, quasiconvex on each S_j. For any componentwise nondecreasing $\Phi : F(S) \to \mathbb{R}$ such that Problem*

$$\max_{x \in S} \Phi(F_1(x), F_2(x), \ldots, F_k(x))$$

has an optimal solution, the union of the set of $k-1$*-faces of all* S_j *also contains an optimal solution.*

In particular, for $F_1, F_2, \ldots, F_k$ linear fractional functions with positive denominators on a polytope S, which are well-known to be quasiconvex (see e.g. [1], p. 165) and $\Phi(s_1, s_2, \ldots, s_k) = s_1 + s_2 + \cdots + s_k$, or $\Phi(s_1, s_2, \ldots, s_k) = \max\{s_1, s_2, \ldots, s_k\}$, we obtain

COROLLARY 23. *The minimum of the sum (respect. the maximum) of k linear fractional functions with positive denominators over a polytope S in dimension* $n \geqslant k-1$ *is attained at some* $k-1$*-face of S.*

This generalizes the results known for the case $k=2$ (see the review of [27] and the references therein). For an application see [25].

REMARK 24. For biobjective problems ($k=2$), since both F_j are quasiconvex on each edge, after embedding such edges as compact intervals of the real line, one can use the results in Section 3 to design an algorithm converging to a weak minimal dominator.

For the case of general k, Proposition 19 seems at the moment to be mainly of theoretical interest: In principle, Algorithm 1 can be generalized to the k-dimensional case, by replacing intervals by e.g. simplices, although the corresponding bounding scheme does not extend to the general case, and less efficient schemes, such as those proposed in [4, 14], should be used.

Nevertheless, this kind of localization results can be used to design new heuristic resolution methods of problems of the form $\min_{x \in S} \Phi(F(x))$, where k, the number of components of F, is very small, and, in particular, much smaller than the dimension n of the space.

We know then that the search for optimal solutions can be reduced to the $k-1$-dimensional faces of S, so that algorithms which alternate a global search in a given low-dimensional face with moves to adjacent low-dimensional faces, can be used.

5. Application: Location of a semi-obnoxious facility

Let $S = S_1 \cup S_2 \cup \cdots \cup S_t$, each S_i being a convex polygon in $\mathbb{R}^2$. Two finite subsets $\mathscr{A}^+$, $\mathscr{A}^-$ of $\mathbb{R}^2$ are given. Associated with each $a \in \mathscr{A}^+$ we have a concave function $g_a : [0, +\infty) \to \mathbb{R}$ and a polyhedral gauge γ_a, [9, 10, 19], i.e., a Minkowski functional whose unit ball is a polytope.

Let $h : [0, +\infty) \to \mathbb{R}$ be a nonincreasing function, and consider the biobjective problem

$$\min_{x \in S} (F_1(x), F_2(x)), \tag{5.22}$$

where

$$F_1(x) = \sum_{a \in \mathcal{A}^+} g_a(\gamma_a(x-a))$$
$$F_2(x) = \max_{a \in \mathcal{A}^-} h(\|x-a\|)\,,$$

$\|\cdot\|$ being the euclidean norm.

This problem has its motivation in Continuous Location of semidesirable facilities, see [17, 23] for an introduction to Continuous Location in general and [6, 24] for semidesirable facility location models: A facility is to be located within region S, and will interact with individuals who want the facility close (those in $\mathcal{A}^+$) and others who want the facility far (those in $\mathcal{A}^-$). Interactions with $\mathcal{A}^+$ provide the first objective in (5.22): the minimization of the total transportation cost $F_1(x)$, where transportation cost from $a \in \mathcal{A}^+$ to x is given by a concave function g_a of the distance from a to x, the latter measured by the polyhedral gauge γ_a, [29].

On the other hand, interactions of the facility with $\mathcal{A}^-$ provide the second objective F_2, which measures the highest damage suffered by points in $\mathcal{A}^-$, where the damage suffered by $a \in \mathcal{A}^-$ is assumed to be given by a nonincreasing function h of the Euclidean distance from a to x, see [11, 24].

In practice, the two objectives of (5.22) are aggregated into a single criterion, yielding a problem of the form

$$\max \Phi\left(-\sum_{a \in \mathcal{A}^+} g_a(\gamma_a(x-a)),\ \max_{a \in \mathcal{A}^-} h(\|x-a\|)\right), \tag{5.23}$$

[6, 24] for some globalizing Φ, and the resulting problem (multimodal, as a rule), can be tackled e.g. by the 2-dimensional Branch and Bound method described in [13]. However, as shown below (Proposition 25), the search of an optimal solution for (5.23) can be restricted to a series of segments, thus (5.23) can be solved by simply using single-variable Global-Optimization techniques, [2, 12], which are usually much faster than their two-variable counterparts.

In order to obtain a dominator for (5.22) one should observe first that, since h is assumed to be nonincreasing, it suffices to obtain a dominator for problem

$$\max_{x \in S} \left(-F_1(x),\ \min_{a \in \mathcal{A}^-} \|x-a\|\right) \tag{5.24}$$

(in fact, if h is decreasing, both problems are equivalent). Let us rewrite now (5.24) within our framework. For polyhedral gauges, using the concept of *elementary convex set* of [10], one can obtain a subdivision $\mathcal{C}$ of the plane into polyhedra in such a way that, within each $C \in \mathcal{C}$, each gauge γ_a is affine, see [9, 10] for further details. For instance, if each γ_a is the l_1 norm, then the polyhedral subdivision of the plane is obtained after constructing horizontal and vertical lines through each $a \in \mathcal{A}^+$, yielding a total of $O(|\mathcal{A}^+|^2)$ cells.

Moreover, defining, for each $a \in \mathcal{A}^-$, the *Voronoi cell* $V(a)$ associated with a as

$$V(a) = \{x \in \mathbb{R}^2 : \|x-a\| \leqslant \|x-b\| \text{ for all } b \in \mathcal{A}^-\}\,,$$

the class $\mathscr{V} = \{V(a) : a \in \mathscr{A}^-\}$ also constitutes a polyhedral subdivision of $\mathbb{R}^2$ in $O(|\mathscr{A}^-|)$ polyhedra, which can be efficiently constructed in $O(|\mathscr{A}^-| \log |\mathscr{A}^-|)$, see e.g. [20, 26].

Consider now the class $\mathscr{Z}$ of all Z of the form

$$S_i \cap C \cap V(a)$$

for some i, $1 \leqslant i \leqslant t$, $C \in \mathscr{C}$ and $a \in \mathscr{A}^-$ which are nonempty. On each $Z \in \mathscr{Z}$, we have that $-F_1$ is convex (it is the composition of the convex function $-\Sigma_{a \in \mathscr{A}^+} g_a$ with the affine functions (within Z!) γ_a, and F_2 is also convex (recall that, for $Z \in \mathscr{Z}$ fixed, there exists some $a^* \in \mathscr{A}^-$ such that $\min_{a \in \mathscr{A}^-} \|x - a\| = \|x - a^*\|$). Hence, rewriting (5.24) as

$$\max_{x \in \cup_{Z \in \mathscr{Z}} Z} \left(-F_1(x), \min_{a \in \mathscr{A}^-} \|x - a\|\right),$$

we can use Corollary 21 to obtain

PROPOSITION 25. *The edges of the sets in $\mathscr{Z}$ constitute a dominator for Problem (5.22).*

After embedding the edges of polytopes in $\mathscr{Z}$ as compact intervals of the real line, one can use the algorithm described in Section 3.3 to reduce the size of such dominator, converging (in case of decreasing h) to a weak minimal dominator.

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