BML model on non-orientable surfaces

Daniel Cámpora [a], Jaime de la Torre [b], Juan Carlos García Vázquez [c,*], Fernando Sancho Caparrini [a]

[a] University of Seville, Department of Computer Science and A. I., 41012, Sevilla, Spain
[b] University of Seville, Sevilla, Spain
[c] University of Seville, Department of Mathematical Analysis, 41012, Sevilla, Spain

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A B S T R A C T

Two-dimensional Biham–Middleton–Levine traffic model on a $N \times N$ square lattice embedded on a Klein bottle and a projective plane is investigated by computer simulations. The behavior of the model with these boundary conditions is compared with the model under toroidal boundary conditions. Our numerical results show that the phase diagram depends strongly on the underlying topology. We also investigate the influence of the ratio between the number of red and blue particles has over the speed of the system and conclude that the projective plane case differs considerably from the other cases reviewed here.

1. Introduction

Models about traffic flow are of importance in a broad range of physical systems and their interest resides not only on specific applications, but also on a theoretical point of view. For a comprehensive review about the topic see Ref. [1].

The traffic model introduced by Biham, Middleton and Levine in Ref. [2] is one of the most studied in recent years. It is a two-dimensional cellular automaton that exhibits self-organization, pattern formation and phase transitions.

The model is defined on a square lattice with $N \times N$ sites with periodic boundary conditions in such a way that the geometric model corresponds to the discrete torus $\mathbb{Z}_N \times \mathbb{Z}_N$. Every site can contain a blue particle, a red particle or can be empty. Initially, particles are placed in random sites. According to a parameter $p \in [0, 1]$, the probability of a site containing a blue particle is $p/2$, a red one is $p/2$ and $1 - p$ is the probability of being empty. In this way, p is the density of particles in the lattice.

The dynamic of the particles is determined by the following rules: (a) in a first step, red particles try to move forward one site in a northward direction; unless the site they wish to occupy is non-empty they move; otherwise, they are blocked. (b) In a second step, blue particles follow the same rules but they try to move in the eastward direction. The process is repeated again and again.

Let us observe that the only random event in the model occurs in the initial distribution of the particles in the lattice.

In every discrete timestep t, the instantaneous speed of the system, v_t, is the ratio between the number of particles that succeed to move and the total number of them. If $v_t = 0$ then no particle has moved in the timestep t; if $v_t = 1$ then all the particles have moved.

In spite of the simplicity of the model very few properties of the system were proven rigorously to date, with almost all published studies reporting the results of numerical experiments. For analytical results we should mention [3–6].

* Corresponding author. Tel.: +34 954559557; fax: +34 954557972.
E-mail address: garcia@us.es (J.C. García Vázquez).

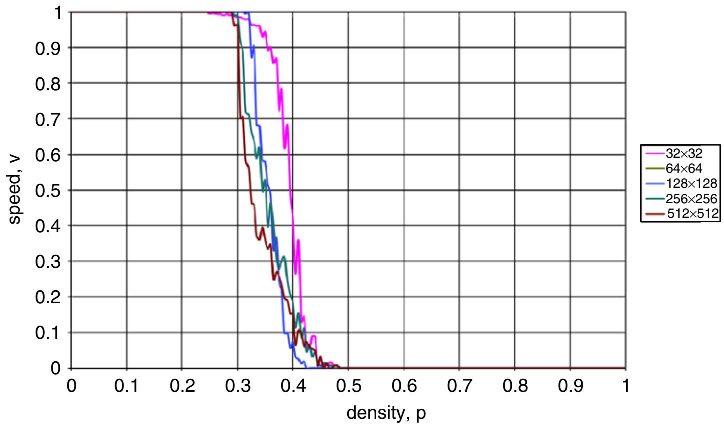


Fig. 1. The particle density p versus speed v corresponding to the BML model on a torus for several sizes of the lattice. Critical density slightly decreases with increasing size of the system.

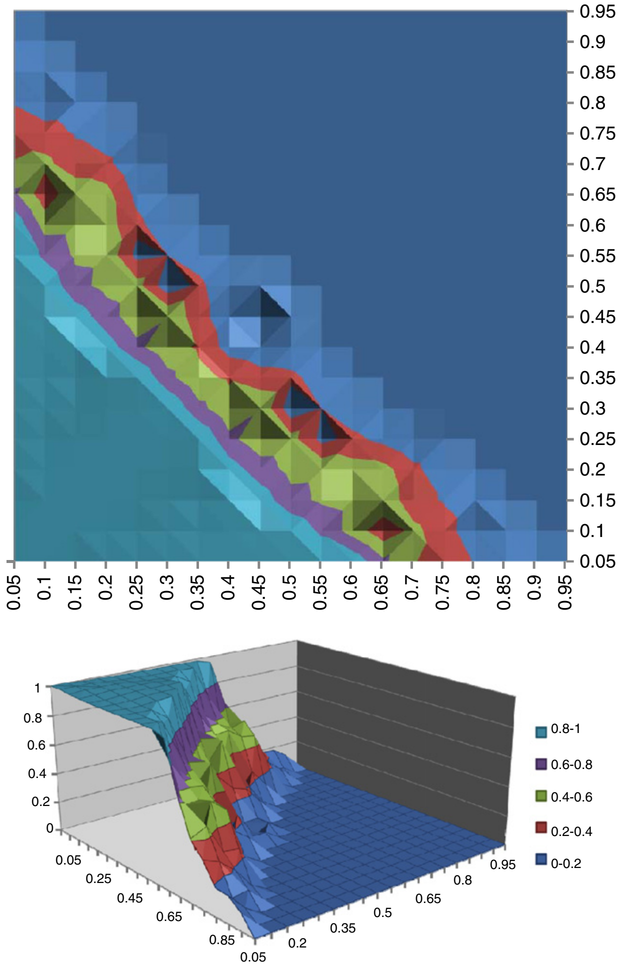


Fig. 2. A two-dimensional graphic of v as a function of p_{red} (density of red particles, in the horizontal axis) and p_{blue} (density of blue particles, in the vertical axis) corresponding to the BML model on a torus. Below, we can see the same graphic in a 3D-picture. The size of the system is 256×256. Disparity among densities does not affect the speed of the system. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Several interesting generalizations of the model have been considered, such as extension to three-dimensional lattice with periodic boundary conditions [7], free boundary conditions [8], non-square aspect ratio of the underlying lattice [9], four-directional traffic [10], etc.

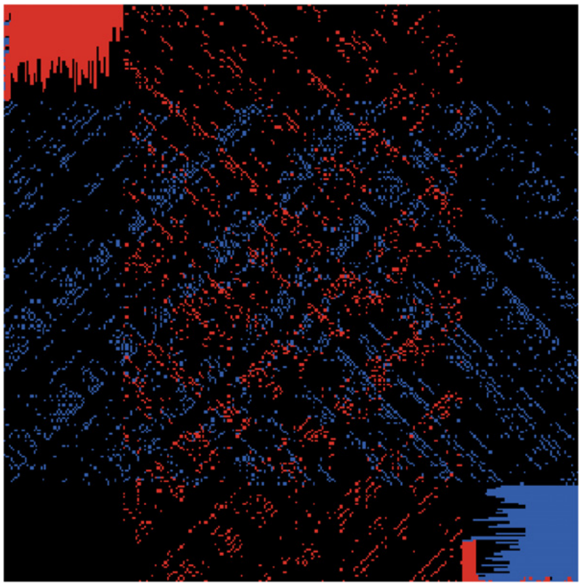


Fig. 3. A typical configuration in a experiment of the BML model on a projective plane. Local jams have appeared on the bottom right and upper left corners and the rest of particles move almost in a free flow, appearing self-organization in diagonal patterns.

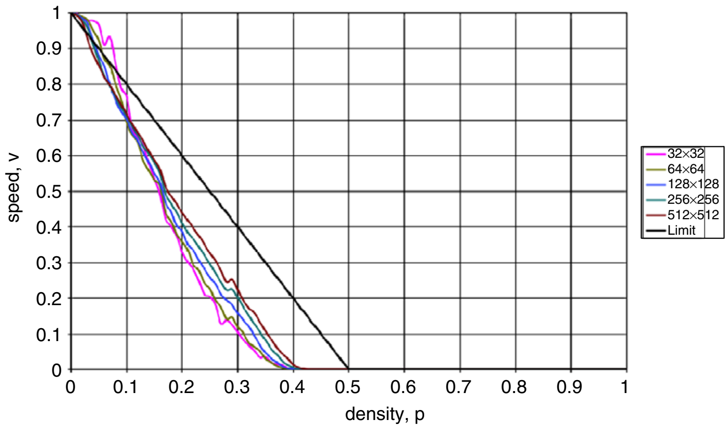


Fig. 4. A graphic of v as a function of p corresponding to the BML model on a projective plane for several sizes of the lattice. Speed starts to decrease as soon as p increase. We can also see clearly that the speed increase when the size of the lattice increase for values of p between 0.2 and 0.4. In black we have the expected limit curve when the size of the lattice tends to infinity.

In Ref. [3] it is remarked that the speed of the system may depend sensitively on the boundary conditions of the model. In this paper, we want to investigate the effect of a change in the boundary conditions, providing a numerical study of the two-dimensional BML model on a $N \times N$ square lattice embedded on a Klein bottle and a projective plane, both non-orientable surfaces. We will focus on the differences between these two models and the one on the torus. Also, we investigate the influence that the ratio between the number of red and blue particles has over the speed of the system, a question considered in Ref. [11].

For the experiments, NVIDIA CUDA technology was used. This technology implements a massive parallel computation over 512 cores, logically arranged into blocks and threads. The strength of this tool lies in the division of the problem into small parts, which are assigned to threads. For more information on this technology see Ref. [12]. Briefly described, in our implementation we divide every step into its horizontal and vertical movements. For the horizontal part, every row is assigned to a block of threads. Thus, every thread processes one or more elements of its row. The same applies to the vertical movement, where every block deals with a column.

2. BML model on a torus

In Ref. [2] numerical simulations show that the BML model on the torus seems to show a sharp transition that separates a low-density dynamical phase in which all particles move at each step of time and a high-density static phase in which

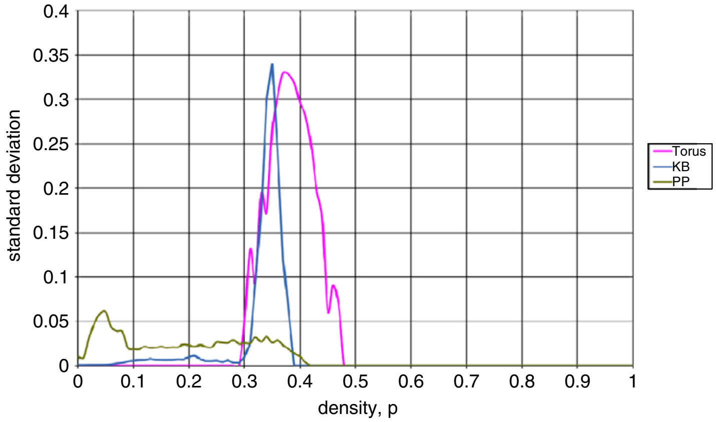
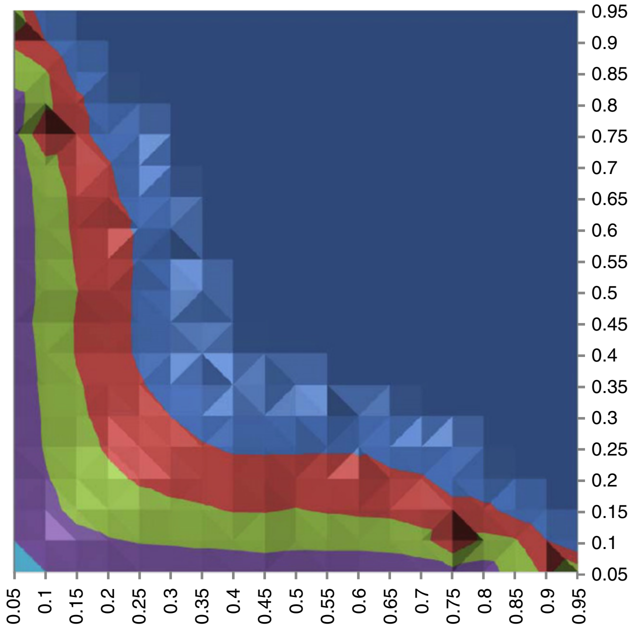


Fig. 5. A comparative of the standard deviation of speeds obtained from 1000 experiments as a function of p corresponding to the BML model on a torus and on a projective plane. The size of the lattice is 256×256. We can observe that for the projective plane case standard deviation is very close to 0, while in the torus and Klein bottle cases we have a peak around the critical probability.



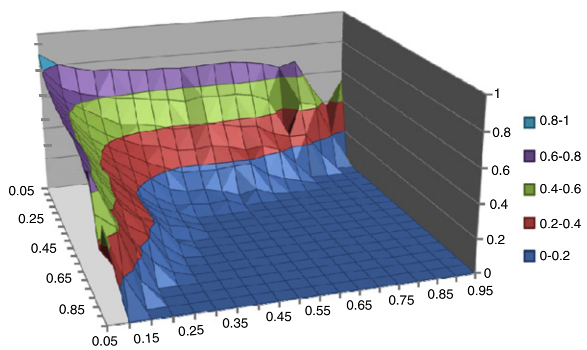


Fig. 6. A two-dimensional graphic of v as a function of p_{red} (density of red particles, in the horizontal axis) and p_{blue} (density of blue particles, in the vertical axis) corresponding to the BML model on a projective plane. Below, we can see the same graphic in a 3D-picture. The size of the system is 256×256. In this case disparity among densities has an influence in the behavior of the system; in fact, speed increases when the difference between densities increases. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

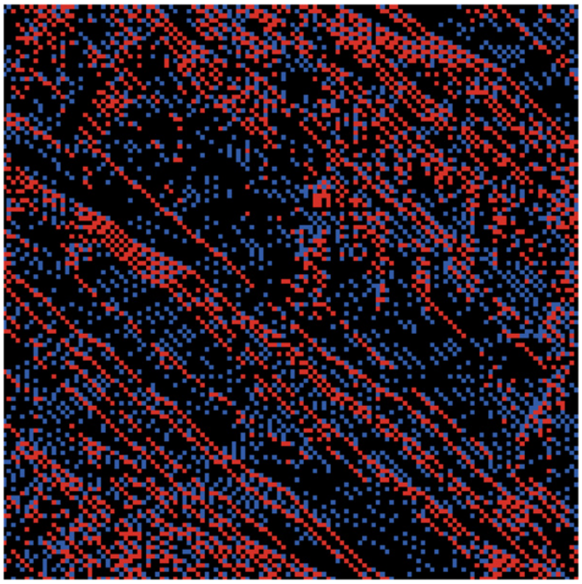


Fig. 7. A typical configuration in a experiment of the BML model on a Klein bottle in the free flow region. Diagonal patterns from left to right are observed as in the torus case, but this time they show a kind on deformation because of the changes on the boundary conditions.

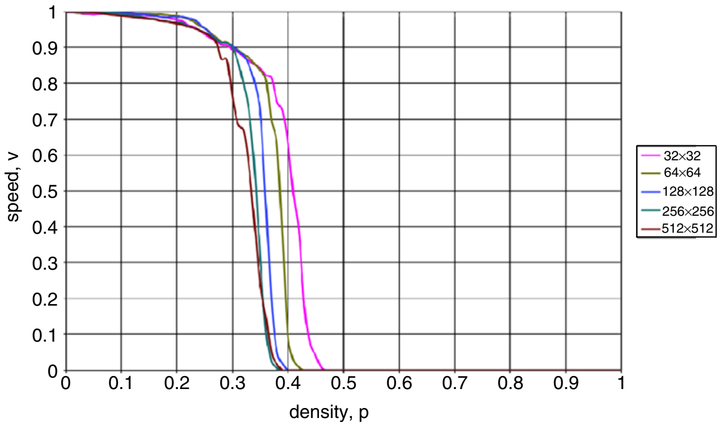


Fig. 8. The particle density p versus speed v corresponding to the BML model on a Klein bottle for several sizes of the lattice. The behavior is similar to the one of the torus case, although speed starts to decrease sooner than in the torus case.

particles are all stuck in a global traffic jam; in the middle there is a range of densities in which both extreme behaviors are found with a non-negligible probability. On the other hand, in Ref. [9] intermediate speed configurations are found in which the system keeps a speed far away from 0 and 1.

We have carried out numerical simulations (see Fig. 1) up to 512×512 square lattice as in Ref. [2]. In the regions of extreme behavior (total free flow or global jam) almost all the experiments we have realized were stabilized before $t = 10\,000$, so the speed was 1 or 0 at that moment, respectively. This is not the case for the intermediate values of p, where $t \sim 10\,000$ is not enough for the system to reach a recurrent state. Even if we allow the experiment to reach 50 000 steps, oscillations in speed continue appearing. In Fig. 1, v is seen as a function of p for several sizes of the lattice. These graphs have been obtained as an average of 1000 experiments for every value of p. The speed for every experiment is computed as the average of the speed of the system between timesteps $t = 50\,000$ and $t = 51\,000$. The step increment of the parameter p has been of 0.01 units. As the reader can check, the plots are similar to those obtained in other papers.

In Ref. [11] several factors that can help in the appearance of jams for p being fixed are investigated. One of these factors is the disparity among the number of red and blue particles. It could be said that a difference among the distributions of colors can make more probable the appearance of jams, and consequently, a lower speed of the system. In Ref. [11] authors report that in 45 experiments they carried out no correlation between these two facts was found.

We have performed a numerical study of the influence of this factor. In Fig. 2 speed is a function of p_{red}, the probability of a site containing a red particle at the initial configuration, and p_{blue}, the probability of a site containing a blue particle at the initial configuration. In this way p_{red} and p_{blue} are the density of red and blue particles, respectively. We have considered

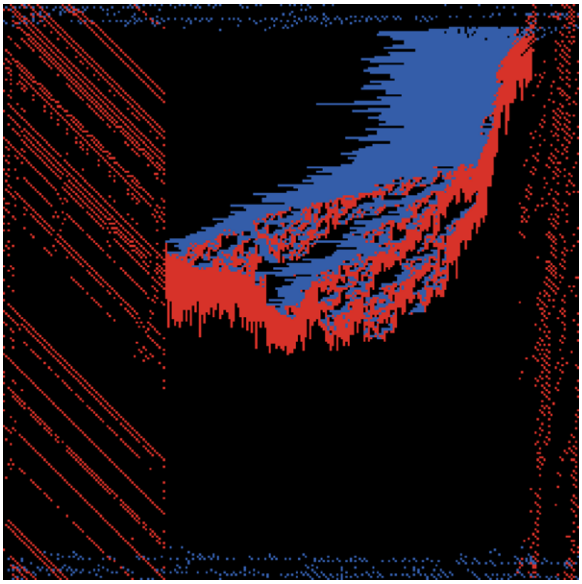


Fig. 9. A periodic intermediate state in the Klein bottle case for a realization on the square lattice with $N = 256$ and $p = 0.22$. The time step is $t = 500\,000$. In contrast to the projective plane case, an stable local jam have appeared on the center and particles on the border move almost in a free flow.

10 experiments for every value of p_{red} and p_{blue}, allowing 5000 steps in every experiment, and taking the average of speeds between timesteps 5000 and 5100. The step increment of density in every color is of 0.05. Because of the symmetry of the parameter space, we have considered the range $p_{\text{red}} \in [0, 1]$, $p_{\text{blue}} \in [0, 1 - p_{\text{red}}]$. The size of the lattice is 256×256.

We can check that speeds for different values of p_{red} and p_{blue}, with total density $p = p_{\text{red}} + p_{\text{blue}}$ constant, are very similar, what it seems to show that disparity among densities does not affect the average speed of the system. In Fig. 2 we can also see a 3D representation of these data.

As we will see in the next section, this is not the case in other topologies, where the disparity among densities is relevant for the behavior of the system.

3. BML model on a projective plane

Let $N \in \mathbb{N}$. Let us consider the finite lattice of size $N \times N$. Every site in the lattice can be represented by a pair (i, j), where $i, j \in \{0, \ldots, N-1\}$. For every $l \in \mathbb{Z}_N$, the boundary site $(N-1, l)$ is connected with the one in $(0, N-l-1)$, in such a way that a blue particle in site $(N-1, l)$ will try to move to the site $(0, N-l-1)$. In a similar way, the site $(l, N-1)$ is connected with the site $(N-l-1, 0)$, so a red particle in $(l, N-1)$ will try to move to site $(N-l-1, 0)$ in the next timestep. Under these geometrical considerations on the border, we obtain a traffic model on a discrete projective plane. In Fig. 3 we can observe an instant of one experiment on a projective plane.

Typically, we can observe how local jams appear on the bottom right and upper left corners. The size of these jams depends on density p. The rest of particles out of this local jam will move almost in a free flow, showing diagonal patterns.

Observe that we can have a stable local traffic jam, a gridlock mechanism as in the case of four directional traffic flow [10], which may occur at any density $p > 0$ when a red particle is on the upper left site at the same time that a blue particle is on the bottom right site of the lattice. This is a fundamental difference between the projective plane and the torus.

In Fig. 4, the average speed as a function of p for different values of the size of the lattice in the projective plane case is shown. These plots, as in the torus case, have been obtained as an average of 1000 experiments for every value of p. The speed for every experiment is computed as the average of the speed of the system between timesteps $t = 50\,000$ and $t = 51\,000$. The step increment of the parameter p has been of 0.01 units.

As we have mentioned before, simulations show that in the torus the speeds obtained from experiments carried out for the same value of p close to a critical probability may be very different. This does not seem to happen in the projective plane, where for every fixed value of p different experiments have very similar results. In Fig. 5, a comparative plot between the standard deviation of speeds obtained in 1000 experiments on the torus, the Klein bottle and on the projective plane is shown. The size of the lattice is 256×256. It is clear that in the torus case we have a peak around the critical value of the density while in the projective case standard deviation is almost constant equal to 0.

Because of the local jams on the corners, as soon as p increases, speed starts to decrease. Fig. 4 suggest the existence of a smooth transition from a phase in which $v > 0$ to a global jam phase where $v = 0$. Another difference with respect to the torus should be mentioned, given $p \geq 0, 1$ the speed of the system as a function of the size. This can be clearly seen in Fig. 4.

The following remark may help us to understand the behavior of the speed in the projective plane case. Let us fix $p \in [0, 1]$. Once the local jam has appeared as a consequence of a red particle being on the upper left site at the same

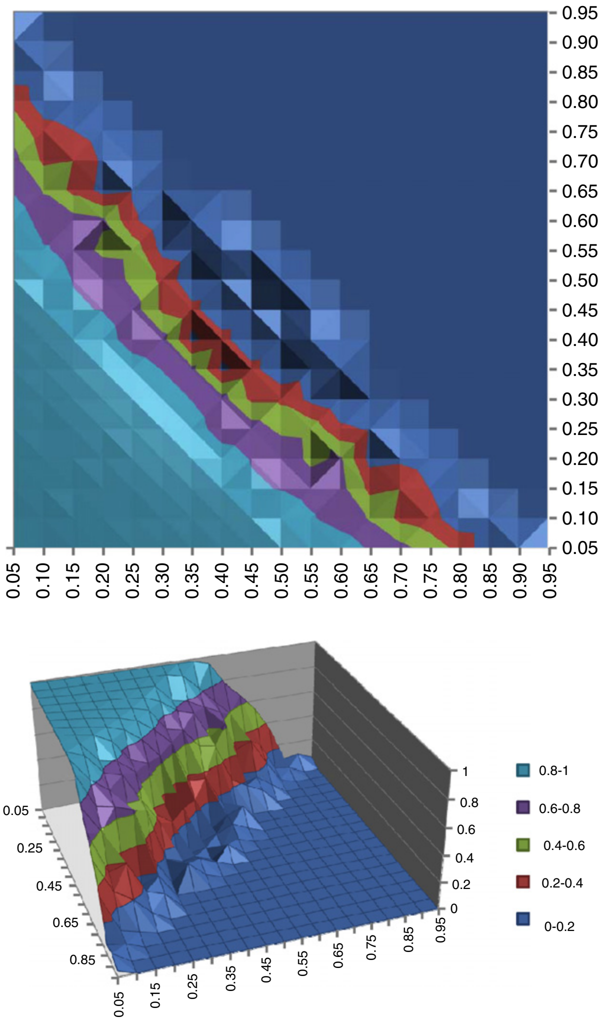


Fig. 10. A two-dimensional graphic of v as a function of p_{red} (density of red particles, in the horizontal axis) and p_{blue} (density of blue particles, in the vertical axis) corresponding to the BML model on a Klein bottle. Below, we can see the same graphic in a 3D-picture. The size of the system is 256×256. Disparity among densities does not affect the speed of the system as in the torus case. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

time that a blue particle is on the bottom right site of the lattice, the expected number of red particles blocked behind the red one is Np. In the same way, the number of blue particles blocked behind the blue one is on average equal to Np. As a consequence two squares of blocked particles, each one of area N^2p^2 are generated so $N^2p - 2N^2p^2 = N^2p(1 - 2p)$ is the number of particles that we expect to see flowing in the center of the lattice. Therefore the speed of the system will be bounded by $(1 - 2p)$ for $0 \leq p \leq 1/2$ and $v = 0$ for $p > 1/2$. Computer simulations up to 512×512 square lattice suggest that particles outside the two local jams are able to self-organize in an almost free-flow phase, consequently, the system should reach speed close to $1 - 2p$. Curves in Fig. 4 seems to show this tendency.

We have also studied in this topology the speed in the case where densities of particles are different. As in the torus case we have considered 10 experiments for every value of p_{red} and p_{blue}, allowing 5000 steps in every experiment and taking the average of speeds between timesteps 5000 and 5100. The step increment of density in every color is of 0.05 again.

In Fig. 6 results are shown. Dependence between speed and disparity of densities can be observed. This is another difference between the projective plane and the torus. Let us observe that the speed of the system tends to increase when densities present big disparity; therefore, a disproportionate density of one color helps us to improve the speed of the system. In Fig. 6, the 3D plot of speed as a function of densities of red and blue particles is also shown.

4. BML model on a Klein bottle

In order to work on the Klein bottle we must again consider the finite lattice of size $N \times N$, with the following boundary conditions. As in the projective plane, for every $l \in \{0, \ldots, N-1\}$ the site $(N-1, l)$ is connected with the one in $(0, N-l-1)$,

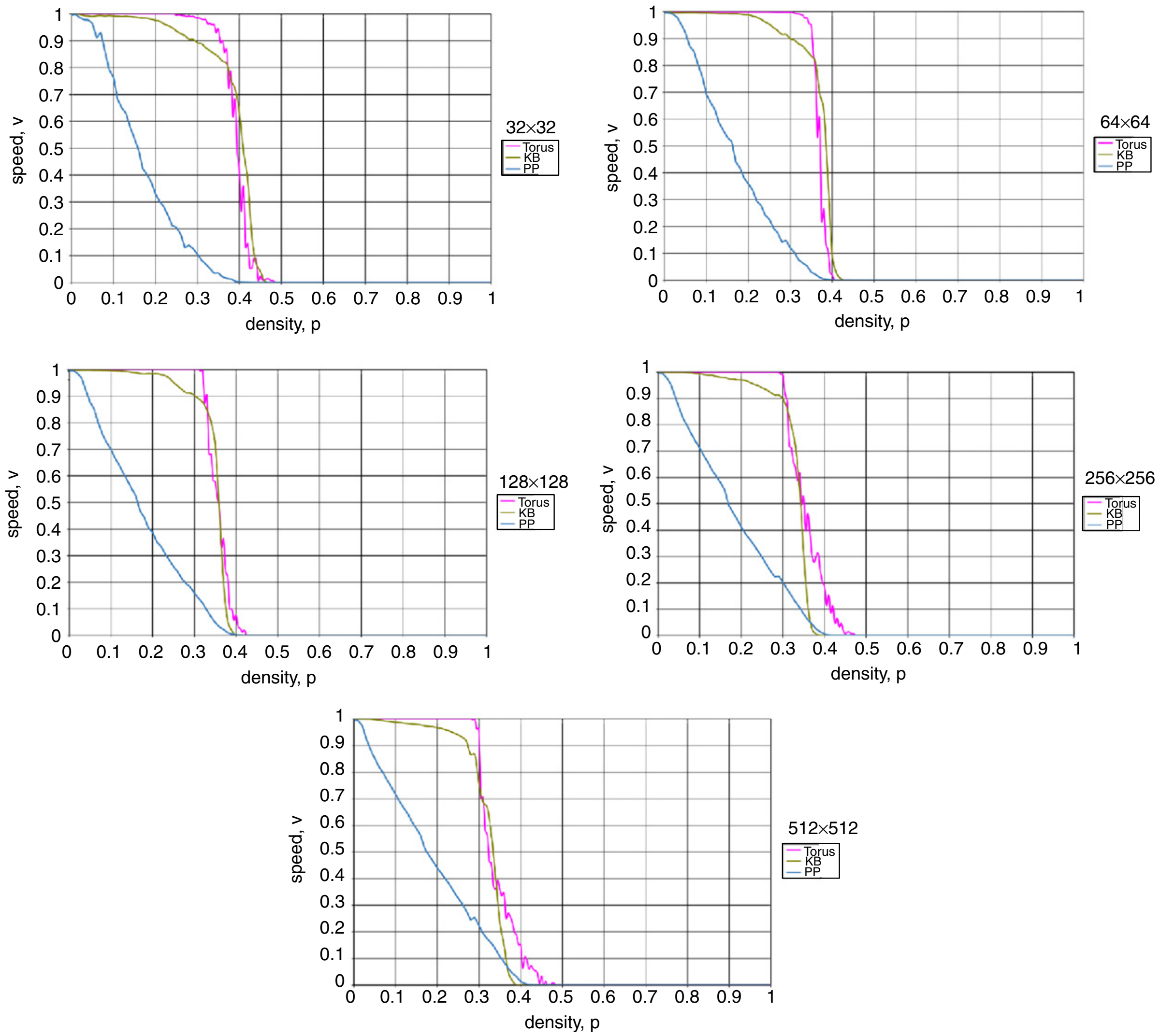


Fig. 11. Differences of speed as a function of p are observed for different topologies in several sizes: torus, Klein bottle (KB) and projective plane (PP).

in such a way that a blue particle in $(N-1, l)$ will try to move to $(0, N-l-1)$. However, the site $(l, N-1)$ is connected with $(l, 0)$, as in the torus case; hence a red particle in $(l, N-1)$ will try to move to site $(l, 0)$ in the next timestep.

In Fig. 7, an instant of a typical experiment for a value p in the region of free flow is shown. The formation of diagonal patterns from left to right is observed, but in this case, the diagonal shows some deformation because of the changes in periodic boundary conditions.

In Fig. 8 the results of the experiments for the Klein bottle are shown. Again, these graphs have been obtained as an average of 1000 experiments for every value of p. The speed for every experiment is computed as the average of the speed of the system between timesteps $t = 50\,000$ and $t = 51\,000$ and the step increment of the parameter p has been of 0.01 units.

The behavior is much more similar to the one in the torus. However, simulations show that in the Klein bottle case the speed starts to decrease sooner than that in the torus case, although after reaching the critical density, the behavior seems to be the same (see Fig. 11).

As in the torus case speeds obtained from experiments carried out for a fixed value of p in a critical region may be very different. In Fig. 5 we have a plot of the standard deviation of speeds obtained in 1000 experiments on the torus, the projective plane and on the Klein bottle. The size of the lattice is 256×256. The plots for the torus and for the Klein bottle are similar.

As in Ref. [11] for the torus case, we also report the existence of intermediate states in the Klein bottle that coexist with global jam states and almost free flow states for values of p close to the critical zone. In Fig. 9 we can see an example obtained for $N = 256$ and $p = 0, 22$. We have let run the realization untill $t = 500\,000$ to observe that speed v stabilizes around the value $v = 0, 3$. A stable local jam appears on the center, and some particles move almost in a free flow on the borders of the world producing a periodic intermediate state, the opposite behavior that we observed on the projective plane case where the local jams appeared on the border and the free flow on the center of the lattice.

To finish we also carried out the study of speeds under disparity of densities among red and blue particles. The behavior is similar to the one in the torus. We can see the results in Fig. 10.

5. Discussion

In this paper the BML model on a projective plane and a Klein bottle has been investigated by computer simulations. We have found differences between these geometries and the torus one, as can be seen in Fig. 11, where speed is shown as a function of density and different topologies are grouped by lattice size.

The existence of stable local jams on the projective plane case which may occur at any density $p > 0$ when a red particle is on the upper left site at the same time that a blue particle is on the bottom right site of the lattice seems to be the reason of the non-existence of phase transition, in contrast with the torus and Klein bottle cases. Also, the speed of the system in the projective plane case depends on the ratio between densities of red and blue particles, another difference with respect to the torus and Klein bottle case, where speed is independent of the ratio between densities of red and blue particles.

We have not studied the case when the aspect ratio of the underlying lattice is non-square; it could be of interest to consider this problem and compare the results for the projective plane and the Klein bottle with the ones obtained for the torus.

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References

[1] T. Nagatani, The physics of traffic jams, Rep. Progr. Phys. 65 (9) (2002) 1331–1386.
[2] O. Biham, A.A. Middleton, D. Levine, Self-organization and a dynamical transition in traffic flow models, Phys. Rev. A 46 (1992) 6124–6127.
[3] H.F. Chau, K.Y. Wan, K.K. Yan, An improved upper bound for critical car density of the two-dimensional Biham–Middleton–Levine traffic model, Physica A 254 (1998) 117–121.
[4] I. Benjamini, O. Gurel-Gurevich, R. Izkovsky, The Biham–Middleton–Levine traffic model for a single junction, in: In and Out Equilibrium: Probability with Physics Flavor II, in: Birkhäuser's Progress in Probability Series, vol. 60, 2008, pp. 97–116.
[5] T. Austin, I. Benjamini, For what number of cars must self-organization occur in the Biham–Middleton–Levine traffic model from any possible starting configuration? 2006. arXiv:math/0607759v3.
[6] O. Angel, A.E. Holroyd, J.B. Martin, The jammed phase of the Biham–Middleton–Levine traffic model, Electron. Comm. Probab. 10 (2005) 167–178.
[7] H.F. Chau, K.Y. Wan, Phase diagram of the Biham–Middleton–Levine traffic model in three dimensions, Phys. Rev. E 60 (1999) 5301–5304.
[8] S. Tadaki, Two dimensional cellular automaton model of traffic flow with open boundaries, Phys. Rev. E 54 (1994) 2409–2413.
[9] R.M. D'Souza, Coexisting phases and lattice dependence of a cellular automata model for traffic flow, Phys. Rev. E 71 (2005) 066112.
[10] D. Huang, W. Huang, Biham–Midleton–Levine model with four-directional traffic, Physica A 370 (2006) 747–755.
[11] N.J. Linesch, R.M. D'Souza, Periodic states, local effects and coexistence in the BML traffic jam model, Physica A 387 (2008) 6170–6176.
[12] http://www.nvidia.com.