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A D.C. biobjective location model

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Abstract. In this paper we address the biobjective problem of locating a semiobnoxious facility, that must provide service to a given set of demand points and, at the same time, has some negative effect on given regions in the plane. In the model considered, the location of the new facility is selected in such a way that it gives answer to these contradicting aims: minimize the service cost (given by a quite general function of the distances to the demand points) and maximize the distance to the nearest affected region, in order to reduce the negative impact. Instead of addressing the problem following the traditional trend in the literature (i.e., by aggregation of the two objectives into a single one), we will focus our attention in the construction of a finite ε-dominating set, that is, a finite feasible subset that approximates the Pareto-optimal outcome for the biobjective problem. This approach involves the resolution of univariate d.c. optimization problems, for each of which we show that a d.c. decomposition of its objective can be obtained, allowing us to use standard d.c. optimization techniques.

Key words: Biobjective Programming; Semi-obnoxious facility location; Univariate D.C. optimization

1. Introduction

We consider the biobjective problem of locating a facility in a region S in the plane, which installation is beneficial for a set of potential users, but, at the same time, it has negative effects on the population or the environment, so we can distinguish a set of negatively affected elements.

A facility with these characteristics, called semiobnoxious, involves two opposed and irreconcilable aims: on the one hand, the new facility must be located as near its potential clients as possible and, on the other hand, it must be placed far from the residents, which are affected in a negative way. In this sense, the model considered here will take into account the following criteria for locating the new facility at $x \in S$:

Criterion 1: Minimize the service cost from the new facility to the clients,

$$\min_{x\in S} T_1(x). \tag{C_1}$$

Criterion 2: Maximize the distance between the new facility and the nearest resident,

$$\max_{x\in S} T_2(x). \tag{C_2}$$

The biobjective problem obtained when both criteria are simultaneously considered can be written as:

$$\min_{x \in S}(T_1(x), -T_2(x)). \tag{1.1}$$

The usual approach for this problem has been the aggregation into a single objective (see Carrizosa and Plastria, 1999; Chen et al., 1992; Maranas and Floudas, 1994; Nickel and Dudenhoffer, 1997 and Tuy et al., 1995 for an updated review on the topic on semiobnoxious facility location). However, since the problem is essentially multiobjective, we propose to construct a finite approximation for the Pareto-optimal outcome, through the concept of ε-dominating set [Carrizosa et al., 1997; Hansen and Thisse, 1981; Lemaire, 1992; White, 1996, 1998). In other words, given $\varepsilon = (\varepsilon_1, \varepsilon_2)$ with $\varepsilon_1 > 0, \varepsilon_2 > 0$, our aim is to obtain a finite subset $S^* \subset S$ such that, for any $x \in S$ we can find $x^* \in S^*$ with:

$$\begin{aligned} T_1(x^*) &\leqslant T_1(x) + \varepsilon_1 \\ -T_2(x^*) &\leqslant -T_2(x) + \varepsilon_2. \end{aligned}$$

If just a singleton is sought, we can use any M.C.D.A. methodology (Vincke, 1992), to choose one element out of this finite set S^*.

In order to obtain a more tractable problem, we make several assumptions that will be introduced in the following section.

2. The Model

The users of the facility are modeled by a set of n points in the plane $A^+ = \{a_1, \ldots, a_n\}$, whereas the negatively affected elements are modeled by means of a set of m regions in the plane $A^- = \{R_1, \ldots, R_m\}$, where each R_i is a compact and convex set, whose boundary can be written (perhaps after an approximation process using splines (Ahlberg et al., 1967) as a finite union of line segments and circumference arcs, i.e.,

$$\operatorname{bd}(R_i) = \bigcup_{j=1}^{n_i} L_j^{(i)}$$

with $L_j^{(i)}$ being a line segment or a circumference arc. Note that these assumptions include the case in which R_i is a polygonal region, decomposed into convex polygons via a triangulation process. Note also that, as a particular case, a region R_i reduced to a point is allowed.

We consider the particular case of (1.1) with T_1 and T_2 defined as

$$T_1(x) = h(D_1(\gamma_1(x - a_1)), \ldots, D_n(\gamma_n(x - a_n))) \tag{2.2}$$

$$T_2(x) = \min_{1 \leqslant i \leqslant m} d_R(x, R_i) \tag{2.3}$$

where:

- x is the (unknown) location for the new facility.
- a_i are the coordinates of the i-th demand point, $i = 1, \dots, n$.
- γ_i is a guage providing a measure of the distance between a_i and any point of the plane, that is, a (convex) function $\gamma_i : \mathbb{R}^2 \mapsto \mathbb{R}$ defined as

  $$\gamma_i(x) = \inf\{t > 0 : x \in tB_i\} \quad x \in \mathbb{R}^2 \tag{2.4}$$

  where B_i is a convex set, the interior of which contains the origin (Michelot, 1993).
- $D_i : \mathbb{R} \mapsto \mathbb{R}_+$ is a convex, non-decreasing and non-constant function providing the service cost to the i-th demand point per unit of distance.
- $h : \mathbb{R}^n \mapsto \mathbb{R}$ is a monotonic norm, (Bauer et al., 1961), i.e., a norm satisfying

  $$(u, v \in \mathbb{R}^n, |u_i| \leqslant |v_i| \,\forall i) \Rightarrow h(u) \leqslant h(v).$$

  In particular, the L_p-norms, $\| \cdot \|_p$, are monotonic for $1 \leqslant p \leqslant \infty$.
- S is the feasible set for locating the new facility, which is assumed to be compact, not necessarily convex, and its boundary is a finite union of arcs of conic curves.
- $d_R(x, R_i) = \min_{x_i \in R_i} \|x - x_i\|_2$

In particular, the first objective includes, among others, the classical criteria in Location Theory (Plastria, 1995: *minisum*$(h = \| \cdot \|_1)$, *minimax* $(h = \| \cdot \|_\infty)$ and *cent-dian* $(h = (1 - \lambda)\| \cdot \|_1 + \lambda \| \cdot \|_\infty)$.

3. The algorithm

In this section we describe a procedure for building a finite ε-dominating set for Problem (1.1). Roughly speaking, such a procedure is based on the search of an ε-dominating set for a finite set of subproblems, obtained by decomposing the feasible set S into pieces within which the objective T_2 has a simpler structure. In this sense, given a region R_j, let V_j denote the *Voronoi cell* associated with R_j:

$$V_j = \{x \in \mathbb{R}^2 : d_R(x, R_j) \leqslant d_R(x, R_i), i \neq j, 1 \leqslant i \leqslant m\}.$$

Hence, since $\{V_j\}_{j=1}^n$ covers the plane, and ε-dominating set can be obtained by merging ε-dominating sets for the subproblems

$$\min_{x \in S \cap V_j} (T_1(x), -T_2(x)). \tag{3.5}$$

From the fact that the boundary of each R_j is made of line segments and circumference arcs, we conclude that the boundary of each V_j will consist of conic arcs, (Okabe et al., 1995).

The construction of an ε-dominating set for Subproblem (3.5) is simplified due to the fact that it can be reduced to solving a finite number of univariate d.c. optimization problems (see Remarks 3.7 and 3.13 below). To do this, the following result, proposed in Blanquero (1999) and Blanquero and Carrizosa (2000), is needed.

PROPOSITION 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a convex set. Let $\gamma : \mathbb{R}^q \to \mathbb{R}$ be a guage in $\mathbb{R}^q$ with unit ball B, let $f = (f_1, \dots, f_q) : \Omega \to \mathbb{R}^q$ be a d.c. mapping, with d.c. decomposition known: $f_i = f_i^+ - f_i^-$, with f_i^+, f_i^- convex. For any $i = 1, \dots, q$, let $M_i \geqslant \max\{\gamma(e_i), \gamma(-e_i)\}$, where e_i is the ith unit vector of $\mathbb{R}^q$. Then, $\gamma \circ f : \Omega \to \mathbb{R}$ is a d.c. function and a d.c. decomposition for it is given by*

$$\gamma \circ f = \left(\gamma \circ f + \sum_{i=1}^{q} M_i(f_i^+ + f_i^-)\right) - \sum_{i=1}^{q} M_i(f_i^+ + f_i^-). \tag{3.6}$$

Now we present a detailed description of the algorithm.

Step 0. Choose $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ and set $\varepsilon = (\varepsilon_1, \varepsilon_2)$.
Step 1. Solve the unconstrained minimization problem associated with the first criterion:

$$\min_{x \in \mathbb{R}^2} T_1(x) := h(D_1(\gamma_1(x - a_1)), \dots, D_n(\gamma_n(x - a_n))). \tag{P_0}$$

The assumptions on D_i and h allow us to ensure that (see Hiriart-Urruty and Lemaréchal, 1993)

PROPOSITION 3.2. *Function T_1 is convex.*

and (see Blanquero, 1999)

PROPOSITION 3.3. *Problem (P_0) has always a finite optimal solution*

An optimal solution for (P_0) can be obtained by the usual techniques in Convex Optimization, although there exist particular cases for which we can use specific methods (see Plastria, 1995 and the references therein).

In what follows, x_0^* will denote an optimal solution for (P_0).

Step 2. Construct the set of Voronoi cells $\mathcal{V}(A^-) = \{V_1, \dots, V_m\}$ associated with the negatively affected regions R_i, $i = 1, \dots, m$.

Once $\mathcal{V}(A^-)$ has been built we must find a Voronoi cell V_k containing the point x_0^*:

$$d_R(x_0^*, R_k) = \min_{1 \leqslant i \leqslant m} d_R(x_0^*, R_i).$$

If x_0^* belongs to more than one Voronoi cell, we select anyone of them.

If x_0^* is feasible, we define the index set $I = \{i : 1 \leqslant i \leqslant m, i \neq k, V_i \cap S \neq \emptyset\}$ and go to Step 3. Otherwise, we define $I = \{i : 1 \leqslant i \leqslant m V_i \cap S \neq \emptyset\}$ and go to Step 4.

Step 3. Obtain a finite ε-dominating set for the set $V_k \cap S$.

The following result asserts that x_0^* ε-dominates every point in a given subset of $V_k \cap S$.

PROPOSITION 3.4. *Given* $E_k = \{x \in V_k : d_R(x, R_k) \leqslant d_R(x_0^*, R_k)\}$, *the point* x_0^*, *optimal solution of* (P_0), *(0,0)-dominates every point* $\tilde{x} \in E_k \cap S$.

Proof. By the choice of x_0^* one has that $T_1(x_0^*) \leqslant T_1(\tilde{x})$ for every $\tilde{x} \in E_k \cap S$. On the other hand:

$$\min_{1 \leqslant i \leqslant m} d_R(\tilde{x}, R_i) = d_R(\tilde{x}, R_k) \leqslant d_R(x_0^*, R_k) = \min_{1 \leqslant i \leqslant m} d_R(x_0^*, R_i)$$

from where it follows that $T_2(x_0^*) \geqslant T_2(\tilde{x})$. Taking into account both inequalities we conclude that $x_0^*(0, 0)$-dominates $\tilde{x}$. □

Now our aim is to obtain a finite ε-dominating set for $(V_k \backslash E_k) \cap S$. We consider the sequence $\{r_j\}_{j=0}^{\infty}$, recursively defined as

$$r_0 = d_R(x_0^*, R_k) \qquad r_j = r_{j-1} + \varepsilon_2 \quad j \in \mathbb{N},$$

as well as the sets

$$\left.\begin{aligned} C_j^k &= \{x \in \mathbb{R}^2 : d_R(x, R_k) = r_j\} \\ D_j^k &= \{x \in \mathbb{R}^2 : r_j \leqslant d_R(x, R_k) \leqslant r_{j+1}\} \end{aligned}\right. \qquad j = 0, 1, \ldots$$

The boundary of R_k is composed by line segments and circumference arcs, from where we conclude that C_j^k can be written as a finite union of these elements. On the other hand, S is bounded and, therefore, we can ensure the existence of an index $J_k \in \mathbb{N}$ in such a way that the set $\left(\bigcup_{j \leqslant J_k} D_j^k\right) \cap V_k \cap S$ covers $(V_k \backslash E_k) \cap S$.

We are now going to describe a procedure for obtaining a point that ε-dominates the set $D_j^k \cap V_k \cap S$, yielding a finite ε-dominating set for $\left(\bigcup_{j \leqslant J_k} D_j^k\right) \cap V_k \cap S$. In order to find these ε-dominating points we consider, for each index $j = 0, \ldots, J_k$, the set $\tilde{C}_j^k$ defined as

$$\tilde{C}_j^k = (C_j^k \cap V_k \cap S) \cup (D_j^k \cap \mathrm{bd}(V_k \cap S))$$

as well as the optimization problem

$$\min\{T_1(x) : x \in \tilde{C}_j^k\}. \qquad (P_j^k)$$

Observe that $\mathrm{bd}(V_k \cap S) = (\mathrm{bd}(V_k) \cap S) \cup (V_k \cap \mathrm{bd}(S))$ since V_k and S are closed.

EXAMPLE 3.5. Consider a biobjective location problem with two negatively affected regions R_1 and R_2, defined as the square of vertices $(-1, 1)$, $(-1, -1)$, $(1, -1)$ and $(1, 1)$, and the point $(5, 0)$, respectively, as well as three demand points $P_1 = (1, 2)$, $P_2 = (3, -1)$ and $P_3 = (4, 1)$ (see Figure 1). The transportation cost from the facility to each demand point is imposed to be proportional to the

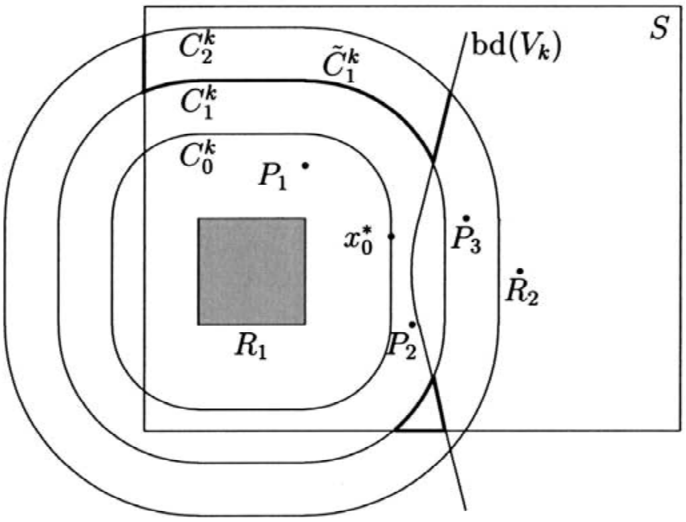

Figure 1. Example of set $\tilde{C}_j^k$

Euclidean distance separating them, the weighting factors being $w_1 = 5$, $w_2 = 4$ and $w_3 = 3$. The feasible region S is assumed to be the rectangle with vertices $(-2, -3)$, $(-2, 5)$, $(8, 5)$ and $(8, -3)$.

The Voronoi cells V_1 and V_2 associated with R_1 and R_2 have a common edge consists of two rays and a parabolic arc connecting them. On the other hand, the optimal solution for (P_0) is the point (2.6030, 0.6673), that belongs to the Voronoi cell V_1, $(k = 1)$, and it is feasible.

In Figure 1 we show, for $\varepsilon_2 = 1$, the sets C_0^1, C_1^1 and C_2^1, as well as $\tilde{C}_1^1$ using wide line. □

The set $\tilde{C}_j^k$ can be expressed as a finite union of U_j^k closed conic arcs, since C_j^k, bd(V_k) and bd(S) satisfy this property. For all of these arcs it is possible to obtain a d.c. parametric representation with known d.c. decomposition (Blanquero, 1999) (for the sake of simplicity, we omit the indices j and k):

$$A_u : t \in [0, 1] \mapsto (x_u(t), y_u(t)) \quad u = 1, \ldots, U_j^k.$$

Hence, the resolution of (P_j^k) reduces to solving a finite number U_j^k of univariate problems of the form

$$\min_{t\in[0,1]} h(D_1(\gamma_1(A_u(t) - a_1)), \ldots, D_n(\gamma_n(A_u(t) - a_n))) \quad u = 1, \ldots, U_j^k. \tag{3.7}$$

The following result asserts that a d.c. decomposition for every component of the objective in (3.7) can be obtained.

PROPOSITION 3.6. *For every index* $i = 1, \ldots, n$ *and* $u = 1, \ldots, U_j^k$, *a d.c. decomposition for the function*

$$G_{iu}(t) = D_i(\gamma_i(A_u(t) - a_i)) \quad t \in [0, 1]$$

can be computed.

Proof. First, note that A_u is d.c. with known d.c. decomposition, since it is a parameterization of a conic arc, and also that $\gamma_i(A_u(t) - a_i)$ is d.c. since it is the composition of a guage with a d.c. function; moreover, by Proposition 3.1, one has a d.c. decomposition for this function:

$$\gamma_i(A_u(t) - a_i) = F_{iu}^+(t) - F_{iu}^-(t).$$

Taking into account the continuity of the functions involved and the compactness of the domain of definition, we can assume without loss of generality that F_{iu}^+ and F_{iu}^- are non-negative. On the other hand, let $L_i \geqslant \max\{\gamma_i(x - a_i) : x \in S\}$, so one has that:

$$0 \leqslant \gamma_i(A_u(t) - a_i) \leqslant L_i \quad \forall t \in [0, 1].$$

Then, Proposition 3.7 in Tuy (1998) provides the following d.c. decomposition for $G_{iu}(t)$:

$$G_{iu}(t) = (G_{iu}(t) + H_{iu}(t)) - H_{iu}(t)$$

where $H_{iu}(t) = K(L_i + F_{iu}^-(t) - F_{iu}^+(t))$ and K is any constant satisfying $K \geqslant D'_{i-}(L_i)$. □

REMARK 3.7. By Proposition 3.1 we have that an optimal solution for (P_j^k) can be obtained by solving a finite number of univariate d.c. optimization problems, with known d.c. decomposition for their objectives.

In order to solve these one-dimensional problems, we can use any d.c. optimization method, such a Branch & Bound or covering algorithms (Baritompa and Cutler, 1994; Blanquero, 1999; Blanquero and Carrizosa, 2000; Breiman and Cutler, 1993).

In the search of an ε-dominating point for the set D_j^k, the following lemma will be needed.

LEMMA 3.8. *For every point* $\bar{x} \in D_j^k \cap V_k \cap S$, *with* $j \geqslant 0$, *one has that* $[x_0^*, \bar{x}] \cap \tilde{C}_j^k \neq \emptyset$

Proof. We can assume that $j > 0$, since $x_0^* \in \tilde{C}_0^k$. Given $\bar{x} \in D_j^k \cap V_k \cap S$, one has that $d_R(x_0^*, R_k) = r_0 < r_j \leqslant d_R(\bar{x}, R_k)$, and then, by continuity of the distance, $[x_0^*, \bar{x}] \cap C_j^k \neq \emptyset$.

Let $\bar{y} \in [x_0^*, \bar{x}] \cap C_j^k$. If $\bar{y} \in V_k \cap S$, it follows immediately that $\bar{y} \in \tilde{C}_j^k$, and the result is shown in that case. Therefore, assume that $\bar{y} \notin V_k \cap S$ and consider a point $\hat{y}$ belonging to $\mathrm{bd}(V_k \cap S) \cap [\bar{x}, \bar{y}]$, which is non-empty, since $\bar{y} \notin V_k \cap S$, $\bar{x} \in V_k \cap S$ and S is robust.

We are now going to show that $\hat{y} \in D_j^k$. In order to achieve this, first recall that the *inf*-distance function $d_R(\cdot, R_i)$ is quasi-convex, (Hiriart-Urruty and Lemaréchal, 1993), so the set $N_\alpha^{(i)} = \{x \in \mathbb{R}^2 : d_R(x, R_i) \leqslant \alpha\}$ is convex for all $\alpha \geqslant 0$. The point $\hat{y}$ must satisfy that $d_R(\hat{y}, R_k) \geqslant r_j$ since, in other case, we consider $d = \max\{d_R(\hat{y}, R_k), r_0\}$, that it is strictly less than r_j. Taking into account that $\bar{y} \in [x_0^*, \hat{y}]$ and that $d_R(\bar{y}, R_k) = r_j > d$, we conclude that $\bar{y} \notin N_d^{(k)}$, and this contradicts the convexity of this set, since $\hat{y} \in N_d^{(k)}$ and $x_0^* \in N_d^{(k)}$.

On the other hand, from $\bar{y} \in C_j^k$, $\bar{x} \in D_j^k$ and the quasi-convexity of the function $d_R(\cdot, R_i)$, it follows that $d_R(\hat{y}, R_k) \leqslant r_{j+1}$ and, therefore, $\hat{y} \in D_j^k$, showing the result. □

Using this lemma, the following proposition provides a finite ε-dominating set for the feasible points in the Voronoi cell V_k located at a distance from R_k between r_j and r_{j+1}. In the sequel, x_{kj}^* will denote an ε_1-optimal solution for Problem (P_j^k).

PROPOSITION 3.9. *Given $j \geqslant 0$, the point x_{kj}^* ε-dominates each point $\tilde{x} \in D_j^k \cap V_k \cap S$.*

Proof. Given a point $\tilde{x} \in D_j^k \cap V_k \cap S$ we have that:

$$\begin{aligned} \min_{1 \leqslant i \leqslant m} d_R(\tilde{x}, R_i) &= d_R(\tilde{x}, R_k) \\ &\leqslant r_{j+1} \qquad (3.8) \\ &= r_j + \varepsilon_2 \\ &\leqslant d_R(x_{kj}^*, R_k) + \varepsilon_2 \qquad (3.9) \\ &= \min_{1 \leqslant i \leqslant m} d_R(x_{kj}^*, R_i) + \varepsilon_2 \qquad (3.10) \end{aligned}$$

where (3.8) is a consequence of $\tilde{x} \in D_j^k$, whereas (3.9) and (3.10) follow from $x_{kj}^* \in D_j^k \cap V_k$. This shows the ε-dominance of x_{kj}^* with regard to the second objective.

On the other hand, Lemma 3.8 asserts the existence of a point $\hat{x} \in [x_0^*, \tilde{x}] \cap \tilde{C}_j^k \subset S$ and, by convexity of the first objective, it follows that:

$$T_1(x_0^*) \leqslant T_1(\hat{x}) \leqslant T_1(\tilde{x}).$$

The point x_{kj}^* is an ε_1-optimal solution of minimizing T_1 over $\tilde{C}_j^k$ and, hence:

$$T_1(x_{kj}^*) - \varepsilon_1 \leqslant T_1(\hat{x}) \leqslant T_1(\tilde{x})$$

completing the proof. □

Step 4. Obtain a finite ε-dominating set for every set $V_l \cap S$, with $l \in I$. Consider the problem

$$\min\{T_1(x) : x \in \mathrm{bd}(V_l \cap S)\} \tag{P_l}$$

which has a finite optimal solution, by the compactness of the feasible set and the continuity of the objective. Taking into account that $\mathrm{bd}(V_l \cap S)$ consists of conic arcs, for which we can obtain a d.c. parameterization, an optimal solution for (P_l) can be found by solving a finite number of univariate d.c. optimization problems. The following result asserts that any ε_1-optimal solution of (P_l) $(\varepsilon_1, 0)$-dominates every point in $V_l \cap S$ not further from R_l than the former. We omit the proof of this and the remaining results proposed in this stage, since they are similar to those provided in Step 3 (see Blanquero, 1999, for details).

PROPOSITION 3.10. *For $l \in I$, let x_l^* be an ε_1-optimal solution for Problem (P_l) and $E_l = \{x \in V_l : d_R(x, R_l) \leqslant d_R(x_l^*, R_l)\}$. Then x_l^* $(\varepsilon_1, 0)$-dominates every point $\tilde{x} \in E_l \cap S$.*

We consider the sequence $\{r_j\}_{j=0}^{\infty}$, recursively defined as

$$r_0 = d_R(x_0^*, R_k) \quad r_j = r_{j-1} + \varepsilon_2 \quad j \in \mathbb{N},$$

as well as the sets

$$C_j^l = \{x \in \mathbb{R}^2 : d_R(x, R_l) = r_j\}$$
$$D_j^l = \{x \in \mathbb{R}^2 : r_j \leqslant d_R(x, R_l) \leqslant r_{j+1}\}.$$

The outline of the procedure consists of obtaining an ε-dominating point for every set $D_j^l \cap V_l \cap S$, $j \geqslant 0$, as in Step 3, for which we consider the set $\bar{C}_j^l$, defined as

$$\bar{C}_j^l = ((C_{j-1}^l \cup C_j^l) \cap V_l \cap S) \cup (D_{j-1}^l \cap \mathrm{bd}(V_l \cap S)),$$

as well as the optimization problem

$$\min\{T_1(x) : x \in \bar{C}_j^l\}. \tag{P_j^l)}$$

EXAMPLE 3.11. In Figure 2 we show the sets C_0^2, C_1^2 and C_2^2, as well $\bar{C}_2^2$ using wide line, for the problem considered in Example 3.5, taking $\varepsilon_2 = 0.8$ in this case. □

The set $\bar{C}_j^l$ can be written as a finite union of U_j^l closed conic arcs, since C_{j-1}^l, C_j^l, $\mathrm{bd}(V_l)$ and $\mathrm{bd}(S)$ satisfy this property. For all of these arcs it is possible to obtain a d.c. parametric representation wit known d.c. decomposition (for the sake of simplicity, we omit the indices j and l):

$$A_u : t \in [0, 1] \mapsto (x_u(t), y_u(t)) \quad u = 1, \ldots, U_j^l.$$

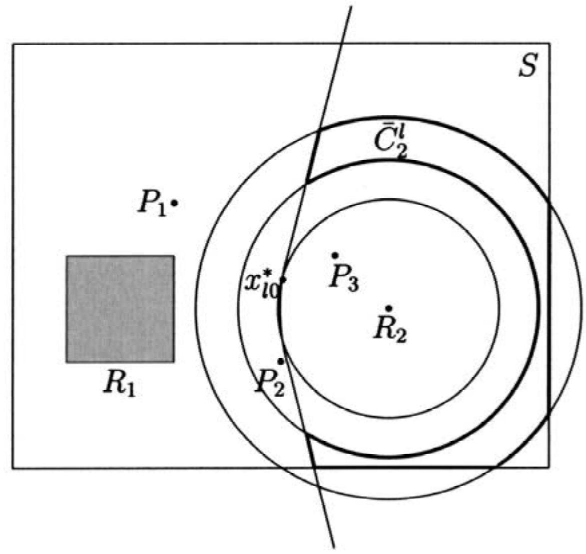

Figure 2. Example of set $\bar{C}_j^l$

Hence, the resolution of (P_j^l) reduces to solving a finite number U_j^l of univariate problems of the form

$$\min_{t\in[0,1]} h(D_1(\gamma_1(A_u(t)-a_1)),\ldots,D_n(\gamma_n(A_u(t)-a_n))) \quad u=1,\ldots,U_j^l. \tag{3.11}$$

As with the set $V_k \cap S$, it is possible to obtain a d.c. decomposition for every component of the objective of (3.11):

PROPOSITION 3.12. *For every index $i = 1,\ldots,n$ and $u = 1,\ldots,U_j^l$, a d.c. decomposition for the function*

$$G_{iu}(t) = D_i(\gamma_i(A_u(t)-a_i)) \quad t \in [0,1]$$

can be computed.

REMARK 3.13. As a consequence of Proposition 3.1 one has that an optimal solution for (P_j^l) can be obtained by solving a finite number of univariate d.c. optimization problems, with known d.c. decomposition for their objectives.

In the sequel, x_{lj}^* will denote an ε_1-optimal solution for problem (P_j^l).

As in Step 3, we present a technical lemma that will be needed in order to build a finite ε-dominating set for D_j^l.

LEMMA 3.14. *For every $\bar{x} \in D_{j-1}^l \cap V_l \cap S$, $j \geqslant 1$, one has that $[x_0^*, \bar{x}] \cap \bar{C}_j^l \neq \emptyset$*

Finally, we provide a result that allows us to obtain a finite ε-dominating set for the feasible points in V_l whose distance from R_l lies between r_{j-1} and r_j.

PROPOSITION 3.15. *Given $j \geqslant 1$, the point x^*_{lj} ε-dominates every point $\tilde{x} \in D^l_{j-1} \cap V_l \cap S$.*

Step 5. Suppress (0,0)-dominated solutions from the set obtained in the previous steps. In this stage we are going to remove from S^* those points (0,0)-dominated by any other element in this set belonging to a different Voronoi cell. In order to achieve this, the following simple procedure can be used:

(1) Sort the elements in S^* by increasing values of T_1.

(2) Examine the list and suppress those points x^*_j with an objective value for T_2 less than or equal to its predecessor, that is, if $T_2(x^*_j) \leqslant T_2(x^*_{j-1})$ we remove x^*_j from the list.

As a summary, we now provide a schematic description of the algorithm.

Algorithm 3.16

Step 0. Initialization

0.1. Set $S^* := \emptyset$.

0.2. Choose $\varepsilon_1 > 0$, $\varepsilon_2 > 0$.

Step 1. Solve the optimization problem given by the first objective.

1.1. Find $x^*_0 \in \arg\ \min\{T_1(x) : x \in \mathbb{R}^2\}$.

Step 2. Build the Voronoi cells

2.1. Build the cover of the plane by Voronoi cells $\mathcal{V}(A^-) = \{V_i\}_{i=1}^m$ generated by the regions R_i.

2.2. Find a Voronoi cell V_k such that $x^*_0 \in V_k$.

2.3. If x^*_0 is feasible, set $I = \{i : 1 \leqslant i \leqslant m, i \neq k, V_i \cap S \neq \emptyset\}$ and go to Step 3. Otherwise, set $I = \{i : 1 \leqslant i \leqslant m, V_i \cap S \neq \emptyset\}$ and go to Step 4.

Step 3. Obtain a finite ε-dominating set for $V_k \cap S$

3.1. Set $r_0 := d_R(x^*_0, R_k)$

3.2. Repeat for $j = 0, 1, \ldots$

3.2.1. Set $r_{j+1} := r_j + \varepsilon_2$.

3.2.2. Set

$$C^k_j := \{x \in \mathbb{R}^2 : d_R(x, R_k) = r_j\}$$
$$D^k_j := \{x \in \mathbb{R}^2 : r_j \leqslant d_R(x, R_k) \leqslant r_{j+1}\}$$
$$\tilde{C}^k_j := (C^k_j \cap V_k \cap S) \cup (D^k_j \cap \mathrm{bd}(V_k \cap S)).$$

3.2.3. If $\tilde{C}^k_j = \emptyset$, go to Step 4.

3.2.4. Find an ε_1-optimal solution x^*_{kj} for Problem $\min\{T_1(x) : x \in \tilde{C}^k_j\}$

Step 4. Obtain a finite ε-dominating set for $V_l \cap S$ with $l \in I$

4.1. Repeat for $l \in I$

Table 1. Negatively affected points (Example 3.1).

(5,17)	(5,54)	(7,14)	(7,61)	(7,67)	(8,14)	(8,20)	(9,42)	(9,74)	(11,76)
(12,76)	(13,20)	(15,10)	(17,10)	(18,91)	(19,37)	(20,33)	(20,53)	(21,58)	(21,92)
(22,65)	(23,31)	(23,38)	(23,49)	(23,72)	(23,95)	(24,18)	(24,77)	(25,45)	(26,40)
(26,67)	(27,52)	(28,70)	(31,44)	(34,82)	(36,58)	(37,25)	(38,61)	(40,9)	(41,12)
(41,60)	(42,25)	(43,12)	(46,63)	(48,11)	(49,34)	(49,51)	(49,57)	(50,52)	(50,54)
(50,63)	(52,89)	(54,8)	(55,95)	(56,91)	(58,69)	(58,93)	(59,60)	(62,70)	(62,79)
(63,37)	(65,12)	(66,89)	(67,79)	(67,83)	(69,18)	(69,81)	(70,6)	(70,12)	(70,25)
(70,37)	(70,44)	(70,54)	(70,57)	(71,17)	(71,42)	(71,53)	(72,56)	(74,79)	(74,95)
(75,58)	(77,54)	(77,93)	(79,41)	(79,53)	(80,38)	(81,21)	(81,69)	(83,53)	(83,56)
(84,92)	(87,49)	(89,36)	(89,38)	(90,71)	(90,84)	(92,9)	(93,23)	(93,75)	(94,47)

4.1.1. Find an ε_1-optimal solution x_l^* for Problem $\min\{T_1(x) : x \in \mathrm{bd}(V_l \cap S)\}$

4.1.2. Set $S^* := S^* \cup \{x_l^*\}$.

4.1.3. Set $r_0 := d_R(x_l^*, R_l)$.

4.1.4. Repeat for $j = 1, 2, \ldots$

4.1.4.1. Set $r_j := r_{j-1} + \varepsilon_2$.

4.1.4.2. Set $\bar{C}_j^l := ((C_{j-1}^l \cup C_j^l) \cap V_l \cap S) \cup (D_{j-1}^l \cap \mathrm{bd}(V_l \cap S))$

4.1.4.3. If $\bar{C}_j^l = \emptyset$, go to Step 5.

4.1.4.4. Find an ε_1-optimal solution x_{lj}^* for Problem $\min\{T_1(x) : x \in \bar{C}_j^l\}$

4.1.4.5. Set $S^* := S^* \cup \{x_{lj}^*\}$

Step 5. Purge the ε-dominating set

5.1. Sort the elements in S^* by increasing values of T_1.

5.2. Examine S^* and remove points $x_j \in S^*$ such that $T_2(x_j) \leqslant T_2(x_{j-1})$.

Note that in Step *4.1.4.* for $j \geqslant 2$, the minimum of $T_1(x)$ over $C_{j-1}^l \cap V_l \cap S$ has already been computed in the previous iteration, so it must not be considered again in the minimization problem of Step *4.1.4.4.*

REMARK 3.17. Note that the algorithm described above can be extended to more general settings. Indeed, the feasible set S can also be any compact and convex set (for which a d.c. representation of its boundary can be obtained using the methodology described in Blanquero and Carrizosa, 2000) or, with more generality, any robust and bounded (not necessarily convex) set with a known d.c. decomposition for its boundary, assumed that the corresponding curves $\tilde{C}_j^k$ can be computed.

Table 2. Demand points and weights (Example 3.1).

P_i	w_i	P_i	w_i
(25,27)	1	(30,20)	1
(31,31)	1	(32,27)	1
(34,31)	1	(35,22)	1
(36,41)	1	(37,43)	1
(38,35)	1	(39,29)	1
(40,29)	1	(45,33)	1
(90,70)	12		

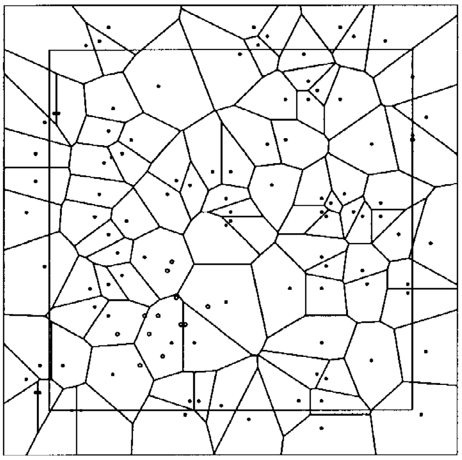

Figure 3. Feasible set and Voronoi diagram (Example 3.1).

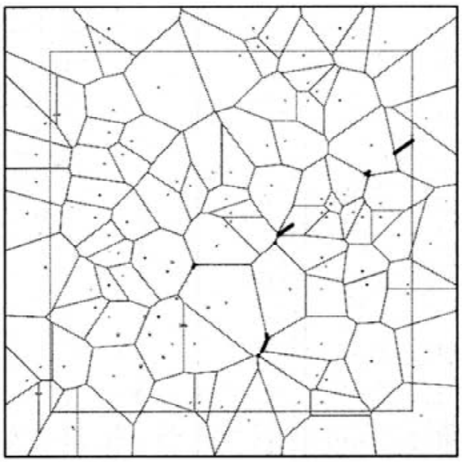

Figure 4. ε-Dominating set (Example 3.1).

Table 3. ε-Dominating set and objective values (Example 3.1).

E_i	T_1	T_2	E_i	T_1	T_2
(41.622,41.919)	839.679	10.824	(41.957,42.500)	839.717	11.039
(56.043,22.304)	984.539	13.653	(56.846,23.308)	973.671	13.262
(56.869,23.361)	973.040	13.233	(57.100,23.901)	966.700	12.946
(57.285,24.331)	961.783	12.733	(57.554,24.959)	954.782	12.446
(57.776,25.478)	949.172	12.233	(57.814,27.036)	930.581	11.233
(57.831,26.954)	931.608	11.298	(57.948,26,410)	938.514	11.733
(59.714,47.286)	815.611	10.798	(60.464,49.000)	814.932	10.767
(60.650,49.217)	814.881	10.503	(61.063,49.508)	814.790	10.003
(61.476,49.799)	814.703	9.503	(61.890,50.091)	814.618	9.003
(62.304,50.384)	814.536	8.503	(62.719,50.678)	814.457	8.003
(63.135,50.973)	814.380	7.503	(63.552,51.269)	814.306	7.003
(79.652,62.599)	812.571	6.542	(79.830,62.298)	812.561	6.410
(80.310,62.240)	812.841	6.795	(80.428,63.118)	812.523	5.910
(86.114,67.237)	812.198	5.410	(86.161,67.071)	812.231	5.500
(86.458,67.471)	812.182	5.000	(86.871,67.766)	812.162	4.500
(87.286,68.062)	812.142	3.000	(88.550,68.963)	812.083	2.500
(88.123,68.659)	812.064	2.000	(89.449,69.605)	812.043	1.500
(90.000,70.000)	812.019	1.000			

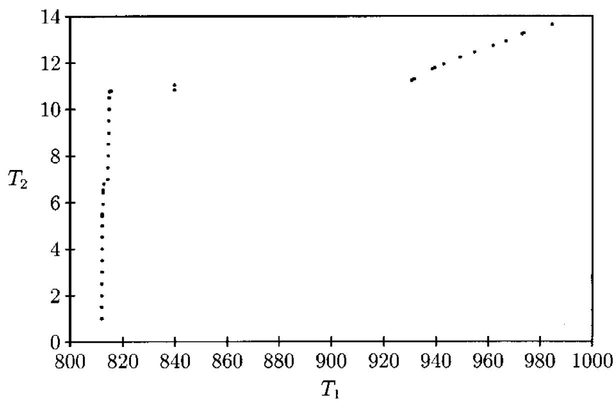

Figure 5. Comparison of objective values over the ε-dominating set (Example 3.1).

3.1. AN ILLUSTRATIVE EXAMPLE

As a step-by-step example we consider the location problem of a semi-obnoxious facility in the square $S = [10, 90]^2$, in which the objective T_1 is given by the sum of the transportation costs from the facility to the demand points, each of which is assumed to be proportional to the Euclidean distance separating them. The set of elements negatively affected by the new facility consists of 100 points, whose coordinates are shown in Table 1; we have also considered a set of 13 demand points, with coordinates and weights given in Table 2. Their position in the plane, as well as the Voronoi diagram generated by the negatively affected points, are shown in Figure 3.

The application of the previous algorithm to this problem, taking $\varepsilon_1 = 10^{-6}$ and $\varepsilon_2 = 0.5$, provided an ε-dominating set of 39 points, whose coordinates and objective values are given in Table 3. Such a set is also shown in Figure 4 (coordinates) and Figure 5 (objective values).

The output given in Table 3 can be of help in the decision-making process. Indeed, note that with this information the original decision problem of choosing a point from the (infinite, non-convex) feasible set S, is reduced to inspecting the finite ε-dominating set, for which the use of a M.C.D.A. methodology may then be appropriate; for instance, if the decision-maker wants to aggregate the objectives T_1 and T_2 by means of an additive utility,

$$\lambda \cdot T_1 + (1 - \lambda) \cdot (-T_2) \tag{3.12}$$

for some λ in [0,1], one just needs to evaluate the ε-dominating set S^* using (3.12): the point in S^* yielding the lowest value is an ε-optimal solution to the problem

$$\min_{x \in S}(\lambda \cdot T_1(x) + (1 - \lambda) \cdot (-T_2)(x))$$

The computational implementation of the algorithm has been performed using C_{++} language and LEDA (Library of Efficient Data types and Algorithms), (Mehlhorn and Näher, 1995, 1999) a development of Max-Planck-Institut für Informatik. □

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References

Ahlberg, J.H., Nilson, E.N. and Walsh, J.L. (1967), *The Theory of Splines and Their Applications*, Academic Press, New York.

Bauer, F.L., Stoer J. and Witzgall, C. (1961), Absolute and monotonic norms, *NBumerische Mathematik* 3, 257–264.

Baritompa, W. and Cutler, A. (1994), Accelerations for global optimization covering methods using second derivatives, *Journal of Global Optimization* 4, 329–341.

Blanquero, R. (1999), *Localización de Servicios en el Plano Mediante Técnicas de Optimización d.c.* Unpublished PhD. thesis. Universidad de Sevilla, Spain.

Blanquero, R. and Carrizosa, E. (2000), On covering methods for dc optimization, *Journal of Global Optimization* 13, 265–274.

Blanquero, R. and Carrizosa, E. (2000), Optimization of the norm of a vector valued dc function and applications, *Journal of Optimization Theory and Applications* 107, 245–271.

Breiman, L. and Cutler, A. (1993), A deterministic algorithm for global optimization, *Mathematical Programming* 58, 179–199.

Carrizosa, E., Conde, E. and Romero-Morales, M.D. (1997), Location of a semiobnoxious facility: a biobjective approach, in *Advances in Multiple Objective and Goal Programming*, R. Caballero, F. Ruiz and R.E. Steuer (Eds.), Springer, Berlin.

Carrizosa, E. and Plastria, F. (1999), Location of semi-obnoxious facilities, *Studies in Locational Analysis* **12**, 1–27.

Chen, P.C., Hansen, P., Jaumard, B. and Tuy, H. (1992), Weber's problem with attraction and repulsion, *Journal of Regional Science* 32, 467–486.

Hansen, P. and Thisse, J.F., The generalized Weber-Rawls problems, in *Operations Research 1981*, J.P. Brans (Ed.), North-Holland, Amsterdam.

Hiriart-Urruty, J.B. and Lemaréchal, C. (1993), *Convex Analysis and Minimization Algorithms I*, Springer, Berlin.

Lemaire, B. (1992), Approximation in multiobjective optimization, *Journal of Global Optimization* 2, 117–132.

Mehlhorn, K. and Näher, S. (1995), LEDA, a platform for combinatorial and geometric computing, *Communications of the ACM* 38, 96–102.

Mehlhorn, K. and Näher, S. (1999), *LEDA, a Platform for Combinatorial and Geometric Computing*, Cambridge University Press, Chichester.

Maranas, C.D. and Floudas, C.A. (1994), A global optimization method for Weber's problem with attraction and repulsion, in *Large Scale Optimization: State of the Art*, W.W. Hager et al. (Eds.), Kluwer Academic Publishers, Dordrecht.

Michelot, C. (1993), The mathematics of continuous location, *Studies in Locational Analysis* 5, 59–83.

Nickel, S. and Dudenhoffer, E.M. (1997), Weber's problem with attraction and repulsion under polyhedral gauges, *Journal of Global Optimization* 11, 409–432.

Okabe, A., Boots, B. and Sugihara, K. (1995), *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams*, John Wiley & Sons, Chichester.

Plastria, F. (1995), Continuous location problems, in *Facility location. A Survey of Applications and Methods*, Z. Drezner (ed.), Springer, New York.

Tuy, H. (1995), D.C. optimization: theory, methods and algorithms, in *Handbook of Global Optimization*, Kluwer Academic Publishers, Dordrecht.

Tuy, H., Al-Khayyal, F. and Zhou, F. (1995), A D.C. optimization method for single facility location problems, *Journal of Global Optimization* 7, 209–227.

Tuy, H. (1998), *Convex Analysis and Global Optimization*, Kluwer Academic Press, Dordrecht.

Vincke, P. (1992), *Multicriteria Decision Aid*, Wiley, New York.

White, D.J. (1996), Epsilon efficiency, *Journal of Optimization Theory and Applications* 49, 319–337.

White, D.J. (1998), Epsilon dominance and constraint partitioning in multiple objective problems, *Journal of Global Optimization* 12, 435–448.