An optimal bound for d.c. programs with convex constraints*

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Abstract. A well-known strategy for obtaining a lower bound on the minimum of a d.c. function $f - g$ over a compact convex set $S \subset \mathbb{R}^n$ consists of replacing the convex function $f$ by a linear minorant at $x_0 \in S$. In this note we show that the $x_0^*$ giving the optimal bound can be obtained by solving a convex minimization program, which corresponds to a Lagrangian decomposition of the problem. Moreover, if $S$ is a simplex, the optimal Lagrangian multiplier can be obtained by solving a system of $n + 1$ linear equations.

Key words: d.c. programs; bounds; Lagrangian decomposition

1 Problem statement

Let $S$ be a nonempty compact convex subset of $\mathbb{R}^n$, and let $f, g$ be convex and finite on $\mathbb{R}^n$. Our aim is to find a lower bound on the optimal value $z^*$ of the d.c. program

$$\min_{x \in S} f(x) - g(x) \quad (1)$$

Obtaining such lower bounds may be a real need when one is solving global optimization problems by a branch-and-bound strategy, [2, 1, 3], both in the bounding process (indeed, one needs to find good lower bounds for subproblems in the form (1) at each stage in the resolution of $\min_{x \in A} f(x) - g(x)$ using polyhedral, — say, simplicial or hyperrectangular — branching schemes) and in order to check feasibility as well (showing that a lower bound

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for \( \min_{x \in S} f(x) - g(x) \) is strictly positive implies the infeasibility of the (sub)-problem \( \min \{ h(x) : f(x) - g(x) \leq 0, x \in S \} \). In both cases, it is important to have the bounds as sharp as possible, since this may considerably reduce the computational effort.

Now we introduce some notation: let \( \text{ext}(S) \) denote the set of extreme points of the convex set \( S \), and, for any \( x_0 \in S \), let \( \partial f(x_0) \) denote the subdifferential of \( f \) at \( x_0 \).

One immediately derives the following result from the definition of sub-gradients and the fact that a concave function attains its minimum on a bounded convex set \( S \) at some point in \( \text{ext}(S) \), e.g. [2].

**Proposition 1.** For any \( x_0 \in S \) and \( u_0 \in \partial f(x_0) \), it follows

\[
    z^* \geq \min_{x \in S} f(x_0) + \langle u_0, x - x_0 \rangle - g(x)
\]

\[
    = \min_{x \in \text{ext}(S)} f(x_0) + \langle u_0, x - x_0 \rangle - g(x)
\]

The bound given in Proposition 1 strongly depends on the choice of the point \( x_0 \in S \). Such bound can then be sharpened if one choses the best possible \( x_0 \).

**Corollary 1.** Define

\[
    z_p = \sup \left\{ \min_{x \in \text{ext}(S)} f(x_0) + \langle u_0, x - x_0 \rangle - g(x) : x_0 \in S, u_0 \in \partial f(x_0) \right\}
\]

Then, \( z_p \leq z^* \).

Finding the optimal lower bound \( z_p \) from the definition amounts to solving a max-min nonlinear problem with nonconvex constraints, thus, at first glance, it does not seem obvious at all that the possible enhancement of the bound \( z_p \) with respect to any of the simple bounds given in Proposition 1 will deserve the resolution of the global-optimization problem (2). It turns out however that (2) can be formulated as a convex program, as shown in Section 2.

**2 A Lagrangian decomposition scheme**

Lower bounds for \( z^* \) can also be obtained via Lagrangian decomposition, [4]. Indeed, (1) can be equivalently rephrased as

\[
    \min \{ f(x) - g(y) : x = y, x \in S, y \in S \}
\]

Dualizing the constraints \( x = y \), one obtains the Lagrangian dual

\[
    z_D = \max_{u \in \mathbb{R}^*} L(u),
\]

(3)
with
\[
L(u) = \min_{x, y \in S} (f(x) - \langle u, x \rangle - g(y) + \langle u, y \rangle)
\]
\[
= \min_{x \in S} (f(x) - \langle u, x \rangle) + \min_{y \in S} (-g(y) + \langle u, y \rangle).
\]

We see that (3) is a bilevel problem, since the mere evaluation of the Lagrangian function \( L \) at a given \( u \in \mathbb{R}^n \) amounts to solving the convex minimization program \( \min_{x \in S} (f(x) - \langle u, x \rangle) \) and the concave minimization program \( \max_{y \in S} (-g(y) + \langle u, y \rangle) \).

Since the latter reduces to vertex enumeration if \( S \) is polyhedral, \( L \) can be evaluated in finite time for particular instances (e.g., when \( f \) is polyhedral or quadratic and \( S \) is a polytope), whilst for general problems \( L \) must be approximated by finding a near-optimal solution of a nonlinear program.

The next result shows that finding an optimal multiplier is equivalent to solving (2).

**Proposition 2.** One has \( z_D = z_P \)

**Proof.** Let \( f_S \) the restriction of \( f \) to \( S \),
\[
f_S(x) = \begin{cases} 
  f(x), & \text{if } x \in S \\
  +\infty, & \text{else}
\end{cases}
\]
and let \( f_S^* \) denote the Fenchel conjunct of \( f_S \), [5]
\[
f_S^*(p) = \sup \{ \langle p, x \rangle - f_S(x) : x \in \mathbb{R}^n \}
\]

We will show now that
\[
L(u) \leq z_P \quad \forall u \in \mathbb{R}^n
\]

(5)

Indeed, for any given \( u \in \mathbb{R}^n \),
\[
L(u) = -f_S^*(u) + \min_{y \in S} \langle u, y \rangle - g(y)
\]

Moreover, there exists \( x_0 \in S \) such that \( f_S^*(u) = \langle u, x_0 \rangle - f_S(x_0) \), thus it follows that \( u \in \partial f_S(x_0) \), see [5]. Hence, by definition of subgradients,
\[
L(u) = f(x_0) - \langle u, x_0 \rangle + \min_{y \in S} \langle u, y \rangle - g(y)
\]
\[
= \min_{y \in S} (-\langle u, x_0 \rangle + f(x_0) + \langle u, y \rangle - g(y))
\]
\[
= \min_{y \in \text{ext}(S)} (-\langle u, x_0 \rangle + f(x_0) + \langle u, y \rangle - g(y))
\]
\[
\leq z_P
\]

Hence, (5) holds.
Conversely, given \( x_0 \in S \) and \( u_0 \in \partial f(x_0) \), one has that \(-f^*_S(u_0) = f_S(x_0) - \langle u_0, x_0 \rangle\), thus

\[
\min_{x \in S} (f(x_0) + \langle u_0, x - x_0 \rangle - g(x)) \\
= f(x_0) - \langle u_0, x_0 \rangle + \min_{x \in S} (\langle u_0, x \rangle - g(x)) \\
= -f^*_S(u_0) + \min_{x \in S} (\langle u_0, x \rangle - g(x)) \\
= L(u_0) \\
\leq z_D
\]

Since, by (4), the function \( L : u \in \mathbb{R}^n \mapsto L(u) = \min_{x \in S} (f(x) - \langle u, x \rangle) + \min_{x \in S} (-g(x) - \langle u, x \rangle) \) is minimum of affine functions, thus concave, Proposition 2 implies that \( z_P \) can be obtained by solving the concave maximization problem

\[
\max_{u \in \mathbb{R}^n} L(u) \tag{6}
\]

Although much simpler than the original expression (1), solving (6) still involves some computational burden, since it is a (nondifferentiable as a rule) nonlinear concave program, the objective function of which has no known analytical expression but must be evaluated by solving a convex minimization problem. This implies that, in practice, finding the optimal multiplier in (6) may be too costly, and, as customary in branch-and-bound approaches to combinatorial problems, see [6], one just performs a few iterations of some concave-maximization algorithm, leading to a lower bound on \( z_P \).

This should be the strategy for an arbitrary compact convex set \( S \). However, branch-and-bound schemes often assume \( S \) to be a simplex in \( \mathbb{R}^n \), see e.g. [2]. In that case, Proposition 2 can be further strengthened, since finding the optimal multiplier for (6) is reduced to solving a linear system of \( n + 1 \) equations. Indeed, one has

**Proposition 3.** Let \( S \) be a simplex in \( \mathbb{R}^n \), with vertices \( v_0, \ldots, v_n \), and let \( \tilde{u} \) be the solution to the system of linear equations

\[
g(v_1) - \langle v_1, u \rangle = g(v_0) - \langle v_0, u \rangle \\
g(v_2) - \langle v_2, u \rangle = g(v_0) - \langle v_0, u \rangle \\
\ldots \\
g(v_n) - \langle v_n, u \rangle = g(v_0) - \langle v_0, u \rangle \tag{7}
\]

Then, \( z_D = L(\tilde{u}) \)

**Proof.** First of all, since \( S \) is assumed to be a simplex, the system of equations (7) has a unique solution, thus \( \tilde{u} \) is well defined. In order to show the result, it
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suffices to show that \( \hat{u} \) is an optimal solution to the convex program \( \min_{u \in \mathbb{R}^n} -L(u) \), by showing that 0 is a subgradient of \(-L\) at \( \hat{u} \). Indeed, since \( f \) is finite at \( S \) and \( S \) is compact, the optimal value of the optimization problem

\[
\max_{x \in S} f(x) - \langle \hat{u}, x \rangle
\]

is attained at some \( x_0 \in S \). In other words, \( x_0 \) satisfies

\[
-fS(x_0) + \langle \hat{u}, x_0 \rangle = f_S^*(\hat{u}).
\]

Hence, by Theorem 23.5 of [5], \( x_0 \in \partial f_S^*(\hat{u}) \). Moreover, since the piecewise linear function \( h : u \in \mathbb{R}^n \mapsto h(u) = \max_{0 \leq i \leq n} -\langle v_i, u \rangle + g(v_i) \) has all its components active at \( \hat{u} \), it follows that

\[
\partial h(\hat{u}) = \text{conv}\{\{-v_0, \ldots, -v_n\}\}
\]

\[
= -S.
\]

Hence \(-x_0 \in \partial h(\hat{u})\), thus

\[
0 \in \partial f_S^*(\hat{u}) + \partial h(\hat{u})
\]

\[
= \partial(-L)(\hat{u}),
\]

showing that \( \hat{u} \) minimizes \(-L\), as asserted. \( \square \)

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