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An optimal bound for d.c. programs with convex constraints*

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Abstract. A well-known strategy for obtaining a lower bound on the minimum of a d.c. function $f - g$ over a compact convex set $S \subset \mathbb{R}^n$ consists of replacing the convex function f by a linear minorant at $x_0 \in S$. In this note we show that the x_0^* giving the optimal bound can be obtained by solving a convex minimization program, which corresponds to a Lagrangian decomposition of the problem. Moreover, if S is a simplex, the optimal Lagrangian multiplier can be obtained by solving a system of $n + 1$ linear equations.

Key words: d.c. programs; bounds; Lagrangian decomposition

1 Problem statement

Let S be a nonempty compact convex subset of $\mathbb{R}^n$, and let f, g be convex and finite on $\mathbb{R}^n$. Our aim is to find a lower bound on the optimal value z^* of the d.c. program

$$\min_{x \in S} f(x) - g(x) \tag{1}$$

Obtaining such lower bounds may be a real need when one is solving global optimization problems by a branch-and-bound strategy, [2, 1, 3], both in the bounding process (indeed, one needs to find good lower bounds for subproblems in the form (1) at each stage in the resolution of $\min_{x \in \Omega} f(x) - g(x)$ using polyhedral, – say, simplicial or hyperrectangular – branching schemes) and in order to check feasibility as well (showing that a lower bound

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for $\min_{x \in S} f(x) - g(x)$ is strictly positive implies the infeasibility of the (sub)-problem $\min\{h(x) : f(x) - g(x) \leq 0, x \in S\}$). In both cases, it is important to have the bounds as sharp as possible, since this may considerably reduce the computational effort.

Now we introduce some notation: let $\text{ext}(S)$ denote the set of extreme points of the convex set S, and, for any $x_0 \in S$, let $\partial f(x_0)$ denote the subdifferential of f at x_0.

One immediately derives the following result from the definition of subgradients and the fact that a concave function attains its minimum on a bounded convex set S at some point in $\text{ext}(S)$, e.g. [2].

Proposition 1. *For any $x_0 \in S$ and $u_0 \in \partial f(x_0)$, it follows*

$$\begin{aligned} z^* &\geq \min_{x \in S} f(x_0) + \langle u_0, x - x_0 \rangle - g(x) \\ &= \min_{x \in \text{ext}(S)} f(x_0) + \langle u_0, x - x_0 \rangle - g(x) \end{aligned}$$

The bound given in Proposition 1 strongly depends on the choice of the point $x_0 \in S$. Such bound can then be sharpened if one choses the best possible x_0,

Corollary 1. *Define*

$$z_P = \sup\left\{ \min_{x \in \text{ext}(S)} f(x_0) + \langle u_0, x - x_0 \rangle - g(x) : x_0 \in S, u_0 \in \partial f(x_0) \right\} \tag{2}$$

Then, $z_P \leq z^*$.

Finding the optimal lower bound z_P from the definition amounts to solving a maxmin nonlinear problem with nonconvex constraints, thus, at first glance, it does not seem obvious at all that the possible enhancement of the bound z_P with respect to any of the simple bounds given in Proposition 1 will deserve the resolution of the global-optimization problem (2). It turns out however that (2) can be formulated as a convex program, as shown in Section 2.

2 A Lagrangian decomposition scheme

Lower bounds for z^* can also be obtained via Lagrangian decomposition, [4]. Indeed, (1) can be equivalently rephrased as

$$\min\{f(x) - g(y) : x = y, x \in S, y \in S\}$$

Dualizing the constraints $x = y$, one obtains the Lagrangian dual

$$z_D = \max_{u \in \mathbb{R}^n} L(u), \tag{3}$$

with

$$L(u) = \min_{x,y \in S}(f(x) - \langle u,x\rangle - g(y) + \langle u, y\rangle)$$

$$= \min_{x \in S}(f(x) - \langle u,x\rangle) + \min_{y \in S}(-g(y) + \langle u, y\rangle). \quad (4)$$

We see that (3) is a bilevel problem, since the mere evaluation of the Lagrangian function L at a given $u \in \mathbb{R}^n$ amounts to solving the convex minimization program $\min_{x \in S}(f(x) - \langle u,x\rangle)$ and the concave minimization program $\max_{y \in S}(g(y) - \langle u, y\rangle)$.

Since the latter reduces to vertex enumeration if S is polyhedral, L can be evaluated in finite time for particular instances (e.g., when f is polyhedral or quadratic and S is a polytope), whilst for general problems L must be approximated by finding a near-optimal solution of a nonlinear program.

The next result shows that finding an optimal multiplier is equivalent to solving (2).

Proposition 2. *One has* $z_D = z_P$

Proof. Let f_S the restriction of f to S,

$$f_S(x) = \begin{cases} f(x), & \text{if } x \in S \\ +\infty, & \text{else} \end{cases}$$

and let f_S^* denote the Fenchel conjungate of f_S, [5]

$$f_S^*(p) = \sup\{\langle p,x\rangle - f_S(x) : x \in \mathbb{R}^n\}$$

We will show now that

$$L(u) \leq z_P \quad \forall u \in \mathbb{R}^n \quad (5)$$

Indeed, for any given $u \in \mathbb{R}^n$,

$$L(u) = -f_S^*(u) + \min_{y \in S}(\langle u, y\rangle - g(y))$$

Moreover, there exists $x_0 \in S$ such that $f_S^*(u) = \langle u, x_0\rangle - f_S(x_0)$, thus it follows that $u \in \partial f_S(x_0)$, see [5]. Hence, by definition of subgradients,

$$L(u) = f(x_0) - \langle u, x_0\rangle + \min_{y \in S}(\langle u, y\rangle - g(y))$$

$$= \min_{y \in S}(-\langle u, x_0\rangle + f(x_0) + \langle u, y\rangle - g(y))$$

$$= \min_{y \in \mathrm{ext}(S)}(-\langle u, x_0\rangle + f(x_0) + \langle u, y\rangle - g(y))$$

$$\leq z_P$$

Hence, (5) holds.

Conversely, given $x_0 \in S$ and $u_0 \in \partial f(x_0)$, one has that $-f_S^*(u_0) = f_S(x_0) - \langle u_0, x_0 \rangle$, thus

$$\begin{aligned}
&\min_{x \in S}(f(x_0) + \langle u_0, x - x_0 \rangle - g(x)) \\
&\qquad = f(x_0) - \langle u_0, x_0 \rangle + \min_{x \in S}(\langle u_0, x \rangle - g(x)) \\
&\qquad = -f_S^*(u_0) + \min_{x \in S}(\langle u_0, x \rangle - g(x)) \\
&\qquad = L(u_0) \\
&\qquad \leq z_D
\end{aligned}$$ □

Since, by (4), the function $L : u \in \mathbb{R}^n \mapsto L(u) = \min_{x \in S}(f(x) - \langle u, x \rangle) + \min_{x \in S}(-g(x) - \langle u, x \rangle)$ is minimum of affine functions, thus concave, Proposition 2 implies that z_P can be obtained by solving the *concave* maximization problem

$$\max_{u \in \mathbb{R}^n} L(u) \tag{6}$$

Although much simpler than the original expression (1), solving (6) still involves some computational burden, since it is a (nondifferentiable as a rule) nonlinear concave program, the objective function of which has no known analytical expression but must be evaluated by solving a convex minimization problem. This implies that, in practice, finding the optimal multiplier in (6) may be too costly, and, as customary in branch-and-bound approaches to combinatorial problems, see [6], one just performs a few iterations of some concave-maximization algorithm, leading to a lower bound on z_P.

This should be the strategy for an arbitrary compact convex set S. However, branch-and-bound schemes often assume S to be a simplex in $\mathbb{R}^n$, see e.g. [2]. In that case, Proposition 2 can be further strengthened, since finding the optimal multiplier for (6) is reduced to solving a linear system of $n+1$ equations. Indeed, one has

Proposition 3. *Let S be a simplex in $\mathbb{R}^n$, with vertices $v_0, \ldots, v_n$, and let $\hat{u}$ be the solution to the system of linear equations*

$$\begin{aligned}
g(v_1) - \langle v_1, u \rangle &= g(v_0) - \langle v_0, u \rangle \\
g(v_2) - \langle v_2, u \rangle &= g(v_0) - \langle v_0, u \rangle \\
\ldots \qquad & \qquad \ldots \\
g(v_n) - \langle v_n, u \rangle &= g(v_0) - \langle v_0, u \rangle
\end{aligned} \tag{7}$$

Then, $z_D = L(\hat{u})$

Proof. First of all, since S is assumed to be a simplex, the system of equations (7) has a unique solution, thus $\hat{u}$ is well defined. In order to show the result, it suffices to show that $\hat{u}$ is an optimal solution to the convex program $\min_{u \in \mathbb{R}^n} -L(u)$, by showing that 0 is a subgradient of $-L$ at $\hat{u}$. Indeed, since f is finite at S and S is compact, the optimal value of the optimization problem

$$\max_{x \in S} f(x) - \langle \hat{u}, x \rangle$$

is attained at some $x_0 \in S$. In other words, x_0 satisfies

$$-f_S(x_0) + \langle \hat{u}, x_0 \rangle = f_S^*(\hat{u}).$$

Hence, by Theorem 23.5 of [5], $x_0 \in \partial f_S^*(\hat{u})$. Moreover, since the piecewise linear function $h : u \in \mathbb{R}^n \mapsto h(u) = \max_{0 \le i \le n} -\langle v_i, u \rangle + g(v_i)$ has all its components active at $\hat{u}$, it follows that

$$\begin{aligned}\partial h(\hat{u}) &= \mathrm{conv}(\{-v_0, \ldots, -v_n\}) \\ &= -S.\end{aligned}$$

Hence $-x_0 \in \partial h(\hat{u})$, thus

$$\begin{aligned}0 &\in \partial f_S^*(\hat{u}) + \partial h(\hat{u}) \\ &= \partial(-L)(\hat{u}),\end{aligned}$$

showing that $\hat{u}$ minimizes $-L$, as asserted. □

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