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Continuous Optimization

A global optimization procedure for the location of a median line in the three-dimensional space ☆

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ABSTRACT

A global optimization procedure is proposed to find a line in the Euclidean three-dimensional space which minimizes the sum of distances to a given finite set of three-dimensional data points.

Although we are using similar techniques as for location problems in two dimensions, it is shown that the problem becomes much harder to solve. However, a problem parameterization as well as lower bounds are suggested whereby we succeeded in solving medium-size instances in a reasonable amount of computing time.

1. Introduction

In this work, we consider the *median line problem* in the Euclidean three-dimensional space, i.e. we seek a line which minimizes the sum of Euclidean distances to some given data or demand points in $\mathbb{R}^3$.

The median line problem in two dimensions and in the context of location theory was first analyzed by Wesolowsky (1975). Therein, it was shown that there exists an optimal line intersecting two data points which leads to a polynomial-time solution algorithm. Many generalizations such as general distance measures, line segments, and restrictions were studied e.g. in Morris and Norback (1983, 1980), Norback and Morris (1980), and Korneenko and Martini (1993) as well as in Schöbel (1999) and references therein.

An overview about locating lines as well as more general dimensional facilities on the plane can be found in Díaz-Báñez et al. (2004). Moreover, also the recent work (Blanquero et al., 2009) addresses the optimal location of structures in the plane by means of d.c. optimization tools. This paper uses a similar approach to the median line location problem in the Euclidean three-dimensional space.

Although the Euclidean two-dimensional median line problem is well-studied and exact polynomial time algorithms are available, the three-dimensional problem becomes much harder and only a few references can be found in the literature. In Brimberg et al. (2002), the authors discussed the problem of locating a vertical line as well as vertical line segments for any ℓ_p norm. It was shown that these problems can be essentially reduced to classical planar Weber problems. The work was extended in Brimberg et al. (2003). Therein, the three-dimensional median line problem was studied with some restrictions, e.g. that all data points and/or the line to be located are contained in a given hyperplane. Furthermore, some heuristics for the general problem were presented, but without any numerical results. Summarizing, to the best of our knowledge no algorithm for the general three-dimensional median line problem has been reported in the literature.

The remainder of this paper is structured as follows. In Section 2, we discuss the problem formulation and some theoretical results are given. Furthermore, we present a problem parameterization which is of fundamental importance for the following sections. Next, geometric branch-and-bound solution methods are briefly summarized in Section 3. To apply this technique to the median line problem, lower bounds are derived in Section 4. Some numerical results can be found in Section 5 where it is shown that the geometric branch-and-bound leads to solutions for the median line problem with data sets of moderate size in a reasonable amount of computing time. Finally, a discussion as well as some further research ideas are given in Section 6.

2. Problem formulation

A line r in $\mathbb{R}^3$ has the form

$$r = r(x, d) = \{x + td : t \in \mathbb{R}\},$$

where $d \in \mathbb{R}^3 \setminus \{0\}$ is the direction of r and $x \in \mathbb{R}^3$. Moreover, we will use the following notation.

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Notation 1. For any $a \in \mathbb{R}^3$ and $x, d \in \mathbb{R}^3$ with $d \neq 0$ denote by

$$\delta_a(x,d) := \min_{t\in\mathbb{R}} \|x + td - a\|_2$$

the Euclidean distance from a to the line $r(x,d)$.

This notation leads to the following analytical expression for the distance from a point to a line.

Lemma 1. *Let $a \in \mathbb{R}^3$ and $x, d \in \mathbb{R}^3$ with $d \neq 0$. Then*

$$\delta_a(x,d) = \left\| x + \left(\frac{d^T(a-x)}{d^Td}\right)\cdot d - a \right\|_2 = \sqrt{\|x-a\|_2^2 - \frac{\left(d^T(a-x)\right)^2}{d^Td}}. \quad (1)$$

Proof. Define the scalar function

$$g(t) := \|x + td - a\|_2^2.$$

Note that g is differentiable, strictly convex, and that $g'(t^*) = 0$ for

$$t^* = \frac{d^T(a-x)}{d^Td}.$$

Hence, t^* minimizes g and we obtain $\delta_a(x,d) = \sqrt{g(t^*)}$. Furthermore, easy calculations lead to

$$((x-a) + t^*d)^T((x-a) + t^*d) = \|x-a\|_2^2 - \frac{\left(d^T(a-x)\right)^2}{d^Td},$$

which proves the claim. □

In the remainder of this paper our goal is to locate a line $r = r(x,d)$ in the three-dimensional Euclidean space which minimizes the sum of distances between r and a given set of data points.

To this end, let $A = \{a_1, \ldots, a_n\} \subset \mathbb{R}^3$ be a set of data points. Then we consider the *median line problem*

$$\min_{\substack{x,d\in\mathbb{R}^3\\ d\neq 0}} \sum_{k=1}^{n} \delta_{a_k}(x,d) = \min_{\substack{x,d\in\mathbb{R}^3\\ d\neq 0}} \sum_{k=1}^{n} \sqrt{\|x-a_k\|_2^2 - \frac{\left(d^T(a_k-x)\right)^2}{d^Td}}. \quad (2)$$

2.1. Properties

Obviously, the line $r(x,d)$ is not uniquely defined by the pair (x,d). Indeed, $r(x,d) = r(x + vd, d)$ for any $v \in \mathbb{R}$. Hence, we can assume without loss of generality that x is the intersection of r with the hyperplane

$$H_d = \{y \in \mathbb{R}^3 : d^Ty = 0\}. \quad (3)$$

Lemma 1 directly leads to the following corollary.

Corollary 2. *For any $a \in \mathbb{R}^3$ and $x, d \in \mathbb{R}^3$ with $d \neq 0$ and $d^Tx = 0$ we have*

$$\delta_a(x,d) = \left\| x + \left(\frac{d^Ta}{d^Td}\right)\cdot d - a \right\|_2 = \sqrt{\|x-a\|_2^2 - \frac{\left(d^Ta\right)^2}{d^Td}}. \quad (4)$$

Next, let us consider the median line problem with fixed direction $d \in \mathbb{R}^3 \setminus \{0\}$ and the hyperplane H_d as defined in (3). We want to show that the median line problem with fixed d is equivalent to a planar Weber problem. This problem is to locate a point in the plane minimizing the sum of distances to a given set of demand points, see Drezner et al. (2001) for an overview. To this end, define the mapping

$$p_d : \mathbb{R}^3 \to H_d \quad \text{with } p_d(x) = x - \frac{d^Tx}{d^Td}\cdot d$$

and note that $p_d(x)$ is the projection of x onto H_d.

Lemma 3. *Consider a fixed direction $d \in \mathbb{R}^3 \setminus \{0\}$. Then*

$$\delta_a(x,d) = \|p_d(x) - p_d(a)\|_2$$

for all $x, a \in \mathbb{R}^3$.

Proof. One has

$$\|p_d(x) - p_d(a)\|_2 = \left\| \left(x - \frac{d^Tx}{d^Td}\cdot d\right) - \left(a - \frac{d^Ta}{d^Td}\cdot d\right) \right\|_2 = \left\| x + \left(\frac{d^T(a-x)}{d^Td}\right)\cdot d - a \right\|_2 = \delta_a(x,d),$$

due to Lemma 1, see Eq. (1). □

We remark that the same result for the special case of a vertical line, i.e. for $d = (0,0,1)$, can also be found in Brimberg et al. (2002). Moreover, Lemma 3 directly leads to the following corollary which is a special case of the results in Martini (1994).

Corollary 4. *The (three-dimensional) median line problem with fixed direction $d \in \mathbb{R}^3 \setminus \{0\}$ is equivalent to a (two-dimensional) Weber problem.*

To be more precise, for any $d \in \mathbb{R}^3 \setminus \{0\}$ one has

$$\min_{x\in\mathbb{R}^3} \sum_{k=1}^{n} \delta_{a_k}(x,d) = \min_{x\in\mathbb{R}^3} \sum_{k=1}^{n} \|p_d(x) - p_d(a_k)\|_2 = \min_{x\in H_d} \sum_{k=1}^{n} \|x - p_d(a_k)\|_2. \quad (5)$$

The following basic property will be important in order to restrict our search to a compact set.

Corollary 5. *There exists an optimal solution $(x^*, d^*) \in \mathbb{R}^6$ to the median line problem such that the line $r = r(x^*, d^*)$ intersects the convex hull of A.*

Proof. Recall that for any fixed $d \in \mathbb{R}^3 \setminus \{0\}$ the median line problem is equivalent to a planar Weber problem, see Corollary 4.

Moreover, it is well-known that there exists an optimal solution x^* to the Weber problem which intersects the convex hull of the (projected) demand points

$$A_d = \{p_d(a_1), \ldots, p_d(a_n)\},$$

see e.g. Drezner et al. (2001), i.e. x^* is the median of $p_d(a_1), \ldots, p_d(a_n) \in H_d$.

Hence, for any fixed $d \in \mathbb{R}^3 \setminus \{0\}$ there exists a $x^* \in H_d$ such that

$$\min_{x\in H_d} \sum_{k=1}^{n} \|x - p_d(a_k)\|_2 = \sum_{k=1}^{n} \|x^* - p_d(a_k)\|_2 = \sum_{k=1}^{n} \|p(x^*) - p_d(a_k)\|_2 = \sum_{k=1}^{n} \delta_{a_k}(x^*, d) = \min_{x\in\mathbb{R}^3} \sum_{k=1}^{n} \delta_{a_k}(x,d),$$

see Eq. (5). To sum up, it exists an optimal line $r = (x^*, d)$ with fixed direction d which intersects the convex hull of A. Since this is true for any $d \in \mathbb{R}^3 \setminus \{0\}$, the statement is shown. □

2.2. Problem parameterization

The six-dimensional problem, i.e. finding $x \in \mathbb{R}^3$ and $d \in \mathbb{R}^3 \setminus \{0\}$, can be reduced to a four-dimensional problem in many ways. In the following we present the parameterization which turns out to be the most efficient one for the solution algorithm proposed in the following sections.

First, we have that $r(x,d) = r(x,\tau d)$ for any $\tau \neq 0$. Thus, we can also assume without loss of generality that $\|d\|_\infty = 1$. Hence, we can parameterize any line $r = r(x,d)$ by its associated pair (x,d) with $\|d\|_\infty = 1$ and $d^Tx = 0$. Moreover, since $r(x,d) = r(x,-d)$, we can assume that

$$\max_{i=1,2,3} |d_i| = \max_{i=1,2,3} d_i = 1. \tag{6}$$

Let $d = (d_1,d_2,d_3) \in \mathbb{R}^3$ satisfying (6) and let us first assume that $d_3 = 1$ is fixed. We only need to consider $x = (x_1,x_2,x_3) \in \mathbb{R}^3$ such that $d^Tx = 0$ as discussed at the beginning of this section. If we do so, we easily obtain

$$x_3 = -(x_1 d_1 + x_2 d_2).$$

With $a_k = (\alpha_k, \beta_k, \gamma_k)$ for $k = 1,\ldots,n$ and making use of Corollary 2, we obtain the objective function (in the case that $d_3 = 1$)

$$f_3(x_1,x_2,d_1,d_2) := \frac{1}{\sqrt{d_1^2+d_2^2+1}} \cdot \sum_{k=1}^n \sqrt{g_3^k(x_1,x_2,d_1,d_2)},$$

where

$$\begin{aligned} g_3^k(x_1,x_2,d_1,d_2) :=& \left((x_1-\alpha_k)^2 + (x_2-\beta_k)^2 + (x_1d_1+x_2d_2+\gamma_k)^2\right) \\ &\cdot (d_1^2+d_2^2+1) - (d_1\alpha_k + d_2\beta_k + \gamma_k)^2. \end{aligned}$$

In the same way we can also fix $d_1 = 1$ and $d_2 = 1$ which yields (renaming the four remaining variables always as x_1, x_2, d_1, and d_2)

$$f_1(x_1,x_2,d_1,d_2) := \frac{1}{\sqrt{d_1^2+d_2^2+1}} \cdot \sum_{k=1}^n \sqrt{g_1^k(x_1,x_2,d_1,d_2)},$$

$$f_2(x_1,x_2,d_1,d_2) := \frac{1}{\sqrt{d_1^2+d_2^2+1}} \cdot \sum_{k=1}^n \sqrt{g_2^k(x_1,x_2,d_1,d_2)},$$

where

$$\begin{aligned} g_1^k(x_1,x_2,d_1,d_2) :=& \left((x_1d_1+x_2d_2+\alpha_k)^2 + (x_1-\beta_k)^2 + (x_2-\gamma_k)^2\right) \\ &\cdot (d_1^2+d_2^2+1) - (\alpha_k + d_1\beta_k + d_2\gamma_k)^2, \end{aligned}$$

$$\begin{aligned} g_2^k(x_1,x_2,d_1,d_2) :=& \left((x_1-\alpha_k)^2 + (x_1d_1+x_2d_2+\beta_k)^2 + (x_2-\gamma_k)^2\right) \\ &\cdot (d_1^2+d_2^2+1) - (d_1\alpha_k + \beta_k + d_2\gamma_k)^2. \end{aligned}$$

To sum up, the six-dimensional problem (2) is equivalent to the four-dimension problem

$$\min_{x_1,x_2,d_1,d_2\in\mathbb{R}} f(x_1,x_2,d_1,d_2) \tag{7}$$

with

$$\begin{aligned} f(x_1,x_2,d_1,d_2) := \min\{&f_1(x_1,x_2,d_1,d_2), f_2(x_1,x_2,d_1,d_2), \\ &f_3(x_1,x_2,d_1,d_2)\}. \end{aligned}$$

3. Geometric branch-and-bound algorithm

To solve the median line problem, we suggest a geometric branch-and-bound algorithm summarized below which is a popular solution technique for non-convex location problems. One of the first geometric branch-and-bound approaches in the area of facility location problems was suggested by Hansen et al. (1985), the *big square small square* technique for some facility location problems on the plane. Plastria (1992) generalized this method to the *generalized big square small square* technique. Using triangles instead of squares, Drezner and Suzuki (2004) proposed the *big triangle small triangle* method. Since all these techniques are branch-and-bound solution methods for problems with two variables, Schöbel and Scholz (2010a) suggested the *big cube small cube* technique for facility location problems with multiple variables.

In general, assume an objective function

$$f : X \to \mathbb{R},$$

where X is a box with sides parallel to the axes, i.e. a Cartesian product of intervals. Moreover, denote by $c(Y)$ the center of any subbox $Y \subset X$ and let $LB(Y)$ be a lower bound for Y, i.e.

$$LB(Y) \leqslant f(z) \quad \text{for all } z \in Y.$$

Then, under certain assumptions on f and the bounding procedure, the following algorithm finds a global minimum of f up to any absolute accuracy of $\varepsilon > 0$, see e.g. Tuy (1998) or Schöbel and Scholz (2010a).

(1) Calculate a lower bound $LB(X)$ and set $UB = f(c(X))$ and $\mathcal{X} = \{X\}$.
(2) Choose a box with the lowest lower bound in $\mathcal{X}$, split it into s congruent smaller boxes $Y_1,\ldots,Y_s$, delete the selected box from $\mathcal{X}$, and add $Y_1,\ldots,Y_s$ to $\mathcal{X}$. Calculate lower bounds $LB(Y_1),\ldots,LB(Y_s)$ and update

$$UB = \min\{UB, f(c(Y_1)),\ldots,f(c(Y_s))\}.$$

Delete all boxes Y from $\mathcal{X}$ with $LB(Y) + \varepsilon \geqslant UB$.
(3) When there are no boxes left, i.e. $\mathcal{X} = \emptyset$, the algorithm terminates and UB is within the absolute accuracy of ε from the global minimum. If there are boxes left, return to step (2).

Before we can apply this geometric branch-and-bound technique to the median line problem, we have to discuss some more details. Note that we consider the four-dimensional parameterization as defined in Eq. (7).

Some lower bounds can be found in the following section. Moreover, we have to ensure that the initial box X contains at least one optimal solution.

Theorem 6. *Without loss of generality assume that $A \subset [-1,1]^3$. Then the initial box*

$$X = [-\sqrt{3},\sqrt{3}] \times [-\sqrt{3},\sqrt{3}] \times [-1,1] \times [-1,1]$$

contains at least one optimal solution to the median line problem using the four-dimensional parameterization given in (7).

Proof. Let $r(x,d)$ be an optimal solution to the median line problem with $x = (x_1,x_2,x_3)$ and $d = (d_1,d_2,d_3)$ such that $d^Tx = 0$. According to Corollary 5 we can further assume that $r(x,d)$ intersects the convex hull of the demand points.

(1) Choose $s \in \{1,2,3\}$ such that $d_s = \max\{|d_1|,|d_2|,|d_3|\}$ and define

$$\tilde{d} = (\tilde{d}_1,\tilde{d}_2,\tilde{d}_3) = \frac{1}{d_s} \cdot (d_1,d_2,d_3).$$

We obtain $\tilde{d}_s = 1$ and $|\tilde{d}_i| \leqslant 1$ for $i = 1, 2, 3$. Since $r(x,d)$ and $r(x,\tilde{d})$ represent the same line, we have shown that there is an optimal solution (x_1,x_2,d_1,d_2) to the median line problem using the parameterization (7) such that $d_1, d_2 \in [-1,1]$.

(2) Next, assume that $x_1 \notin [-\sqrt{3}, \sqrt{3}]$ or $x_2 \notin [-\sqrt{3}, \sqrt{3}]$. We know that $d^T x = 0$. Hence, by Corollary 2, the Euclidean distance from $0 \in \mathbb{R}^3$ to the line $r(x,d)$ is given by

$$\delta_0(x,d) = \|x\|_2 = \sqrt{x_1^2 + x_2^2 + (x_1 d_1 + x_2 d_2)^2} > \sqrt{3}.$$

However, since $\|a\|_2 \leqslant \sqrt{3}$ for all $a \in [-1,1]^3$, the line $r(x,d)$ does not intersect the convex hull of the demand points, a contradiction. □

4. Calculating lower bounds

Before we present lower bounds for the median line problem, we recall some general concepts for the calculation of lower bounds.

4.1. Natural interval extension

We assume that the reader is familiar with interval analysis, see Hansen (1992) or Ratschek and Rokne (1988), which leads to simple but in general not very sharp lower bounds. Applications of this bounding procedure to location problems can be found for example in Fernández et al. (2007), Fernández et al. (2006), and Tóth et al. (2009) where some competition location models were solved.

Let $g : \mathbb{R}^m \to \mathbb{R}$ be a function such that the natural interval extension exists. For any box $Y = X_1 \times \cdots \times X_m \subset \mathbb{R}^m$ we then obtain the lower bound

$$LB(Y) = G(Y)^L,$$

where $G(Y) = G(X_1,\ldots,X_m)$ is the natural interval extension of $g(x)$ and the superindex ^{L} denotes the left endpoint of the interval $G(Y)$.

For a second, more sophisticated lower bound, we will use the general bounding operation of order two as introduced in Schöbel and Scholz (2010b) which is summarized in the following subsection.

4.2. General bounding operation

Assume a differentiable function $g : \mathbb{R}^m \to \mathbb{R}$ and calculate some lower bounds on the partial derivatives using the natural interval extension, i.e. calculate the vector

$$L(Y) := (G_1(Y)^L, \ldots, G_m(Y)^L),$$

where $G_k(Y)$ is the natural interval extension of

$$g_k(x) := \frac{\partial g}{\partial x_k}(x) \quad \text{for } k = 1,\ldots,m.$$

Furthermore, let $\ell = \ell(Y) = (X_1^L, \ldots, X_m^L)$ be the left point of $Y = X_1 \times \cdots \times X_m \subset \mathbb{R}^m$ and define the linear function

$$m(x) := g(\ell) + L(Y)^T \cdot (x - \ell).$$

As shown in Schöbel and Scholz (2010b), we obtain $m(x) \leqslant g(x)$ for all $x \in Y$. Hence, we get the lower bound

$$LB(Y) = \min_{v \in V(Y)} m(v),$$

where $V(Y)$ it the set of the 2^m vertices of Y.

4.3. Lower bounds for the median line problem

Recall that for any subbox

$$Y = X_1 \times X_2 \times D_1 \times D_2 \subset \mathbb{R}^4,$$

we want to find a lower bound on the median line objective function

$$f(x_1,x_2,d_1,d_2) = \min\{f_1(x_1,x_2,d_1,d_2), f_2(x_1,x_2,d_1,d_2), f_3(x_1,x_2,d_1,d_2)\},$$

where

$$f_i(x_1,x_2,d_1,d_2) = \frac{1}{\sqrt{d_1^2 + d_2^2 + 1}} \cdot \sum_{k=1}^n \sqrt{g_i^k(x_1,x_2,d_1,d_2)}$$

for $i = 1, 2, 3$ as defined before.

One obtains a first lower bound for this problem using the natural interval extension, i.e.

$$LB_1(Y) := F(Y)^L, \tag{8}$$

where $F(Y) = F(X_1,X_2,D_1,D_2)$ is the natural interval extension of $f(x_1,x_2,d_1,d_2)$.

For a second lower bound, we make use of the general bounding operation as follows. Note that for $i = 1, 2, 3$ and $k = 1,\ldots,n$ the functions g_i^k are differentiable, define the linear function

$$m_i^k(x_1,x_2,d_1,d_2) := g_i^k(\ell) + L_i^k(Y)^T \cdot ((x_1,x_2,d_1,d_2) - \ell)$$

derived from the general bounding operation, and define

$$M_i^k(Y) := \min_{v \in V(Y)} m_i^k(v).$$

Using these definitions, we obtain the following result.

Lemma 7. *For $i = 1, 2, 3$ and $k = 1,\ldots,n$, the functions*

$$h_i^k(x_1,x_2,d_1,d_2) := \begin{cases} \sqrt{m_i^k(x_1,x_2,d_1,d_2)} & \text{if} \quad M_i^k(Y) \geqslant 0 \\ 0 & \text{if} \quad M_i^k(Y) < 0 \end{cases}$$

are concave in Y and satisfy

$$h_i^k(x_1,x_2,d_1,d_2) \leqslant \sqrt{g_i^k(x_1,x_2,d_1,d_2)}$$

for all $(x_1,x_2,d_1,d_2) \in Y$.

Proof. Obviously, 0 is a concave function. Next, if $M_i^k(Y) \geqslant 0$ then

$$m_i^k(x_1,x_2,d_1,d_2) \geqslant 0 \quad \text{for all} \quad (x_1,x_2,d_1,d_2) \in Y,$$

since m_i^k is linear. Moreover, since the scalar function $u(t) = \sqrt{t}$ is concave and monotone increasing for $t \geqslant 0$, we know that also

$$h_i^k(x_1,x_2,d_1,d_2) = u(m_i^k(x_1,x_2,d_1,d_2))$$

is concave. Finally, since

$$m_i^k(x_1,x_2,d_1,d_2) \leqslant g_i^k(x_1,x_2,d_1,d_2) \quad \text{for all } (x_1,x_2,d_1,d_2) \in Y$$

and since u is monotone increasing, we know that

$$0 \leqslant h_i^k(x_1,x_2,d_1,d_2) \leqslant \sqrt{g_i^k(x_1,x_2,d_1,d_2)},$$

which proves the claim. □

With the help of Lemma 7 we obtain the following second lower bound for the median line problem.

Theorem 8. *Define the functions*

$$h_i(x_1,x_2,d_1,d_2) := \frac{1}{\sqrt{d_1^2 + d_2^2 + 1}} \cdot \sum_{k=1}^n h_i^k(x_1,x_2,d_1,d_2)$$

for $i = 1, 2, 3$ and let

$$h(x_1,x_2,d_1,d_2) : = \min\{h_1(x_1,x_2,d_1,d_2), h_2(x_1,x_2,d_1,d_2), h_3(x_1,x_2,d_1,d_2)\}.$$

Then

$$LB_2(Y) := \min_{v \in V(Y)} h(v) \tag{9}$$

is a lower bound where V(Y) is the set of the 16 vertices of Y.

Proof. First of all define $q(x_1, x_2, d_1, d_2) := \sqrt{d_1^2 + d_2^2 + 1}$ and

$$s_i(x_1, x_2, d_1, d_2) := \sum_{k=1}^{n} h_i^k(x_1, x_2, d_1, d_2)$$

for $i = 1, 2, 3$. Then, q is a strictly positive and convex function and the functions s_i are positive and concave for $i = 1, 2, 3$ by Lemma 7. Hence, we conclude that

$$h_i(x_1, x_2, d_1, d_2) = \frac{s_i(x_1, x_2, d_1, d_2)}{q(x_1, x_2, d_1, d_2)}$$

are quasiconcave functions for $i = 1, 2, 3$, see e.g. Avriel et al. (1987). Moreover, since the minimum of quasiconcave functions is quasiconcave again, h is quasiconcave on Y and we therefore obtain

$$\min_{x \in Y} h(x) = \min_{v \in V(Y)} h(v).$$

Lemma 7 furthermore states that

$$h(x_1, x_2, d_1, d_2) \leqslant f(x_1, x_2, d_1, d_2) \quad \text{for all} \quad (x_1, x_2, d_1, d_2) \in Y$$

and the theorem is shown. □

5. Numerical results

In this section we present some numerical experiences solving the median line problem. To this end, we employed the geometric branch-and-bound technique as well as the lower bounds presented in the previous sections.

We randomly generated some demand points $a_k \in \{-1.0, -0.9, \ldots, 0.9, 1.0\}^3$ and all selected boxes were split into $s = 2$ congruent small subboxes, i.e. all selected boxes were bisect perpendicular to the direction of the maximum width component, see Section 3.

Furthermore, in our algorithm we used three initial boxes as follows. We started with $\mathcal{X} = \{X_1, X_2, X_3\}$ where

$$X_i = [-1.74, 1.74] \times [-1.74, 1.74] \times [-1, 1] \times [-1, 1],$$

see Theorem 6, and each box X_i for $i = 1, 2, 3$ was only assigned to the function f_i.

Our code was written in Fortran, compiled by Intel Visual Fortran Compiler Professional 11.1.051, and ran on a 2.67 GHz computer with 8 GB of memory under Windows 7. In the following, we present three different studies.

5.1. Randomly input data

For various values of n, we solved 10 problem instances with randomly generated input data as given above and $\varepsilon = 10^{-6}$. As lower bound, we used the maximum of the lower bounds $LB_1(Y)$ and $LB_2(Y)$ as suggested in Section 4, i.e. we calculated

$$LB_3(Y) := \max\{LB_1(Y), LB_2(Y)\}$$

for all subboxes $Y \subset X$.

Our results are illustrated in Table 1. Therein, the minimum, maximum, and average run times as well as iterations throughout the branch-and-bound algorithm are reported. Moreover, Fig. 1 shows the run times for all solved problem instances.

As can be seen, all problem instances with up to $n = 100$ demand points could be solved in less than a few minutes of computing time. However, it should be mentioned that the standard deviation in the run times is quite high. For example, although nine out of ten problem instances with $n = 5$ demand points were solved in less than 2 s, there was one instance with a run time of 14.91 s. Similar observations can also be found for other values of n.

5.2. Comparison of lower bounds

In this subsection our aim is to compare the suggested lower bounds. To this end, we consider problem instances with $n = 5$ demand points which were solved twice. In the first run, we made use of the lower bound LB_1, i.e. of the natural interval extension. In the second run, we employed the lower bound LB_2. Table 2 presents the run times as well as the number of iterations throughout the algorithm for 20 randomly generated problem instances and $\varepsilon = 10^{-1}$.

Furthermore, we remark that we could not solve any instances for some smaller values of ε. Using e.g. $\varepsilon = 10^{-2}$, the lower bound LB_2 yields almost the same results as presented in Table 2. But no instance could be solved with $\varepsilon = 10^{-2}$ and LB_1 since the list of boxes filled up with our limit of 24,000,000 boxes without convergence.

To sum up, our results demonstrate unequivocally that the natural interval extension alone does not yield sharp lower bounds such that LB_1 should not to be used throughout the algorithm. Hence, only the suggested second lower bound makes it possible to solve the median line problem in an efficient way.

5.3. Solving one particular problem instance

Finally, we present a particular problem instance with $n = 50$ demand points. Using the data given in Table 3 and $\varepsilon = 10^{-6}$ again, we obtained after 1,223,403 iterations and a run time of 47.62 s the optimal line

$$r = r(x^*, d^*) = \begin{pmatrix} 1.087929 \\ 1.106126 \\ 1.129687 \end{pmatrix} + t \cdot \begin{pmatrix} -0.980392 \\ 1.000000 \\ -0.153610 \end{pmatrix}$$

with an objective value of 36.893231, see Fig. 2.

6. Discussion

In this paper, we studied the median line problem in three dimensions. Some theoretical results as well as a specific

Table 1
Numerical results for the median line problem with randomly generated input data and $\varepsilon = 10^{-6}$.

n	Run time (sec.)			Iterations		
	Min	Max	Ave.	Min	Max	Ave.
5	0.39	14.91	2.21	96,784	3,274,910	486,212.6
10	1.54	56.05	20.75	185,437	6,649,657	2,498,832.4
15	2.25	32.35	12.25	185,568	2,589,609	1,009,505.1
20	3.18	61.31	22.46	198,271	3,788,279	1,415,919.5
25	3.67	28.80	15.47	184,857	1,442,695	777,971.8
30	5.73	30.73	15.60	248,481	1,302,031	663,204.2
35	11.95	83.57	35.28	438,693	3,009,059	1,277,984.5
40	9.31	49.75	30.14	298,652	1,578,879	977,898.0
45	16.33	124.96	34.71	465,717	3,741,455	1,018,784.2
50	13.23	78.89	34.72	346,308	2,045,602	899,292.3
55	15.83	80.65	37.14	376,639	1,874,345	873,764.9
60	20.14	83.57	41.23	440,004	1,869,197	910,500.1
65	19.61	80.39	46.97	393,906	1,627,484	943,908.1
70	17.67	81.90	44.56	330,381	1,535,738	833,716.2
75	22.99	67.27	43.91	405,202	1,185,531	768,827.1
80	37.02	111.06	68.90	603,655	1,872,381	1,133,722.3
85	19.39	92.04	54.66	297,282	1,411,662	836,951.5
90	33.32	161.76	75.43	498,420	2,342,423	1,107,803.5
95	39.70	154.27	78.53	549,138	2,162,391	1,096,360.5
100	25.68	192.65	76.62	336,837	2,481,556	999,837.8

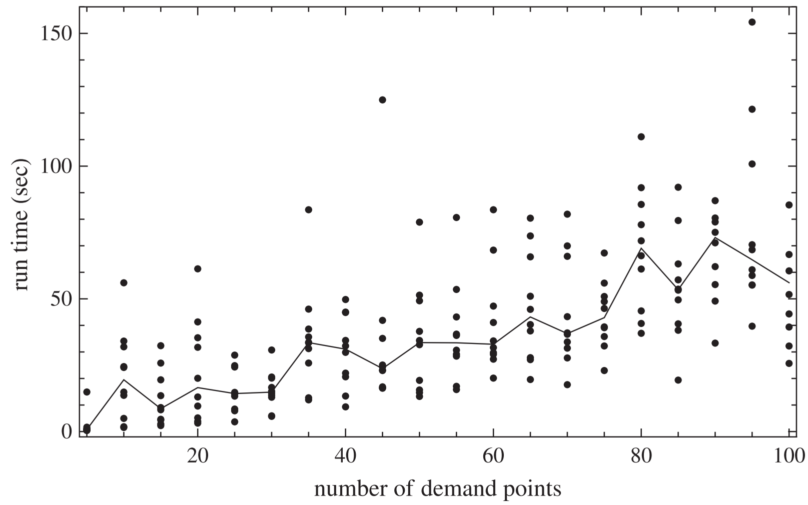

Fig. 1. Run times for all problem instances of the median line problem with randomly generated input data and $\varepsilon = 10^{-6}$. The line represents the median of these values.

Table 2
Numerical results for the comparison of the lower bounds.

	Run time (sec.)			Iterations		
	Min	Max	Ave.	Min	Max	Ave.
LB_1	2.79	64.37	17.19	1,025,080	20,538,265	5,827,158
LB_2	0.17	0.55	0.39	45,145	110,137	84,100

Table 3
Input data $A = \{a_1, \ldots, a_{50}\}$ for the particular problem instance discussed in Section 5.3.

(1.6,0.2,0.0)	(0.5,0.4,1.0)	(0.3,1.8,1.8)	(0.7,1.4,1.5)	(1.5,1.8,0.7)
(0.8,2.0,1.2)	(2.0,1.8,0.0)	(1.3,0.6,0.5)	(1.7,0.1,1.6)	(0.4,1.4,0.2)
(1.4,1.2,0.1)	(1.7,0.3,1.2)	(0.7,2.0,1.1)	(0.8,1.2,0.8)	(1.6,1.7,0.8)
(0.1,1.5,0.2)	(1.9,0.6,1.6)	(1.9,0.9,1.0)	(2.0,0.2,0.1)	(2.0,0.6,1.2)
(0.0,0.4,0.8)	(1.6,1.0,0.8)	(0.7,1.0,2.0)	(1.7,0.1,1.9)	(0.3,1.5,1.1)
(1.0,1.9,1.4)	(0.5,1.5,0.9)	(0.4,0.7,1.1)	(0.8,0.9,2.0)	(1.9,0.2,1.6)
(0.8,1.3,1.4)	(1.8,1.8,0.6)	(1.5,1.1,1.6)	(0.3,0.9,2.0)	(0.8,0.1,2.0)
(0.8,1.1,0.3)	(2.0,1.8,1.6)	(1.6,1.5,0.8)	(0.2,2.0,1.2)	(1.2,1.6,0.7)
(1.8,1.4,1.8)	(0.1,1.2,1.1)	(1.1,0.3,0.6)	(1.9,1.4,0.3)	(0.0,0.9,0.1)
(0.7,1.5,1.1)	(1.5,1.2,1.6)	(1.6,0.0,1.3)	(1.3,1.7,1.3)	(0.5,0.0,0.3)

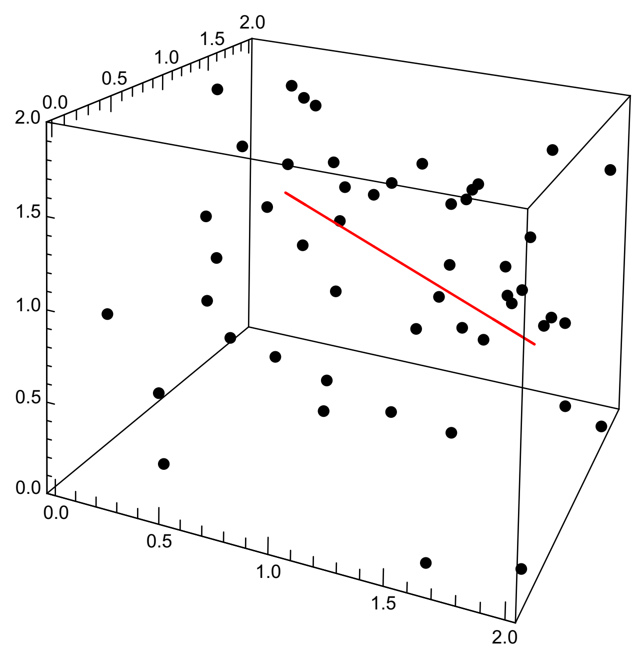

Fig. 2. Optimal line for the particular problem instance discussed in Section 5.3.

four-dimensional problem parameterization were discussed and a geometric branch-and-bound method as solution procedure was suggested. To be more precise, we derived some lower bounds as well as an initial box which contains at least one optimal solution. In the numerical results reported, it was shown that we succeeded in solving medium-size problem instances. Although we only solved the unweighted median line problem, note that the problem parameterization as well as the proposed lower bounds are still valid for weighted demand points with non-negative weights.

Furthermore, we only considered the median line problem for the Euclidean norm. It is further research to investigate some general distance functions. The main task here is to derive a closed formula for other distance functions similar to the formula (1) for the Euclidean case.

We remark that other parameterizations of the median line problem are possible, e.g. spherical coordinates as suggested in Blanquero et al. (2009). We also implemented several other lower bounds using e.g. techniques from d.c. programming, the centered interval bounding operation, or making use of bound procedures similar to those ones given in Blanquero and Carrizosa (2009) and Schöbel and Scholz (2010b). However, all other parameterizations as well as all other lower bounds we tried were worse compared to the parameterization as given in Section 2 and the lower bounds presented in Section 4.

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