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A characterization of efficient points in constrained location problems with regional demand

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Abstract

In this paper we characterize the set of efficient points in the planar point-objective location problem under a convex locational constraint, when distances are measured by a strictly convex norm in $\mathbb{R}^2$ and the set of demand points is a compact set.

It is shown that, under these assumptions, the efficient set coincides with the closest-point projection of the convex hull of the demand points onto the feasible set.

Keywords: Location problems; Efficiency; Convex analysis; Strictly convex norms

1. Introduction

Let A and S be nonempty sets in $\mathbb{R}^2$, and let γ be a norm in $\mathbb{R}^2$. Consider the vector-optimization problem $\mathbf{P}(\gamma,A,S)$,

$$\mathbf{P}(\gamma,A,S): \quad \min_{x\in S}(\gamma_a(x):\ a\in A),$$

where, for each $a\in A$, γ_a is the function

$$\gamma_a: x\in\mathbb{R}^2 \to \gamma_a(x)=\gamma(x-a)$$

measuring the distance up to a.

A point $x\in S$ is said to be *efficient* for problem $\mathbf{P}(\gamma,A,S)$ iff there exists no $y\in S$ such that $\gamma_a(y)\leqslant\gamma_a(x)$ for all $a\in A$, with at least one strict inequality. A point $x\in S$ is *weakly efficient* iff there exists no $y\in S$ such that $\gamma_a(y)<\gamma_a(x)$ for all $a\in A$.

Throughout this note, the set of efficient and weakly efficient points for $\mathbf{P}(\gamma,A,S)$ will be denoted, respectively, $\mathbf{E}(\gamma,A,S)$ and $\mathbf{WE}(\gamma,A,S)$.

A number of papers (see e.g. [2,4,6,10,14,15]) have been devoted to the search of efficient points of the problem above, known in the literature as the *point-objective location problem* (see [13]), but mostly in the unconstrained case, i.e., under the assumption that the facility can be placed at any point in the plane, i.e., $S=\mathbb{R}^2$.

Although this assumption has been widely questioned (see, e.g., [5]), only some partial results have been obtained in the presence of constraints. For instance, in [5, 9] necessary conditions for a point to be efficient are derived, e.g., the points in $\mathbf{E}(\gamma,A,S)$ are *visible* from the set $\mathbf{E}(\gamma,A,\mathbb{R}^2)$ of efficient points for the unconstrained problem [9]. However, a full characterization of the set of (weakly) efficient points has

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only been obtained in [1] for the classical case when γ is the euclidean norm and A is finite, showing that $\mathbf{E}(\gamma, A, S)$ and $\mathbf{WE}(\gamma, A, S)$ coincide with the orthogonal projection onto S of $\mathbf{E}(\gamma, A, \mathbb{R}^2)$, known to equal the convex hull of A [13].

In this paper we characterize $\mathbf{E}(\gamma, A, S)$ and $\mathbf{WE}(\gamma, A, S)$ when A is compact, S is a closed convex set and γ is a *strictly convex norm*, (i.e., γ is a norm such that the boundary of its unit ball does not contain nondegenerate line segments), showing that both $\mathbf{E}(\gamma, A, S)$ and $\mathbf{WE}(\gamma, A, S)$ equal the closest-point projection (with respect to γ) of the convex hull of A (i.e., $\mathbf{E}(\gamma, A, \mathbb{R}^2) = \mathbf{WE}(\gamma, A, \mathbb{R}^2)$ [13]).

Since the euclidean norm is strictly convex, the characterization given in [1] is extended here in two ways: A is allowed to be infinite, and γ is an arbitrary strictly convex norm.

The proofs make use of rather well-known results of Convex Analysis, which may be found, e.g., in [12].

2. The results

In what follows, S is a nonempty closed and convex set in $\mathbb{R}^2$, A is a nonempty compact subset of $\mathbb{R}^2$, and γ is a strictly convex norm, whose unit ball is denoted by B; the dual norm of γ is denoted by γ^0, and its unit ball by B^0.

Given a set $X \subset \mathbb{R}^2$, let conv(X) denote its convex hull, and $bd(X)$ its boundary; for any $x \in X$, let $X^*(x)$ be the convex cone

$$X^*(x) = \{u \in \mathbb{R}^2 \colon \langle u, y - x \rangle \geqslant 0 \text{ for all } y \in X\},$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product. In other words, $X^*(x) = -N_X(x)$, where $N_X(x)$ is the normal cone of X at x (see [12]).

Given $x \in S$, it is well-known that, since γ is a strictly convex norm, saying that no $y \in S$ verifies $\gamma_a(y) < \gamma_a(x)$ for all $a \in A$ is equivalent to saying that no $y \in S$ verifies ($\gamma_a(y) \leqslant \gamma_a(x)\ \forall a \in A$, with some inequality strict) [10,13], thus the concepts of efficiency and weak efficiency coincide:

$$\mathbf{WE}(\gamma, A, S) = \mathbf{E}(\gamma, A, S).$$

In order to characterize $\mathbf{E}(\gamma, A, S)$, some properties of strictly convex norms are needed. These properties are stated in Lemmas 1–4: Lemma 1 is a consequence of a more general result given in [11], and the proof is not repeated here. Lemmas 2–4 are new, and the proofs can be found in the Appendix.

Lemma 1. *For any $x \in S$, the following statements are equivalent:*

(i) *There exists no $y \in S$ such that $\gamma_a(y) < \gamma_a(x)$ for all $a \in A$.*

(ii) $S^*(x) \cap \text{conv}(\bigcup_{a \in A} \partial\gamma_a(x)) \neq \emptyset$.

Lemma 2. *The following statements are equivalent:*

(i) $0 \in \text{conv}(\bigcup_{a \in A} \partial\gamma(a))$,

(ii) $0 \in \text{conv}(A)$,

(iii) $0 \in \bigcup_{a \in \text{conv}(A)} \partial\gamma(a)$.

Recall that, if $x = 0$, then $\partial\gamma(x) = B^0$, whilst, for $x \neq 0$, $\partial\gamma(x)$ is an exposed face of B^0, see e.g. [3].

Lemma 3. *Let $a, b, c \in bd(B)$, $u, v, w \in bd(B^0)$ be such that*

$$u \in \partial\gamma(a), \qquad v \in \partial\gamma(b), \qquad w \in \partial\gamma(c).$$

(i) *If $a \neq \pm b$ and $c = \lambda a + \mu b$ for some $\lambda, \mu > 0$, then there exist $\alpha, \beta > 0$ such that $w = \alpha u + \beta v$.*

(ii) *If $a \neq -b$ and $w = \alpha u + \beta v$ for some $\alpha, \beta \geqslant 0$, then there exist $\lambda, \mu \geqslant 0$ such that $c = \lambda a + \mu b$.*

Lemma 4. *For any closed convex cone C with vertex at 0, the following statements are equivalent:*

(i) $C \cap \text{conv}(\bigcup_{a \in A} \partial\gamma(a)) \neq \emptyset$,

(ii) $C \cap (\bigcup_{a \in \text{conv}(A)} \partial\gamma(a)) \neq \emptyset$.

Given a point $x \in \mathbb{R}^2$, denote by $\text{proj}_{\gamma,S}(x)$ the point in S closest to x with respect to γ, i.e.,

$$\text{proj}_{\gamma,S}(x) = \arg\min_{y \in S} \gamma_x(y).$$

Since S is closed and convex and γ is a strictly convex norm, $\text{proj}_{\gamma,S}$ is always well-defined.

For any set $X \subset \mathbb{R}^2$, denote also by $\text{proj}_{\gamma,S}(X)$, the set

$$\text{proj}_{\gamma,S}(X) = \{\text{proj}_{\gamma,S}(x) \colon x \in X\}.$$

With this notation, we are in position to characterize the set $\mathbf{E}(\gamma, A, S)$ of efficient points for problem $\mathbf{P}(\gamma, A, S)$, showing that the characterization given in

[1] for the euclidean norm remains valid for general strictly convex norms.

Theorem 1. *Let S be a nonempty closed convex set in $\mathbb{R}^2$, and let γ be a strictly convex norm. Then, for any nonempty compact set $A \subset \mathbb{R}^2$,*

$$\mathbf{E}(\gamma, A, S) = \mathbf{WE}(\gamma, A, S) = \mathrm{proj}_{\gamma,S}(\mathrm{conv}(A))$$

Proof. Let $x \in S$; by Lemma 1, $x \in \mathbf{WE}(\gamma, A, S)$ $(=\mathbf{E}(\gamma, A, S))$ iff

$$S^*(x) \cap \mathrm{conv}\Big(\bigcup_{a\in A} \partial\gamma_a(x)\Big) \neq \emptyset.$$

As $\mathrm{conv}(\bigcup_{a\in A} \partial\gamma_a(x)) = \mathrm{conv}(\bigcup_{b\in x-A} \partial\gamma(b))$, $S^*(x)$ is a closed convex cone with vertex at 0, and $x - A$ is compact, Lemma 4 applies, and we have

$$x \in \mathbf{WE}(\gamma, A, S) \text{ iff } S^*(x) \cap \Big(\bigcup_{b\in \mathrm{conv}(x-A)} \partial\gamma(b)\Big) \neq \emptyset.$$

In other words, $x \in \mathbf{WE}(\gamma, A, S)$ iff $\exists b \in \mathrm{conv}(x-A) = x - \mathrm{conv}(A)$ such that $S^*(x) \cap \partial\gamma(b) \neq \emptyset$, which occurs iff $\exists a^* \in \mathrm{conv}(A)$ such that $S^*(x) \cap \partial\gamma_{a^*}(x) \neq \emptyset$. By Lemma 1 (with $A = \{a^*\}$), nonvoidness of $S^*(x) \cap \partial\gamma_{a^*}(x)$ is equivalent to x being equal to $\mathrm{proj}_{\gamma,S}(a^*)$. Hence, $x \in \mathbf{WE}(\gamma, A, S)$ iff $x \in \mathrm{proj}_{\gamma,S}(\mathrm{conv}(A))$. □

Remark 1. As a consequence of Theorem 1 in [7], when A is finite, $\mathbf{WE}(\gamma, A, S)$ equals the set of Weber points, i.e., the optimal solutions to problems of the form

$$\min_{x\in S} \sum_{a\in A} \lambda_a \gamma_a(x)$$

when $\lambda = (\lambda_a)_{a\in A}$ varies in the set of nonnegative nonzero vectors.

Hence, Theorem 1 implies that the set of Weber points for constrained problems equals the closest-point projection (with respect to γ) of the set of Weber points without constraints, a result more precise than those given in [5].

Remark 2. The proof of Theorem 1 (in fact, the technical precedent lemmas) heavily relies on the fact that γ is a norm, thus its ball B is symmetric with respect to the origin. Extensions of Theorem 1 to general strictly convex gauges with asymmetric balls, seem to require different tools than those used in this paper. See also the recent paper [8] for a different approach to the problem.

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Appendix

Proof of Lemma 2 (part (i) ⇔ (ii)). Since γ is a norm, it follows that $\partial\gamma(-a) = -\partial\gamma(a)$ for all $a \in \mathbb{R}^2$. Hence,

$$0 \in \mathrm{conv}\Big(\bigcup_{a\in A} \partial\gamma(a)\Big) \text{ iff } 0 \in \mathrm{conv}\Big(\bigcup_{a\in A} \partial\gamma(-a)\Big)$$

By Lemma 1 as $(\mathbb{R}^2)^*(x) = \{0\}$, one has

$$0 \in \mathrm{conv}\Big(\bigcup_{a\in A} \partial\gamma(-a)\Big)$$
$$\text{iff } (\mathbb{R}^2)^*(0) \cap \mathrm{conv}\Big(\bigcup_{a\in A} \partial\gamma_a(0)\Big) \neq \emptyset,$$

iff there exists no $y \in \mathbb{R}^2$ such that $\gamma_a(y) < \gamma_a(0)$ $\forall a \in A$, which is equivalent to $0 \in \mathbf{WE}(\gamma, A, \mathbb{R}^2)$. Hence,

$$0 \in \mathrm{conv}\Big(\bigcup_{a\in A} \partial\gamma(a)\Big) \text{ iff } 0 \in \mathbf{WE}(\gamma, A, \mathbb{R}^2).$$

By Corollary 1 of [11],

$$0 \in \mathbf{WE}(\gamma, A, \mathbb{R}^2) \text{ iff } 0 \in \mathbf{WE}(\gamma, A', \mathbb{R}^2) \text{ for some finite } A' \subset A$$

Since $0 \in \mathbf{WE}(\gamma, A', \mathbb{R}^2) = \mathrm{conv}(A')$ for all finite A' (see [13]), and $\mathrm{conv}(A) = \bigcup\{\mathrm{conv}(A')\colon A' \subset A,\ A' \text{ is finite}\}$, it follows that $0 \in \mathrm{conv}(\bigcup_{a\in A} \partial\gamma(a))$ iff $0 \in \mathrm{conv}(A)$, as asserted. □

Proof of Lemma 2 (part (ii) ⇔ (iii)). $0 \in \mathrm{conv}(A)$ iff $\exists a^* \in \mathrm{conv}(A)$ such that 0 minimizes in $\mathbb{R}^2$ the function γ_{a^*}. As γ_{a^*} is convex, 0 minimizes γ_{a^*} iff $0 \in \partial\gamma(-a^*) = -\partial\gamma(a^*)$. Hence, $0 \in \mathrm{conv}(A)$ iff $0 \in \bigcup_{a\in \mathrm{conv}(A)} \partial\gamma(a)$, as asserted. □

Proof of Lemma 3. We only prove part (i); part (ii) can be proven with similar arguments.

As $u \in \partial\gamma(a)$, $v \in \partial\gamma(b)$, $\pm a, \pm b \in bd(B)$, and B is symmetric with respect to 0, one has:

(1) $\langle a, u\rangle = 1$,

(2) $\langle b, v\rangle = 1$,

(3) $|\langle x, u\rangle| \leqslant 1$ and $|\langle x, v\rangle| \leqslant 1$ for all $x \in B$.

In particular, $|\langle a, v\rangle| \leqslant 1$ and $|\langle b, u\rangle| \leqslant 1$. Furthermore, these inequalities are strict; indeed, if $\langle a, v\rangle = 1$ (respect. $\langle a, v\rangle = -1$), the line $\langle v, x\rangle = 1$ would support B at a and b (resp. $-a$ and b), implying, because of the convexity of B, that the whole segment with extreme points a and b (resp. $-a$ and b) is contained in $bd(B)$, contradicting the assumption that γ is a strictly convex norm. Hence, one has:

(4) $|\langle a, v\rangle| < 1$,

(5) $|\langle b, u\rangle| < 1$.

As, by assumption, $c = \lambda a + \mu b \in B$, (1)–(2) imply (by (3), for $x = c$):

(6) $|\lambda + \mu\langle b, u\rangle| \leqslant 1$,

(7) $|\lambda\langle a, v\rangle + \mu| \leqslant 1$.

The vectors u and v are linearly independent; indeed, otherwise, as $\gamma^0(u) = \gamma^0(v) = 1$, we would have that $u = \pm v$, contradicting (1)–(4). Hence, $\{u, v\}$ is a basis in $\mathbb{R}^2$, thus there exist $\alpha, \beta \in \mathbb{R}$ such that $w = \alpha u + \beta v$.

Observe that $w \in bd(B^0)$; hence, the line $\langle w, x\rangle = 1$ supports B, thus

(8) $|\alpha\langle b, u\rangle + \beta| \leqslant 1$,

(9) $|\alpha + \beta\langle a, v\rangle| \leqslant 1$.

Furthermore, as $w \in \partial\gamma(c)$ and $\langle w, c\rangle = 1$, we also have:

(10) $\alpha\{\lambda + \mu\langle b, u\rangle\} + \beta\{\lambda\langle a, v\rangle + \mu\} = 1$.

Define the segment Γ in $\mathbb{R}^2$,

$$\Gamma = \{(\overline{\alpha}, \overline{\beta}) \in \mathbb{R}^2\colon\ (\overline{\alpha}, \overline{\beta}) \text{ verifies (8), (9) and (10)}\}$$

which, of course, contains the point (α, β).

In order to show that α and β are strictly positive, we first show that Γ is included in the nonnegative quadrant $\Gamma' = \{(\overline{\alpha}, \overline{\beta}) \in \mathbb{R}^2\colon\ \overline{\alpha} \geqslant 0, \overline{\beta} \geqslant 0\}$, by showing that the vertices of Γ are in Γ'. The vertices of Γ are among the points obtained by replacing one of the inequalities in (8) or (9) by an equality.

Let us study separately the different cases:

Case 1: $\overline{\alpha}\langle b, u\rangle + \overline{\beta} = 1$

It leads to the values:

$$\overline{\alpha} = (1 - \mu - \lambda\langle a, v\rangle)/(\lambda(1 - \langle a, v\rangle\langle b, u\rangle)),$$

$$\overline{\beta} = (\lambda - (1 - \mu)\langle b, u\rangle)/(\lambda(1 - \langle a, v\rangle\langle b, u\rangle)).$$

By (4), (5) and (7), it immediately follows that $\overline{\alpha} \geqslant 0$. On the other hand, (9) implies that $\lambda \geqslant |1 - \mu|$; indeed, by (9),

$$\begin{aligned} 1 &\geqslant |\overline{\alpha} + \overline{\beta}\langle a, v\rangle| \\ &= |((1 - \mu - \lambda\langle a, v\rangle) \\ &\qquad + \langle a, v\rangle(\lambda - (1 - \mu)\langle b, u\rangle))/(\lambda(1 - \langle a, v\rangle\langle b, u\rangle))| \\ &= |(1 - \mu)(1 - \langle a, v\rangle\langle b, u\rangle)/(\lambda(1 - \langle a, v\rangle\langle b, u\rangle))| \\ &= |(1 - \mu)/\lambda|, \text{ thus } |1 - \mu| \leqslant |\lambda| = \lambda, \end{aligned}$$

as asserted.

Hence, by (5),

$$\lambda - (1 - \mu)\langle b, u\rangle \geqslant \lambda - |1 - \mu| \geqslant 0.$$

As $\lambda > 0$, (4) and (5) imply that $\lambda(1 - \langle a, v\rangle\langle b, u\rangle) > 0$; hence, $\overline{\beta} \geqslant 0$.

Case 2: $\overline{\alpha}\langle b, u\rangle + \overline{\beta} = -1$

It leads to the values

$$\overline{\alpha} = (1 + \mu + \lambda\langle a, v\rangle)/\lambda(1 - \langle a, v\rangle\langle b, u\rangle),$$

$$\overline{\beta} = -(\lambda + (1 + \mu)\langle b, u\rangle)/\lambda(1 - \langle a, v\rangle\langle b, u\rangle).$$

First, (5) and (6) imply that

$$1 \geqslant \lambda + \mu\langle b, u\rangle \geqslant \lambda - \mu|\langle b, u\rangle| > \lambda - \mu,$$

thus $\lambda < 1 + \mu$. On the other hand, $(\overline{\alpha}, \overline{\beta})$ must verify (9), which is readily seen to be equivalent to $\lambda \geqslant |1 + \mu| = 1 + \mu$, what is a contradiction.

Hence, the solution of (10) and $\overline{\alpha}\langle b, u\rangle + \overline{\beta} = -1$ does not give a feasible point.

Due to the symmetry in $\overline{\alpha}, \overline{\beta}, \lambda, \mu$ in Γ and constraints (1)–(7), similar results are obtained for the cases

$$\overline{\alpha} + \overline{\beta}\langle a, v\rangle = \pm 1.$$

Hence, all the extreme points of Γ are contained in Γ', thus $\Gamma \subset \Gamma'$; as $(\alpha, \beta) \in \Gamma$, it follows that $\alpha, \beta \geqslant 0$.

Furthermore, α and β are both strictly positive; otherwise, $w = \alpha u$ for some $\alpha > 0$ or $w = \beta v$ for some $\beta > 0$; suppose that $w = \alpha u$ for some $\alpha > 0$; as $1 = \gamma^0(w) = \gamma^0(u)$, we would have that $w = u$. Hence, the line $\langle u, x\rangle = 1$ would support B both at a and c, thus the nontrivial segment with extreme points a and c would be contained in $bd(B)$, which is a contradiction (recall that γ is a strictly convex norm).

With this we have shown part (i). □

Proof of Lemma 4 (part (i) ⇒ (ii)). As $0 \in C$, the result follows from Lemma 2 if $0 \in \text{conv}(\bigcup_{a\in A} \partial\gamma(a))$. Hence, we further assume that $0 \notin \text{conv}(\bigcup_{a\in A} \partial\gamma(a))$ (or, equivalently, $0 \notin \text{conv}(A)$).

Consider the planar convex cone Λ,

$$\Lambda = \left\{ u \in \mathbb{R}^2 \colon\ u = \lambda\xi \text{ for some } \xi \in \text{conv}\Big(\bigcup_{a\in A} \partial\gamma(a)\Big), \lambda \geqslant 0 \right\}.$$

Λ is a planar closed convex cone, and $\Lambda \neq \mathbb{R}^2$; indeed, it is easily seen that Λ is a convex cone; closedness follows from the fact that $\text{conv}(\bigcup_{a\in A} \partial\gamma(a))$ is compact (see, e.g. [11]); as, by assumption, $0 \notin \text{conv}(\bigcup_{a\in A} \partial\gamma(a))$ and $\text{conv}(\bigcup_{a\in A} \partial\gamma(a))$ is compact and convex, it follows that $\Lambda \neq \mathbb{R}^2$. Furthermore, it is straightforward to check that the extreme rays of Λ are necessarily elements of $\bigcup_{a\in A} \partial\gamma(a)$, i.e., there exist $a_1, a_2 \in A$, $\xi_1 \in \partial\gamma(a_1)$, $\xi_2 \in \partial\gamma(a_2)$ such that

$$\Lambda = \{u \in \mathbb{R}^2 \colon\ u = t_1\xi_1 + t_2\xi_2 \text{ for some } t_1, t_2 \geqslant 0\}.$$

Furthermore, $(1/\gamma(a_1))a_1 \neq -(1/\gamma(a_2))a_2$ because, otherwise, $0 \in \text{conv}(A)$.

Let $d \in C \cap \text{conv}(\bigcup_{a\in A} \partial\gamma(a))$. As $d \in \Lambda$, and $d \neq 0$, it follows that

$$d = \lambda_1\xi_1 + \lambda_2\xi_2 \text{ for some } \lambda_1, \lambda_2 \geqslant 0, \lambda_1 + \lambda_2 > 0$$

Let $d' = d/\gamma^0(d)$. One has:

- $d' = \alpha\xi_1 + \beta\xi_2$ for some $\alpha, \beta \geqslant 0$, $\alpha + \beta > 0$.
- There exists $c \in bd(B)$ such that $d' \in \partial\gamma(c)$ (because $d' \in bd(B^0)$).

As $\xi_i \in \partial\gamma(a_i/\gamma(a_i))$, $(i = 1, 2)$, $a_1/\gamma(a_1) \neq -a_2/\gamma(a_2)$, and $a_i/\gamma(a_i) \in bd(B)$, $(i = 1, 2)$, by Lemma 3 (part ii),

$$c = \lambda a_1 + \mu a_2 \quad \text{for some } \lambda, \mu \geqslant 0, \lambda + \mu > 0$$

Let c' be the vector $c' = c/(\lambda + \mu) \in \text{conv}(A)$.

It follows that $d' \in \partial\gamma(c') \subset \bigcup_{a\in \text{conv}(A)} \partial\gamma(a)$, and $d' \in C$.

Hence, $\bigcup_{a\in \text{conv}(A)} \partial\gamma(a) \cap C \neq \emptyset$, as asserted. □

Proof of Lemma 4 (part (ii) ⇒ (i)). The result follows from Lemma 2 if $0 \in \bigcup_{a\in \text{conv}(A)} \partial\gamma(a)$. Hence, we further assume that $0 \notin \bigcup_{a\in \text{conv}(A)} \partial\gamma(a)$, i.e., $0 \notin \text{conv}(A)$. Consider the planar convex cone Γ,

$$\Gamma = \{u \in \mathbb{R}^2 \colon u = \lambda a \text{ for some } \lambda > 0, a \in \text{conv}(A)\}.$$

Then, there exist $a_1, a_2 \in A$ such that

$$\Gamma = \{u \colon u = t_1 a_1 + t_2 a_2 \text{ for some } t_1, t_2 \geqslant 0, t_1 + t_2 > 0\}.$$

Furthermore, $(1/\gamma(a_1))a_1 \neq -(1/\gamma(a_2))a_2$ (else, $0 \in \text{conv}(A)$).

Let $d \neq 0$, $d \in C \cap \bigcup_{a\in \text{conv}(A)} \partial\gamma(a)$. There exists $a^* \in \text{conv}(A)$ such that $d \in \partial\gamma(a^*)$. Furthermore, as $0 \notin \text{conv}(A)$ and $a^* \neq 0$, it follows that $d \in bd(B^0)$.

As $\text{conv}(A) \subset \Gamma$, there exist $\lambda, \mu \geqslant 0$, $\lambda + \mu > 0$ such that

$$a^* = \lambda a_1 + \mu a_2.$$

If $(1/\gamma(a^*))a^* = (1/\gamma(a_i))a_i$ for some $i = 1, 2$, $(1/\gamma(a^*))a^* = (1/\gamma(a_1))a_1$, say, the result holds because $d \in C$ and

$$d \in \partial\gamma(a^*) = \partial\gamma(a^*/\gamma(a^*)) = \partial\gamma(a_1/\gamma(a_1)) = \partial\gamma(a_1) \subset \text{conv}\Big(\bigcup_{a\in A} \partial\gamma(a)\Big)$$

If it is not the case, we have that $\lambda > 0$, $\mu > 0$, and

$$(1/\gamma(a_1))a_1 \neq (1/\gamma(a_2))a_2.$$

Let $\xi_1 \in \partial\gamma(a_1)$, $\xi_2 \in \partial\gamma(a_2)$. By Lemma 3, part (i), there exist $\alpha, \beta > 0$ such that $d' = \alpha\xi_1 + \beta\xi_2$. Hence, the vector $d'' = d'/(\alpha + \beta)$ verifies:

- $d'' \in C$,
- $d'' \in \text{conv}(\partial\gamma(a_1) \cup \partial\gamma(a_2)) \subset \text{conv}(\bigcup_{a\in A} \partial\gamma(a))$, thus (i) holds, as asserted. □

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