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Discrete Optimization

A fractional model for locating semi-desirable facilities on networks ☆

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Abstract

In this paper, we address the problem of locating a series of facilities on a network maximizing the average distance to population centers (assumed to be distributed in the plane) per unit transportation cost (a function of the network distances to users). A finite dominating set is constructed, allowing the resolution of the problem by standard integer programming techniques. We also discuss some extensions of the model (including, in particular, the Weber problem with attraction and repulsion in networks), for which (ε-) dominating sets are derived. © 2002 Elsevier Science B.V. All rights reserved.

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1. The model

Locational analysis studies were to locate one or several facilities which will interact with some individuals. The classical models assume that interactions are positive, in that all affected individuals want the facility as close as possible, and then the *transportation cost* minimization is the main concern. Transportation is associated with a network, thus the feasible region is usually modeled as a (planar) network, and transportation costs are assumed to be proportional to network distances, e.g., [13,14,31].

In the last decade, there has been an increasing interest in considering, together with these transportation costs, the *environmental impact* caused by the facilities over individuals such as population centers, landfill, waters and the like. These individuals, also interacting with the facilities, see such interactions as negative, and want the facilities as far as possible. Since these negative effects usually spread throughout the plane, most papers assume the feasible region to be a (non-degenerate) subset of the plane, and consider the

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negative effects to be a function of the *planar* distance from the facility to the affected individuals [12,15,19,25].

When both transportation costs and environmental impact are present, the facilities, called then *semi-desirable* or *semi-obnoxious* [26] should be located at points optimizing a certain compromise measure between the two above-mentioned criteria (see [4] for a review on semi-desirable facility location models). Most existing papers address fully planar-based models (both transportation costs and environmental impact are functions of planar distances) [5,10,22] or fully network-based models (both transportation costs and environmental impact are functions of network distances) [8], and only recently both network and planar models are being combined, e.g., [30].

The model addressed in this paper, described in what follows, is of this latter type. A set of p semi-desirable facilities, $F_1, F_2, \ldots, F_p$, is to be located at points $\hat{x}^k$, $k \in K = \{1, \ldots, p\}$ within a given network $G := (N, E)$ embedded in the plane, whose edges are straight-line segments, and equipped with a metric d_G defined in the usual way, e.g., [18].

Such facilities must cover the demand of a set A of users, identified with a subset of nodes of the network (thus points of the Euclidean plane). The total transportation cost $\mathrm{TC}(\hat{x}^1, \ldots, \hat{x}^p)$ incurred, if the facilities are located at $\hat{x}^1, \ldots, \hat{x}^p$, is given by the optimal value of a linear program, namely,

$$\begin{aligned} \mathrm{TC}\left(\hat{x}^1, \ldots, \hat{x}^p\right) = \min \quad & \sum_{a \in A, k \in K} \omega_a d_G\left(a, \hat{x}^k\right) \hat{y}_{ak} \\ \text{s.t.} \quad & \sum_{k \in K} \hat{y}_{ak} = 1 \quad \forall a \in A, \\ & \sum_{a \in A} \omega_a \hat{y}_{ak} \leqslant c_k \quad \forall k \in K, \\ & \hat{y}_{ak} \geqslant 0 \quad \forall a \in A, \ k \in K, \end{aligned} \tag{1}$$

where ω_a is the demand (which must be fulfilled) of user at $a \in A$; $\hat{y}_{ak}$ is the allocation variable giving the fraction of demand ω_a shipped from F_k to user at a; c_k is the capacity of plant F_k, assumed to be big enough to make the problem feasible, i.e., $\sum_{a \in A} \omega_a \leqslant \sum_{k \in K} c_k$.

It is assumed hereafter that the cardinality of A is strictly greater than p so that TC is positive at any feasible solution.

Since the facilities are assumed to be semi-desirable, they also negatively affect some individuals. We model the set of negatively affected individuals (also called in what follows as *repelling points*) by a finite subset R of points in the plane. These points want the facilities as far as possible, then we aggregate the individual utilities into an environmental utility EU as follows:

$$\mathrm{EU}\left(\hat{x}^1, \ldots, \hat{x}^p\right) = \sum_{r \in R} u_r \left\{ \sum_{k \in K} \Phi_k\left(\left\| r - \hat{x}^k \right\|_r \right) \right\}, \tag{2}$$

where $\hat{x}^1, \ldots, \hat{x}^p$ are the locations of the facilities; u_r is the weight (fraction of population or strength) of affected individual $r \in R$; we assume without loss of generality that $\sum_{r \in R} u_r = 1$; $\| \cdot \|_r$ is an arbitrary finite gauge [11,24], such as the Euclidean norm $\| \cdot \|_2$, or any skewed gauge $\|x\| = \|x\|_2 - a'x$, as introduced by [23]; $\Phi_k : [0, +\infty) \to [0, +\infty)$, is assumed to be upper-semi-continuous and non-decreasing, and $\Phi_k(\|r - \hat{x}^k\|_r)$ gives the utility for individuals r in R if facility F_k is located at $\hat{x}^k$.

Remark 1.1. Oldest models assumed each Φ_k to be affine [10], whereas in more recent papers, non-linear decay effects are modeled through, e.g., exponential functions, $\Phi_k(t) = 1 - \exp(-\beta_k t)$ [7,32]. Above, the functions Φ_k are not even obliged to be continuous, thus our model also accommodates *coverage* problems, e.g., [30], in which a reward π_k exists if (and only if) the distance from facility F_k to a user is greater than a threshold value δ_k: Just define Φ_k as the lower-semi-continuous non-increasing function (see Fig. 1)

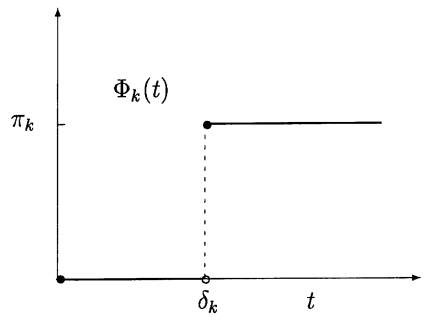

Fig. 1. Coverage function $\Phi_k(t)$.

$$\Phi_k(t) = \begin{cases} 0 & \text{if } t < \delta_k, \\ \pi_k & \text{otherwise.} \end{cases}$$

TC (to be minimized) and EU (to be maximized) are conflicting objectives, aggregated into a single objective by means of an aggregating function AF (to be maximized), which should be non-increasing in its first argument and non-decreasing in the second. This gives rise to the optimization problem

$$\max_{\hat{x}^1,\ldots,\hat{x}^p \in G} \mathrm{AF}\Big(\mathrm{TC}\Big(\hat{x}^1,\ldots,\hat{x}^k\Big),\ \mathrm{EU}\Big(\hat{x}^1,\ldots,\hat{x}^p\Big)\Big). \tag{3}$$

In the existing literature, AF is considered to be linear, $\mathrm{AF}(\mathrm{TC},\mathrm{EU}) = \mathrm{EU} - \mathrm{TC}$ [4,7,10,32]. However, this is not the only possible choice as detailed in Section 3.

The remaining of the paper is structured as follows. In Section 2, we derive a localization result for problem (3) for a quite general AF and piecewise linear function Φ_k. In Section 3, we particularize AF to represent the environmental utility per unit transportation cost. The localization result of the preceding section will enable us to solve (3) by means of integer programming techniques. An illustrative example is presented in Section 4. The paper concludes discussing some extensions to models with non-affine Φ_k or non-fractional utilities in Section 5.

2. The general case – localization results

In this section, we address problem (3) under the extra assumption that each function Φ_k is piecewise linear. In practice, this should be considered as almost the general case, since arbitrary Φ_k functions can be approximated with arbitrary accuracy by piecewise linear (even piecewise constant) functions, see also Section 5. This class of functions may also be especially useful in the (realistic) case in which the functional form of Φ_k is not known, but obtained by interpolating a finite set of values. This is the case of the well-known procedure called the midpoint method, which, in words of Chankong and Haimes, is probably most commonly used for continuous attributes, see [6, p. 186].

To solve (3) global optimization techniques are required due to the fact that functions $\Phi_k(\|r - x^k\|_r)$ are in general neither concave nor convex on each rectilinear edge of G.

Since its origins with the seminal work of Hakimi [13,14], a standard strategy in coping with location problems on networks has consisted of reducing the set of candidate locations to a finite set, called in [16] a *finite dominating set*, see also [21,31] and the references therein.

For each edge $e \in E$, $r \in R$ and s, t, with $0 \leqslant s \leqslant t$, the set

$$\left\{x \in \mathbb{R}^2 : \ s \leqslant \|r - x\|_r \leqslant t\right\} \cap E$$

is either empty or consists of at most two (eventually degenerate) closed segments. Let $B(r, e; s, t)$ denote the (eventually empty) class of endpoints of such segments.

We state the result for a problem more general than (9) since utility functions other than the fractional AF given in (8) are allowed.

Proposition 2.1. *Let* AF: $(0, +\infty) \times (0, +\infty) \to \mathbb{R}$ *be non-increasing in its first argument and non-decreasing in the second, and quasiconvex function. Suppose that, for each* k, $1 \leqslant k \leqslant p$, *the function* Φ_k *in* (2) *is also piecewise linear with breakpoints at* $t_{k_1} < t_{k_2} < \cdots < t_{k_{n_k}}$. *Then there exists an optimal solution for* (3) *with each facility* F_k *at* B_k,

$$B_k = N \cup \bigcup_{\substack{e \in E, r \in R \\ 1 \leqslant j \leqslant n_k - 1}} B(r, e; t_{k_j}, t_{k_{j+1}}). \tag{4}$$

Proof. Let $\hat{x} = (\hat{x}^1, \ldots, \hat{x}^p)$ be a feasible solution. If $\hat{x}^i \in B_i$, we are done. Else, let $\hat{x}^i \notin B_i$. Then there exist two points in B_i, endpoints of a closed subedge l of E, such that $\hat{x}^i$ (but no point in B_i) is in the relative interior of l.

Parameterize l by the interval $[0, 1]$, in such a way that $x(t)$ represents the point in l at distance fraction t of one of its endpoints.

Since Φ_i is upper semi-continuous and non-decreasing, it follows that

$$\lim_{t \to 0^+} \Phi_i(\|r - x(t)\|_r) \leqslant \Phi_i(\|r - x(0)\|_r), \tag{5}$$

$$\lim_{t \to 1^-} \Phi_i(\|r - x(t)\|_r) \leqslant \Phi_i(\|r - x(1)\|_r). \tag{6}$$

Moreover, the function $t \in [0, 1] \to \Phi_i(\|r - x(t)\|_r)$ is convex on $(0, 1)$, thus by Eqs. (5) and (6), it is also convex on the closed interval $[0, 1]$. Hence, for $\hat{x}^1, \ldots, \hat{x}^{i-1}, \hat{x}^{i+1}, \ldots, \hat{x}^p$ fixed, the function

$$t \in [0, 1] \to \mathrm{EU}\left(\hat{x}^1, \ldots, \hat{x}^{i-1}, x(t), \hat{x}^{i+1}, \ldots, \hat{x}^p\right)$$

is convex on $[0, 1]$. On the other hand, the function

$$t \in [0, 1] \mapsto \mathrm{TC}\left(\hat{x}^1, \ldots, \hat{x}^{i-1}, x(t), \hat{x}^{i+1}, \ldots, \hat{x}^p\right)$$

is known to be concave and continuous on $[0, 1]$, thus due to the quasiconvexity of AF and its monotonic character, the function

$$t \to \mathrm{AF}\left(\mathrm{TC}\left(\hat{x}^1, \ldots, \hat{x}^{i-1}, x(t), \hat{x}^{i+1}, \ldots, \hat{x}^p\right),\ \mathrm{EU}\left(\hat{x}^1, \ldots, \hat{x}^{i-1}, x(t), \hat{x}^{i+1}, \ldots, \hat{x}^p\right)\right)$$

is quasiconvex on $[0,1]$ [2], thus attains its maximum at an endpoint of $[0,1]$. Hence, we have found another feasible solution with better or equal objective value and its ith component in B_i. By iteration, the results hold. □

Remark 2.1. The assumption that edges are straight-line segments has been used in Proposition 2.1 to show the (piecewise) quasiconvexity of the objective function of our model on each edge.

If the real-world transportation network does not have rectilinear edges, they should be approximated by an open polygonal curve, thus adding its breakpoints to the set of nodes.

In particular, when each Φ_k is linear, $\Phi_k(s) = \alpha_k - \beta_k s$, with $\beta_k \geqslant 0$, no breakpoints exist, thus each set B_k in (4) reduces to the set N of nodes of the network.

This yields:

Corollary 2.1. *Under the assumptions of Proposition 2.1, if each Φ_k is affine and non-decreasing, then there exists an optimal solution for* (3) *with all the facilities located at nodes of G.*

Setting AF as

$$\mathrm{AF}(\mathrm{TC}, \mathrm{EU}) = \mathrm{EU} - \mathrm{TC}, \tag{7}$$

problem (3) becomes the (generalized) p-median problem with attraction and repulsion, extensively considered in the planar case (i.e., when transportation distances are induced also by metrics in the plane instead of network distances) for $p = 1$ [10]. Since such function AF also satisfies the assumptions of Proposition 2.1, one obtains the following:

Corollary 2.2. *For the p-median problem with attraction and repulsion and affine functions Φ_k, an optimal solution exists with its p facilities located at nodes of the network.*

3. The fractional case

Citing [28], the efficiency of a system is sometimes characterized by a ratio of technical and/or economical terms, such as maximization of productivity, of return of investment, of return/risk, of reward per unit time or maximization of the output per input unit, see, e.g., [28,29] and references therein.

In this section, we propose to take AF as the environmental utility per unit transportation cost, i.e.,

$$\mathrm{AF}(\mathrm{TC}, \mathrm{EU}) = \frac{\mathrm{EU}}{\mathrm{TC}}, \tag{8}$$

that is

$$\mathrm{AF}\left(\mathrm{TC}\left(\hat{x}^1, \ldots, \hat{x}^p\right),\ \mathrm{EU}\left(\hat{x}^1, \ldots, \hat{x}^p\right)\right)$$
$$= \frac{\sum_{r\in R,k\in K} u_r \Phi_k\left(\left\|r - \hat{x}^k\right\|_r\right)}{\min\left\{\sum_{a\in A,k\in K} \omega_a d_G\left(a, \hat{x}^k\right)\hat{y}_{ak}:\ \sum_{k\in K} \hat{y}_{ak} = 1,\ \sum_{a\in A} \omega_a \hat{y}_{ak} \leqslant c_k,\ \hat{y}_{ak} \geqslant 0\ \forall a, k\right\}}.$$

The resulting problem (3) is, in this case, a bilevel program equivalent to

$$
\begin{aligned}
\max \quad & \frac{\sum_{r\in R,k\in K} u_r \Phi_k\left(\left\|r-\hat{x}^k\right\|_r\right)}{\sum_{a\in A,k\in K} \omega_a d_G\left(a,\hat{x}^k\right)\hat{y}_{ak}} \\
\text{s.t.} \quad & \sum_{k\in K} \hat{y}_{ak} = 1 \quad \forall a \in A, \\
& \sum_{a\in A} \omega_a \hat{y}_{ak} \leqslant c_k \quad \forall k \in K, \\
& 0 \leqslant \hat{y}_{ak} \quad \forall a \in A,\ k \in K, \\
& \hat{x}^1,\ldots,\hat{x}^p \in G,
\end{aligned}
\tag{9}
$$

where the average environmental utility per unit transportation cost is maximized.

Proposition 2.1 proves the existence of a finite set $B_k = \{v_1,\ldots,v_n\}$ of candidate points containing an optimal solution for facility F_k according to problem (9). In this section, problem (9) is reformulated as an integer linear program using the following notation:

- For each $k \in K$ and $v \in B_k$, let x_{vk} be the Boolean variable defined as

$$
x_{vk} = \begin{cases} 1 & \text{if plant } F_k \text{ is located at } v, \\ 0 & \text{in other case.} \end{cases}
$$

- For each $a \in A$ and $v \in V = \bigcup_{k\in K} B_k$, let y_{av} represent the fraction of demand of consumer at $a \in A$ served by some plant located at $v \in V$.
- For each $v \in V$ and $k \in K$ define the constant q_{vk} as

$$
q_{vk} = \sum_{r\in R} u_r \Phi_k(\|r-v\|_r).
$$

Then (9) can be rewritten as the following capacitated p-facility discrete location model with a fractional objective

$$
\begin{aligned}
\max \quad & \frac{\sum_{k\in K,\, v\in B_k} q_{vk} x_{vk}}{\sum_{a\in A,v\in V} \omega_a d_G(a,v) y_{av}} \\
\text{s.t.} \quad & \sum_{v\in B_k} x_{vk} = 1 \quad \forall k \in K, \\
& \sum_{v\in V} y_{av} = 1 \quad \forall a \in A, \\
& \sum_{a\in A} \omega_a y_{av} \leqslant \sum_{k\in K} c_k x_{vk} \quad \forall v \in V, \\
& 0 \leqslant y_{av} \quad \forall a \in A,\ v \in V, \\
& x_{vk} \in \{0,1\} \quad \forall v \in B_k,\ k \in K.
\end{aligned}
\tag{P}
$$

The non-linear mixed integer program (P) will be transformed into an equivalent linear mixed integer problem by means of a change of variables similar to that proposed in [20]. This allows us to solve exactly at least small instances of (P) with standard optimization packages like spreadsheet solvers.

Define

$$\begin{aligned} t &= \frac{1}{\sum_{a\in A, v\in V} \omega_a d_G(a,v) y_{av}}, \\ \bar{x}_{vk} &= t\cdot x_{vk}, \quad v\in B_k,\ k\in K, \\ \bar{y}_{av} &= t\cdot y_{av}, \quad a\in A,\ v\in V. \end{aligned} \tag{10}$$

Define also the positive constant L as

$$L = \frac{1}{(|A|-p)\min\{w_a d_G(a,v):\ a\in A,\ v\in V\setminus A\}}, \tag{11}$$

where $|A|$ denotes the cardinality of the set A.

Remark 3.1. Constant $1/L$ is a lower bound of the denominator of the objective function of (P) in its feasible set. Indeed, since by assumption the number $|A|$ of attracting points is greater than p, there exist at least $(|A|-p)$ points that must fulfill their demand at strictly positive cost. Hence, the transportation cost is bounded by $(|A|-p)$ times the smallest positive transportation cost.

We now introduce a mixed integer linear problem (P_I), solvable by standard packages, which turns out to be equivalent to (P), as stated in the following result:

$$\begin{aligned} \max \quad & \sum_{k\in K, v\in B_k} q_{vk}\bar{x}_{vk} \\ \text{s.t.} \quad & \sum_{a\in A, v\in V} \omega_a d_G(a,v)\bar{y}_{av} = 1, \\ & \sum_{v\in V} \bar{x}_{vk} = t \quad \forall k\in K, \\ & \sum_{v\in V} \bar{y}_{av} = t \quad \forall a\in A, \\ & \sum_{a\in A} \omega_a \bar{y}_{av} \leqslant \sum_{k\in K} c_k \bar{x}_{vk} \quad \forall v\in V, \\ & \bar{x}_{vk} \leqslant L - L\bar{z}_{vk} \quad \forall v\in B_k,\ k\in K, \\ & \bar{x}_{vk} \geqslant t - L\bar{z}_{vk} \quad \forall v\in B_k,\ k\in K, \\ & \bar{z}_{vk} \in \{0,1\} \quad \forall v\in B_k,\ k\in K, \\ & \bar{x}_{vk} \geqslant 0 \quad \forall v\in B_k,\ k\in K, \\ & \bar{y}_{av} \geqslant 0 \quad \forall a\in A,\ v\in V, \\ & t\geqslant 0. \end{aligned} \tag{P_I}$$

Proposition 3.1. *Problems* (P) *and* (P_I) *are equivalent.*

Proof. It is immediate to see that if (x,y) is feasible to (P), then $(\bar{x},\bar{y},\bar{z},t) = (tx, ty, \mathbf{1}-x, t)$ is feasible to (P_I) with the same objective-function value, where $\mathbf{1}$ is a vector of 1's and t is given by (10).

To show the converse note that the blocks of constraints

$$\bar{x}_{vk} \leqslant L - L\bar{z}_{vk} \quad \forall v,k \tag{12}$$

and

$$\bar{x}_{vk} \geqslant t - L\bar{z}_{vk} \quad \forall v, k \tag{13}$$

force every feasible solution of (P_I) to have $\bar{x}_{vk} \in \{0, t\}$ according to $\bar{z}_{vk} = 0$ or $\bar{z}_{vk} = 1$. Indeed, we distinguish two cases:

- If $\bar{z}_{vk} = 0$, then (12) and (13) imply that $t \leqslant \bar{x}_{vk} \leqslant L$, thus, since $\sum_{v \in V} \bar{x}_{vk} = t \ \forall k$, then $\bar{x}_{vk} = t$.
- If $\bar{z}_{vk} = 1$, then (12) implies $\bar{x}_{vk} = 0$, and (13) is redundant by definition of L.

On the other hand $t > 0$ because $t = 0$ implies $\sum_{v \in V} \bar{y}_{av} = 0$ for all $a \in A$ and then $\bar{y}_{av} = 0$ for all a and v, which is not compatible with the first constraint of (P_I). Hence, taking $x = \frac{1}{t}\bar{x}$, $y = \frac{1}{t}\bar{y}$ one has a feasible point of (P) with the same objective value. □

Remark 3.2. Problem (P_I) contains $\sum_{k \in K} |B_k|$ Boolean variables $\bar{z}_{vk}$. Hence, the larger the number of breakpoints of the environmental utilities, the larger the size of the problem.

This means that, from a computational viewpoint, whereas for very complex models, heuristic methods will be the only options [1,9], the exact method described in this section applies for the case in which linear environmental utilities are generalized to the (much more realistic) case of functions with a few breakpoints.

4. A numerical example

Let us consider a numerical example, in which two plants will be located at the coordinates x^1 and x^2 on the graph of Fig. 2. The new services must satisfy the demand of the nodes $V = A = \{v_1, \ldots, v_5\}$, and simultaneously will have to *minimize* the negative effects caused over the population, concentrated at the points in $R = \{v_2, v_3, v_4, v_5, v_6\}$.

For the sake of simplicity, we measure this environmental impact through the same utility and gauge functions for the whole set of repelling points.

In Fig. 3, the (non-continuous) piecewise linear environmental utility is depicted. The balls of the gauge determined by points at 0.5, 1, and 2 distance units from the origin are shown in Fig. 4. Their appearance seems to indicate that this function would be suitable (as suggested in [23]) to model situations with steady wind conditions.

Once these elements of the problem have been defined, we can determine the finite dominating set B. The points obtained by intersecting the nested balls of Fig. 4 (centered at every repelling point) with the edges of G are collected in Table 1 (see also Fig. 5).

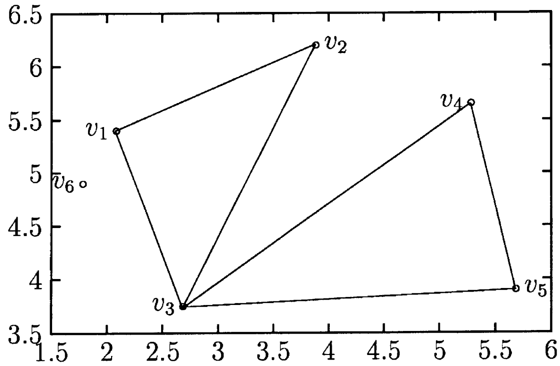

Fig. 2. The graph $G = (N, E)$, $A = \{v_1, \ldots, v_5\}$ and $R = \{v_2, \ldots, v_6\}$.

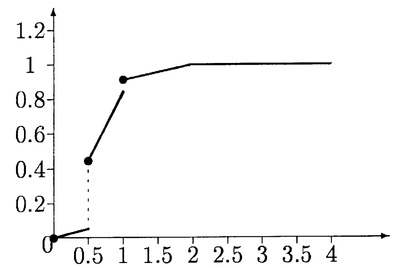

Fig. 3. Environmental utility $\Phi(t)$: $\Phi(t) = \begin{cases} 0.1t & \text{if } t < 0.5, \\ 0.05 + 0.8t & \text{if } 0.5 \leqslant t < 1, \\ 0.85 + 0.075t & \text{if } 1 \leqslant t < 2, \\ 1 & \text{if } t \geqslant 2. \end{cases}$

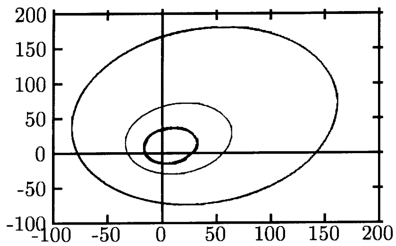

Fig. 4. Level sets of $\|\cdot\|$ at 0.5, 1, 2: $\|x\| = \sqrt{x_1^2 + x_2^2} - 0.3x_1 - 0.4x_2$.

Table 1
Coordinates of the finite dominating set of points

	x_1	x_2		x_1	x_2		x_1	x_2
v_1	2.00000	5.40000	v_{14}	3.26992	5.96441	v_{27}	4.98152	5.49034
v_2	3.80000	6.20000	v_{15}	3.54556	6.08691	v_{28}	5.09076	5.57017
v_3	2.60000	3.75000	v_{16}	3.67278	6.14345	v_{29}	2.89357	3.76467
v_4	5.20000	5.65000	v_{17}	2.77289	4.10300	v_{30}	3.18715	3.77935
v_5	5.60000	3.90000	v_{18}	2.94579	4.45600	v_{31}	4.06789	3.82339
v_6	1.70000	4.90000	v_{19}	3.46449	5.51500	v_{32}	4.84314	3.86215
v_7	2.12967	5.04340	v_{20}	3.50502	5.59775	v_{33}	5.29725	3.88486
v_8	2.21276	4.81488	v_{21}	3.68200	5.95910	v_{34}	5.44862	3.89243
v_9	2.41186	4.26736	v_{22}	3.74100	6.07955	v_{35}	5.23368	5.50263
v_{10}	2.41428	4.26072	v_{23}	2.90947	3.97615	v_{36}	5.26736	5.35527
v_{11}	2.50593	4.00868	v_{24}	3.21895	4.20231	v_{37}	5.27081	5.34017
v_{12}	2.22546	5.50020	v_{25}	4.14739	4.88078	v_{38}	5.36841	4.91319
v_{13}	3.16390	5.91729	v_{26}	4.65381	5.25086	v_{39}	5.46832	4.47606
						v_{40}	5.53416	4.18803

Taking the set of points in B as the candidate solutions (which does not depend on the rest of parameters to be fixed in the model) we have solved some instances of (P_I). The transportation costs were set, in each instance, to be proportional to the Euclidean length of the shortest path on G. Different distributions of demands and populations are collected in Table 2, showing the sensitivity of the optimal locations with respect to these changes (see Fig. 6).

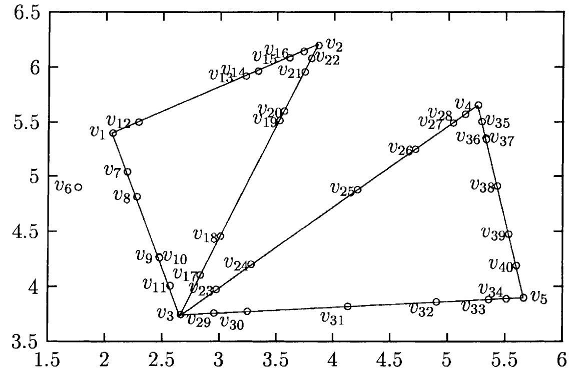

Fig. 5. Finite dominating set B.

5. ε-dominating sets

Proposition 2.1 shows the existence of a dominating set for the particular case of each Φ_k piecewise affine. For arbitrary non-decreasing upper-semi-continuous Φ_k (not necessarily piecewise linear), it is shown in the following how to construct, for any $\varepsilon > 0$, a set which always contains an ε-optimal solution.

To do this, we propose a procedure which, though perhaps inefficient in case the functions Φ_k have a rich structure to be exploited, is general enough to cope with arbitrary non-decreasing upper-semi-continuous functions: We approximate each Φ_k by a piecewise constant (thus piecewise linear) non-decreasing function $\tilde{\Phi}_k$, in such a way that the total error is bounded by some ε given in advance.

Define the constant D as

$$D = \max_{r \in R,\, x \in E} \|r - x\|_r.$$

Table 2
Four numerical examples with their optimal solutions

	v_1	v_2	v_3	v_4	v_5	v_6	c_1	c_2
w	100	100	100	100	100	–	383	217
u	0	1/5	1/5	1/5	1/5	1/5		
y_{a,v_4}	1	1	1	0	0	–		
y_{a,v_2}	0	0	0	1	1	–		
w	100	100	600	100	100	–	433	767
u	0	1/12	8/12	1/12	1/12	1/12		
$y_{a,v_{12}}$	1	1	0.055	0	0	–		
$y_{a,v_{23}}$	0	0	0.945	1	1	–		
w	200	200	200	100	100	–	347	653
u	0	5/17	5/17	1/17	1/17	5/17		
$y_{a,v_{39}}$	0	0	0	1	1	–		
y_{a,v_1}	1	1	1	0	0	–		
w	1000	300	300	300	1000	–	1257	2223
u	0	1/39	1/39	1/39	21/39	15/39		
$y_{a,v_{39}}$	1	1	1	0.173	0	–		
y_{a,v_1}	1	1	1	0.857	1	–		

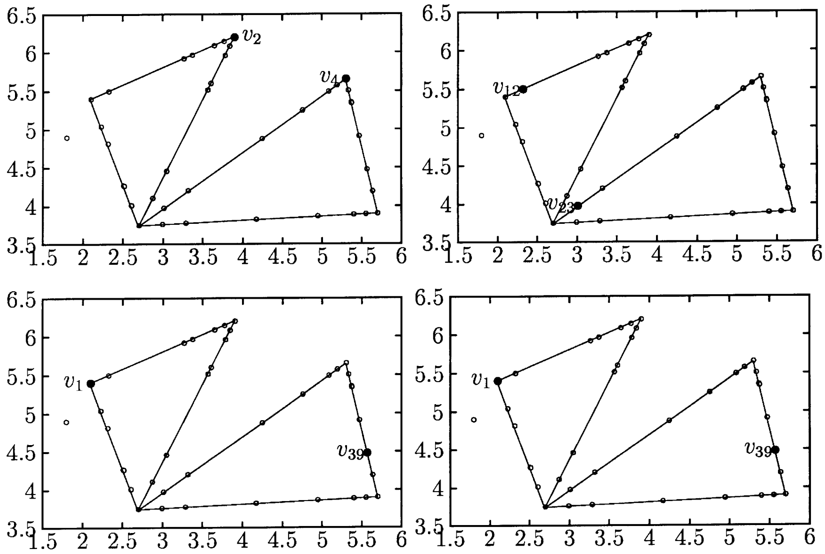

Fig. 6. Solutions for the four data-set of Table 2.

Let $\varepsilon > 0$; for each $k \in K$ define n_k as the lowest integer greater than or equal to $(\Phi_k(D) - \Phi_k(0))/(2\varepsilon)$. Define also the scalars:

$$
\begin{aligned}
t_{k,0} &= 0,\\
t_{k,1} &= \min\{t: \ \Phi_k(t) \geqslant \Phi_k(0) + 2\varepsilon\},\\
t_{k,2} &= \min\{t: \ \Phi_k(t) \geqslant \Phi_k(0) + 4\varepsilon\},\\
&\ \vdots\\
t_{k,n_k-1} &= \min\{t: \ \Phi_k(t) \geqslant \Phi_k(0) + 2(n_k - 1)\varepsilon\},\\
t_{k,n_k} &= D,
\end{aligned}
$$

which, since Φ_k is upper-semi-continuous, are well defined.

Define the upper-semi-continuous piecewise constant function

$$\tilde{\Phi}_k^\varepsilon : [0, t_{k,n_k}] \to \mathbb{R}$$

as

$$
\tilde{\Phi}_k^\varepsilon = \begin{cases}
\Phi_k(0) + \varepsilon & \text{if } t_{k,0} \leqslant t < t_{k,1},\\
\Phi_k(0) + 3\varepsilon & \text{if } t_{k,1} \leqslant t < t_{k,2},\\
\Phi_k(0) + 5\varepsilon & \text{if } t_{k,2} \leqslant t < t_{k,3},\\
\vdots & \vdots\\
\Phi_k(0) + (2n_k - 1)\varepsilon & \text{if } t_{k,n_k-1} \leqslant t \leqslant t_{k,n_k}.
\end{cases} \tag{14}
$$

Moreover, since Φ_k is non-decreasing, one immediately obtains:

Proposition 5.1. *For any $k \in K$,*

$$|\Phi_k(t) - \tilde{\Phi}_k^{\varepsilon}(t)| \leqslant \varepsilon \quad \forall t \in [0, D]. \tag{15}$$

The following result shows that, under Lipschitz-continuity of the aggregating function AF, once a piecewise linear uniform approximation $\tilde{\Phi}_k$ for each Φ_k, such as $\tilde{\Phi}_k^{\varepsilon}$, is known, the problem of finding an ε-optimal solution for problem (3) is reduced to finding an optimal solution for the problem with functions $\tilde{\Phi}_k$ replacing Φ_k, and hence, by Proposition 2.1, reduced to inspecting a finite number of points.

Proposition 5.2. *Suppose there exists a positive constant L such that, for any $s \geqslant 0$, the function* $\mathrm{AF}(s, \cdot)$ *is Lipschitz-continuous with Lipschitz constant L. Then, any optimal solution for*

$$\min_{\hat{x}^1,\ldots,\hat{x}^p \in G} \mathrm{AF}\left(\mathrm{TC}\left(\hat{x}^1,\ldots,\hat{x}^p\right), \sum_{r\in R} u_r \sum_{k\in K} \tilde{\Phi}_k^{\varepsilon}\left(\left\|r - \hat{x}^k\right\|_r\right)\right) \tag{16}$$

is an $Lp\varepsilon$-optimal solution for problem (3).

Proof. Define $\widetilde{\mathrm{EU}}$ as

$$\widetilde{\mathrm{EU}}\left(\hat{x}^1,\ldots,\hat{x}^p\right) = \sum_{r\in R} u_r \left(\sum_{k\in K} \tilde{\Phi}_k\left(\left\|r - \hat{x}^k\right\|_r\right)\right).$$

Then, by assumption, for any $(\hat{x}^1,\ldots,\hat{x}^p)$,

$$\begin{aligned}
\left|\mathrm{EU}\left(\hat{x}^1,\ldots,\hat{x}^p\right) - \widetilde{\mathrm{EU}}\left(\hat{x}^1,\ldots,\hat{x}^p\right)\right| &= \left|\sum_{r\in R} u_r \sum_{k\in K} \Phi_k\left(\left\|r - \hat{x}^k\right\|_r\right) - \sum_{r\in R} u_r \sum_{k\in K} \tilde{\Phi}_k\left(\left\|r - \hat{x}^k\right\|_r\right)\right| \\
&\leqslant \sum_{r\in R} u_r \sum_{k\in K} \left|\Phi_k\left(\left\|r - \hat{x}^k\right\|_r\right) - \tilde{\Phi}_k\left(\left\|r - \hat{x}^k\right\|_r\right)\right| \\
&\leqslant \sum_{r\in R} u_r \sum_{k\in K} \varepsilon = p\varepsilon, \qquad (17)
\end{aligned}$$

where (17) follows from (15) and the assumption $\sum_{r\in R} u_r = 1$. Hence, for any $\hat{x}^1,\ldots,\hat{x}^p$,

$$\begin{aligned}
&\left|\mathrm{AF}\left(\mathrm{TC}\left(\hat{x}^1,\ldots,\hat{x}^p\right),\ \mathrm{EU}\left(\hat{x}^1,\ldots,\hat{x}^p\right)\right) - \mathrm{AF}\left(\mathrm{TC}\left(\hat{x}^1,\ldots,\hat{x}^p\right),\ \widetilde{\mathrm{EU}}\left(\hat{x}^1,\ldots,\hat{x}^p\right)\right)\right| \\
&\quad \leqslant L\left|\mathrm{EU}\left(\hat{x}^1,\ldots,\hat{x}^p\right) - \widetilde{\mathrm{EU}}\left(\hat{x}^1,\ldots,\hat{x}^p\right)\right| \\
&\quad \leqslant Lp\varepsilon.
\end{aligned}$$

Hence, if $(\hat{x}_*^1,\ldots,\hat{x}_*^p)$ denotes an optimal solution for (16), one has for any $(\hat{x}^1,\ldots,\hat{x}^p)$ that

$$\begin{aligned}
\mathrm{AF}\left(\mathrm{TC}\left(\hat{x}^1,\ldots,\hat{x}^p\right),\ \mathrm{EU}\left(\hat{x}^1,\ldots,\hat{x}^p\right)\right) &\geqslant -Lp\varepsilon + \mathrm{AF}\left(\mathrm{TC}\left(\hat{x}^1,\ldots,\hat{x}^p\right),\ \widetilde{\mathrm{EU}}\left(\hat{x}^1,\ldots,\hat{x}^p\right)\right) \\
&\geqslant -Lp\varepsilon + \mathrm{AF}\left(\mathrm{TC}\left(\hat{x}_*^1,\ldots,\hat{x}_*^p\right),\ \widetilde{\mathrm{EU}}\left(\hat{x}_*^1,\ldots,\hat{x}_*^p\right)\right),
\end{aligned}$$

thus any optimal solution for (16) is $Lp\varepsilon$-optimal for (3). □

Remark 5.1. In order to avoid an astronomical increase in the cardinality of the sets B_k in (4), it is convenient to approximate each Φ_k by some $\tilde{\Phi}_k$ satisfying (15) with as few breakpoints as possible.

The construction above is applicable to any upper-semi-continuous non-decreasing Φ_k, yielding $O(1/\varepsilon)$ breakpoints. If further assumptions are made, smaller dominating sets may be obtained. For instance, if Φ_k is assumed to be convex, as suggested, among others, in [12,25], one could use, e.g., the sandwich algorithm described in [3,27], to obtain piecewise linear approximations with $O(1/\sqrt{\varepsilon})$ breakpoints.

Even if case Φ_k is already piecewise constant, it may be convenient to replace it by an approximation $\tilde{\Phi}_k^{\varepsilon}$ with a (much) smaller number of pieces. The algorithm given in [17] could be used to perform efficiently such approximation.

6. Conclusions

In this paper, we have addressed a p-facility location problem for semi-desirable facilities whose location is restricted to the edges of a planar network with rectilinear edges. When the aggregating function AF is assumed to be quasiconvex, the search of (ε-) optimal solutions can be reduced to inspecting a finite set of candidates.

Hence, solving the problem amounts to solving an integer problem, which becomes linear when AF is linear, namely, for the classical Weber problem with attraction and repulsion. For the case of rational AF, the problem can also be transformed into a mixed integer linear problem. For broader classes of functions AF or a large number of facilities to locate metaheuristic procedures might be the unique realistic option.

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