mathematics

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Generalized Sasakian Space Forms Which Are Realized as Real Hypersurfaces in Complex Space Forms

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Received: 11 May 2020; Accepted: 26 May 2020; Published: 29 May 2020

Abstract: We prove a classification theorem of the generalized Sasakian space forms which are realized as real hypersurfaces in complex space forms.

Keywords: real hypersurfaces; complex space forms; generalized Sasakian space forms

MSC: 53B20; 53C15; 53C25

1. Introduction

A complex space form $\widetilde{M}_n(c)$ is an n-dimensional manifold with Kählerian structure $(J, \widetilde{g})$ and constant holomorphic sectional curvature c. A complete and simply connected complex space form must be a complex projective space $P_n\mathbb{C}$ (if $c > 0$), a complex Euclidean space $E_n\mathbb{C}$ (if $c = 0$), or a complex hyperbolic space $H_n\mathbb{C}$ (if $c < 0$).

If M is an oriented real hypersurface in a complex space form $\widetilde{M}_n(c)$, then M accepts an almost contact metric structure (η, ϕ, ξ, g) (see Section 2.2). Many authors have studied how to classify real hypersurfaces in complex space forms. For instance, Takagi classified the homogeneous real hypersurfaces in $P_n\mathbb{C}$ [1,2] and Cecil and Ryan [3] investigated real hypersurfaces whose structure vector field ξ is principal, that is, an eigenvector of the shape operator A of M.

A special class of real hypersurfaces are the Hopf hypersurfaces, whose structure vector field is a principal curvature vector field. Hopf hypersurfaces in complex hyperbolic spaces $H_n\mathbb{C}$ with constant principal curvatures were classified by Berndt [4]. In the case of Hopf hypersurfaces in complex projective spaces $P_n\mathbb{C}$ with constant principal curvatures, M. Kimura provided a classification [5].

On the other hand, Alegre, Blair and Carriazo introduced [6] the generalized Sasakian space forms, which are almost contact metric manifolds whose curvature tensor generalizes the expression of the curvature tensor of Sasakian space forms. A generalized Sasakian space form has many remarkable geometric properties: (1) the ϕ-sectional curvature is pointwise constant; (2) the Ricci operator S commutes with the structure operator ϕ; and (3) the structure Jacobi operator R_ξ commutes with the structure operator ϕ.

In these circumstances, it is very interesting to give a classification of the generalized Sasakian space forms which are realized as real hypersurfaces in complex space forms. Namely, we prove:

Theorem 1. *Let M be a connected generalized Sasakian space form which is realized as a real hypersurface in a complex space form $\widetilde{M}_n(c)$, $n \geq 2$.*

(I) *If $\widetilde{M}_n(c) = P_n\mathbb{C}$, then M is locally congruent to one of the following:*

   (i) *a geodesic hypersphere $G(r)$ of radius $0 < r < \frac{\pi}{2}$; or*
   (ii) *a non-homogeneous real hypersurface in $P_2\mathbb{C}$, with $A\xi = 0$.*

(II) *If $\widetilde{M}_n(c) = H_n\mathbb{C}$, then M is locally congruent to one of the following:*

   (i) *a horosphere;*
   (ii) *a geodesic hypersphere $G(r)$ of radius $r > 0$;*
   (iii) *a tube of radius $r > 0$ around a totally geodesic $H_{n-1}\mathbb{C}$; or*
   (iv) *a non-homogeneous real hypersurface in $H_2\mathbb{C}$, with $A\xi = 0$.*

(III) *If $\widetilde{M}_n(c) = E_n\mathbb{C}$, then M is locally congruent to one of the following:*

   (i) *a hyperplane $\mathbb{R}^{2n-1}$;*
   (ii) *a sphere $S^{2n-1}(r)$ of radius $r > 0$; or*
   (iii) *a cylinder over a plane curve $\gamma \times \mathbb{R}^{2n-2}$.*

2. Preliminaries

In this paper, all manifolds are supposed to be connected and of class C^∞, and all real hypersurfaces oriented.

2.1. Generalized Sasakian Space Forms

In this subsection, we include some definitions and results on almost contact metric geometry (for more details, see [7]). In particular, we present the definition of generalized Sasakian space forms and some of their properties (see [6]).

An almost contact manifold is an odd-dimensional differentiable manifold M that admits a $(1,1)$-tensor field ϕ, a vector field ξ (usually called the structure vector field or Reeb vector field), and a 1-form η satisfying

$$\phi^2 = -I + \eta \otimes \xi, \; \eta(\xi) = 1. \tag{1}$$

On an almost contact manifold, there exists a compatible Riemannian metric, i.e., a metric that satisfies

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2}$$

for all vector fields X, Y on M. Then, (η, ϕ, ξ, g) is said to be an almost contact metric structure and the manifold $M = (M; \eta, \phi, \xi, g)$ is called an almost contact metric manifold. On such a manifold, it follows from Equations (1) and (2) that

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X, \xi), \tag{3}$$

for every vector field X on M.

If an almost contact metric manifold M also satisfies $d\eta = \Phi$, where $\Phi(X, Y) = g(\phi X, Y)$ is the fundamental 2-form, then M is called a contact metric manifold. An almost contact metric manifold is called normal if $[\phi, \phi](X, Y) = -d\eta(X, Y)\xi$, for any vector fields X, Y on M, where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ. If the contact metric manifold is normal, then it is called a Sasakian manifold (see [7] for other different characterizations of those manifolds). Sasakian manifolds with constant ϕ-sectional curvature F are called Sasakian space forms and they are the analog on contact metric geometry to complex space forms in complex geometry. Every Sasakian space form has constant curvature and its curvature tensor R can be written in terms of its curvature. Moreover, any complete, simply connected Sasakian space form must be one of three known models, which have $F > -3$, $F = -3$, or $F < -3$ ([7]).

Generalized Sasakian space forms were introduced by Alegre, Blair and Carriazo [6] as almost contact metric manifolds M whose curvature tensor R can be written as

$$\begin{aligned} R(X,Y)Z = & f_1\{g(Y,Z)X - g(X,Z)Y\} \\ & + f_2\{g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\} \\ & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(X,Z)\eta(Y)\xi\}, \end{aligned} \tag{4}$$

for any vector fields X, Y, Z on M, where f_1, f_2 and f_3 are functions on M. These manifolds, denoted by $M(f_1, f_2, f_3)$, include the Sasakian space forms for $f_1 = \frac{F+3}{4}$ and $f_2 = f_3 = \frac{F-1}{4}$, where F is the constant ϕ-sectional curvature. There are also many other examples of generalized Sasakian space forms (see [6]).

On a generalized Sasakian space form $M^{2n-1}(f_1, f_2, f_3)$, given a unit vector field X, orthogonal to ξ, the ϕ-sectional curvature $K(X,\phi X) = g(R(X,\phi X)\phi X, X)$ is independent of the direction X, since $K(X,\phi X) = f_1 + 3f_2$. Moreover, the self-adjoint Jacobi operator $R_\xi = R(\cdot,\xi)\xi$ can be written as

$$R_\xi = (f_1 - f_3)(I - \eta\otimes\xi), \tag{5}$$

and the Ricci operator S as

$$S = \Big((2n-2)f_1 + 3f_2 - f_3\Big)I + \Big(-3f_2 - (2n-3)f_3\Big)\eta\otimes\xi, \tag{6}$$

where I denotes the identity.

2.2. Real Hypersurfaces in Complex Space Forms

In this subsection, we present some known results about real hypersurfaces in n-dimensional complex space forms $\widetilde{M}_n(c)$ with constant holomorphic sectional curvature c.

On a complex space form $\widetilde{M}_n(c)$, we denote by J the Hermitian structure tensor and by $\tilde{g}$ its metric, which satisfy

$$J^2 = -I, \ \tilde{g}(JX, JY) = \tilde{g}(X,Y), \ \tilde{\nabla}J = 0.$$

If M is an oriented real hypersurface in a complex space form $\widetilde{M}_n(c)$ and N is a unit normal vector on M, then we can write

$$JX = \phi X + \eta(X)N, \quad JN = -\xi, \tag{7}$$

for any any vector field X tangent to M, where ϕ is a $(1,1)$-tensor, η is a 1-form and ξ is a unit vector field on M. Let us call g the Riemannian metric on M. Then, it follows from Equation (7) that Equations (1) and (2) are satisfied, and thus (ϕ,ξ,η,g) is an almost contact metric structure on M.

Given an oriented real hypersurface (M,g) in a complex space form $(\widetilde{M}_n(c),\widetilde{g})$, we denote by $\widetilde{\nabla}$ and ∇ their respective Levi–Civita connections and write Gauss and Weingarten formulas the following way:

$$\begin{aligned} \widetilde{\nabla}_X Y &= \nabla_X Y + g(AX,Y)N, \\ \widetilde{\nabla}_X N &= -AX, \end{aligned} \tag{8}$$

for any X,Y tangent vector fields on M, where A is the shape operator. It follows from Equations (7) and (8) and $\widetilde{\nabla}J = 0$ that

$$\begin{aligned} (\nabla_X\phi)Y &= \eta(Y)AX - g(AX,Y)\xi, \\ \nabla_X\xi &= \phi AX. \end{aligned} \tag{9}$$

Therefore, Gauss and Codazzi equations are:

$$\begin{aligned} R(X,Y)Z = \frac{c}{4}\{&g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\} \\ &+ g(AY,Z)AX - g(AX,Z)AY, \end{aligned} \tag{10}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi\}. \tag{11}$$

It follows from Equation (10) that the the Ricci operator S can be written as

$$SX = \frac{c}{4}\{(2n+1)X - 3\eta(X)\xi\} - A^2X + HAX, \tag{12}$$

where H is the trace of the shape operator A. Moreover, Equation (10) also implies that the self-adjoint Jacobi operator $R_\xi = R(\cdot,\xi)\xi$ can be written as

$$R_\xi(X) = \frac{c}{4}(X - \eta(X)\xi) + g(A\xi,\xi)AX - \eta(AX)A\xi. \tag{13}$$

The homogeneous real hypersurfaces in $P_n\mathbb{C}$ were classified by Takagi [1,2]. On the other hand, Cecil and Ryan [3] studied real hypersurfaces in $P_n\mathbb{C}$ on which the structure vector field ξ is principal. Later, Kimura [5] completed the classification of Hopf hypersurfaces in $P_n\mathbb{C}$ with constant principal curvatures.

Theorem 2 ([5])**.** *Let M be a Hopf hypersurface in $P_n\mathbb{C}$, $n \geq 2$, with constant principal curvatures. Then, M must be locally congruent to one of the following:*

(A_1) *a geodesic hypersphere of radius $0 < r < \frac{\pi}{\sqrt{c}}$;*
(A_2) *a tube of radius $0 < r < \frac{\pi}{\sqrt{c}}$ around a totally geodesic $P_k\mathbb{C}$, with $1 \leq k \leq n-2$;*
(B) *a tube of radius $0 < r < \frac{\pi}{2\sqrt{c}}$ around a complex quadric Q^{n-1} and $P_n\mathbb{R}$;*
(C) *a tube of radius $0 < r < \frac{\pi}{2\sqrt{c}}$ around $P_1\mathbb{C} \times P_{\frac{n-1}{2}}\mathbb{C}$, where $n \geq 5$ and odd;*
(D) *a tube of radius $0 < r < \frac{\pi}{2\sqrt{c}}$ around the complex Grassmannanian $G_{2,5}\mathbb{C}$, with $n = 9$; or*
(E) *a tube of radius $0 < r < \frac{\pi}{2\sqrt{c}}$ around a Hermitian symmetric space $SO(10)/U(5)$, with $n = 15$.*

On the other hand, Hopf hypersurfaces in complex hyperbolic spaces $H_n\mathbb{C}$ whose principal curvatures are all constant were classified by J. Berndt [4].

Theorem 3 ([4])**.** *Let M be a Hopf hypersurface in $H_n\mathbb{C}$, $n \geq 2$, with constant principal curvatures. Then, M must be locally congruent to one of the following:*

(A_0) *a horosphere;*
$(A_{1,0})$ *a geodesic hypersphere of radius $r > 0$;*
$(A_{1,1})$ *a tube of radius $r > 0$ around a complex hyperbolic hyperplane $H_{n-1}\mathbb{C}$;*
(A_2) *a tube of radius $r > 0$ around a totally geodesic $H_k\mathbb{C}$, with $1 \leq k \leq n-2$; or*
(B) *a tube of radius $r > 0$ around a totally real hyperbolic space $H_n\mathbb{R}$.*

In the following, real hypersurfaces of type (A) denote either real hypersurfaces of type (A_1) or (A_2) in $P_n\mathbb{C}$ or real hypersurfaces of type (A_0), (A_1), or (A_2) in $H_n\mathbb{C}$. These real hypersurfaces of type (A) have many interesting geometric properties and have been studied by many authors (see, e.g., [8–12]).

3. Real Hypersurfaces in $P_n\mathbb{C}$ or $H_n\mathbb{C}$

It was shown in [13] that real hypersurfaces in non-flat complex space forms cannot be totally umbilical. This led some authors to investigate real hypersurfaces in $P_n\mathbb{C}$ or $H_n\mathbb{C}$ that are totally η-umbilical, i.e., whose shape operator A has the form

$$A = \lambda I + \mu\eta \otimes \xi,$$

for some smooth functions λ, μ on M.

Totally η-umbilical real hypersurfaces in $P_n\mathbb{C}$ and $H_n\mathbb{C}$ have been classified, as the next theorem shows.

Theorem 4 ([2,3,14,15])**.** *Let M be a real hypersurface in a non-flat complex space form $\widetilde{M}_n(c)$, $c \neq 0$. Then, M is totally η-umbilical if and only if M is locally congruent to*

(A_1) a geodesic hypersphere of radius $0 < r < \frac{\pi}{\sqrt{c}}$ in $P_n\mathbb{C}$;
(A_0) a horosphere in $H_n\mathbb{C}$;
$(A_{1,0})$ a geodesic hypersphere of radius $r > 0$ in $H_n\mathbb{C}$; or
$(A_{1,1})$ a tube of radius $r > 0$ around a complex hyperbolic hyperplane $H_{n-1}\mathbb{C}$ in $H_n\mathbb{C}$.

Thus, we have:

Proposition 1. *A totally η-umbilical real hypersurface in a non-flat complex space form $\widetilde{M}_n(c)$, $c \neq 0$, is a generalized Sasakian space form with $f_1 = \frac{c}{4} + \lambda^2$, $f_2 = \frac{c}{4}$, $f_3 = -\lambda\mu$, and ϕ-sectional curvature $F = c + \lambda^2$.*

Proof. Suppose that a real hypersurface M in a non-flat complex space form $\widetilde{M}_n(c)$, $c \neq 0$, is totally η-umbilical, that is, $A = \lambda I + \mu\eta \otimes \xi$ for smooth functions λ, μ in M. Then, from Gauss Equation (10), we have

$$\begin{aligned} R(X,Y)Z =& (\frac{c}{4} + \lambda^2)\Big(g(Y,Z)X - g(X,Z)Y\Big) \\ &+ \frac{c}{4}\Big(g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\Big) \\ &- \lambda\mu\Big(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\Big), \end{aligned} \tag{14}$$

for any vector fields X, Y, Z tangent to M. Then, from Equation (14), we easily get $f_1 = \frac{c}{4} + \lambda^2$, $f_2 = \frac{c}{4}$, $f_3 = -\lambda\mu$, and we compute the ϕ-sectional curvature $F = c + \lambda^2$. □

Among totally η-umbilical real hypersurfaces, only the following ones are Sasakian space forms of constant ϕ-sectional curvature F:

- a horosphere in $H_n\mathbb{C}$, with $F = -3$;
- a geodesic hypersphere with $r = \frac{2}{\sqrt{c}} \tan^{-1}(\frac{\sqrt{c}}{2})$ in $P_n\mathbb{C}$, with $F > 1$;
- a geodesic hypersphere with $r = \frac{2}{\sqrt{-c}} \tanh^{-1}(\frac{\sqrt{-c}}{2})$ in $H_n\mathbb{C}$, with $-3 < F < 1$; and
- a tube of radius $r = \frac{2}{\sqrt{-c}} \coth^{-1}(\frac{\sqrt{-c}}{2})$ around a complex hyperbolic hyperplane $H_{n-1}\mathbb{C}$, which is a real hypersurface in $H_n\mathbb{C}$ with $F < -3$.

The shape operator of all those hypersurfaces is $A = I - \frac{c}{4}\eta \otimes \xi$.

Lemma 1 ([16])**.** *Let M be a real hypersurface in $P_n\mathbb{C}$ and $H_n\mathbb{C}$. If M satisfies $\phi S = S\phi$ and $\phi R_\xi = R_\xi\phi$ at the same time, then $A\xi = 0$ or M is locally congruent to one of the real hypersurfaces of type (A).*

Very recently, J. T. Cho and M. Kimura [17] gave a nice geometric description of real hypersurfaces in a complex projective space when the hypersurfaces are of constant ϕ-sectional curvature. Besides geodesic hyperspheres, such real hypersurfaces are obtained as the image of either a curve or a surface in a complex projective space under the polar map.

4. Real Hypersurfaces in $E_n\mathbb{C}$

Kon [18] introduced a pseudo-Einstein (or η-Einstein) real hypersurface M in a Kählerian manifold by the condition $S = aI + b\eta \otimes \xi$, where a, b are some constants. Then, he proved:

Theorem 5 ([18])**.** *Let M be a connected complete pseudo-Einstein real hypersurface in $E_n\mathbb{C}$, $n \geq 3$. Then, M is congruent to a hyperplane $\mathbb{R}^{2n-1}$, a sphere $S^{2n-1}(r)$, a cylinder over $(2n-2)$-sphere $S^{2n-2}(r) \times \mathbb{R}$, or a cylinder over a complete plane curve $\gamma \times \mathbb{R}^{2n-2}$.*

We prove in the next result that one of those cases is not possible.

Proposition 2. *A cylinder over $(2n-2)$-sphere $S^{2n-2}(r) \times \mathbb{R}$ does not admit a pseudo-Einstein structure.*

Proof. For $S^{2n-2}(r) \times \mathbb{R} \subset \mathbb{R}^{2n-1} \times \mathbb{R} = \mathbb{C}^{n-1} \times \mathbb{R} \times \mathbb{R} = \mathbb{C}^n$, we have a position vector $x \in ((\cos\theta p, r\sin\theta), t)$ for $p \in S^{2n-3}(r) \subset \mathbb{C}^{n-1}$ and $t \in \mathbb{R}$. Then, we have a unit normal vector field $N = (-(\cos\theta p, r\sin\theta), 0)$, and we have $\xi = -JN = ((\cos\theta ip, 0), r\sin\theta)$. Then, we have:

- when $\theta = 0$, $\xi = ((ip, 0), 0) \in T_{(p,0)}S^{2n-2}(r) \oplus \{0\}$;
- when $\theta = \frac{\pi}{2}$, $\xi = ((0, 0), r) \in \{0\} \oplus T_t\mathbb{R}$; and
- when $0 < \theta < \frac{\pi}{2}$, $\xi = ((\cos\theta ip, 0), r\sin\theta) \notin (T_{(p,0)}S^{2n-2}(r) \oplus \{0\}) \cup (\{0\} \oplus T_t\mathbb{R})$.

On the other hand, it is known that, for $S^{2n-2}(r) \times \mathbb{R} \subset \mathbb{C}^n$,

$$A(X, 0) = \frac{1}{r}(X, 0), \quad A(0, Y) = (0, 0), \tag{15}$$

where $(X, 0) \in T_{(\tilde{p},t)}M = T_{\tilde{p}}S^{2n-2} \oplus T_t\mathbb{R}$. Hence, we have

$$A\xi = (\frac{1}{r}(\cos\theta ip, 0), 0). \tag{16}$$

We have:

- when $\theta = 0$, $A\xi = \frac{\xi}{r}$;
- when $\theta = \frac{\pi}{2}$, $A\xi = 0\xi$; and
- when $0 < \theta < \frac{\pi}{2}$, $A\xi = (\frac{1}{r}(\cos\theta ip, 0), 0)$ (which is not parallel to ξ).

Thus, we find that $S^{2n-2}(r) \times \mathbb{R} \subset \mathbb{C}^n$ is not a Hopf hypersurface. From Equations (12) and (15), we compute the Ricci operator:

$$S(X, 0) = \frac{2n-3}{r^2}(X, 0), \quad S(0, Y) = (0, 0). \tag{17}$$

Then, we have:

- when $\theta = 0$, $S\xi = \frac{\xi}{r^2}$;
- when $\theta = \frac{\pi}{2}$, $S\xi = 0\xi$; and
- when $0 < \theta < \frac{\pi}{2}$, $S\xi = (\frac{2n-3}{r^2}(\cos\theta ip, 0), 0)$ (which is not parallel to ξ).

Therefore, M is not a pseudo-Einstein hypersurface. □

The other spaces that appeared in Theorem 5 become generalized Sasakian space forms:

- hyperplanes $M = \mathbb{R}^{2n-1}$: $A = 0$; $f_1 = f_2 = f_3 = 0$;
- spheres $S^{2n-1}(r)$: $A = \frac{1}{r}I$; $f_1 = \frac{1}{r^2}, f_2 = f_3 = 0$; and

- cylinders over complete plane curves $\gamma \times \mathbb{R}^{2n-2}$: $A = \lambda I_1 \oplus 0$ for some function on λ; $f_1 = f_2 = f_3 = 0$.

Hence, we have:

Proposition 3. *Pseudo-Einstein real hypersurfaces of a complex Euclidean space $E_n\mathbb{C}$ are generalized Sasakian space forms.*

We can now prove our main theorem.

Proof of the Theorem 1. We divide our arguments into two parts: (I) $\widetilde{M}_n(c) = P_n\mathbb{C}$ or $H_n\mathbb{C}$; and (II) $\widetilde{M}_n(c) = E_n\mathbb{C}$.

(I) Suppose that a real hypersurface M in $P_n\mathbb{C}$ and $H_n\mathbb{C}$ is a generalized Sasakian space form $M(f_1, f_2, f_3)$. Then, from Equations (5) and (6), we easily find that the Ricci operator S and the structure Jacobi operator R_ξ simultaneously commute with ϕ. Then, due to Lemma 1, we have that either $A\xi = 0$ or that M is locally congruent to one of the real hypersurfaces of type (A).

In Theorem 4 and Proposition 1, we already find that a real hypersurface of type (A_1) in $P_n\mathbb{C}$ and a real hypersurface of type (A_0), $(A_{1,0})$, and $(A_{1,1})$ are generalized Sasakian space forms. Taking a look at the case of type (A_2) in $P_n\mathbb{C}$ or $H_n\mathbb{C}$, we see that a real hypersurface M of type (A_2) is not of constant ϕ-sectional curvature (cf. [17,19]).

Now, we look at the case of Hopf hypersurfaces with $\alpha = 0$. From Equation (6), we find that $S = aI + b\eta \otimes \xi$, where a, b are smooth functions on M. Due to results in [3,18,20] (for the case $P_n\mathbb{C}$) and [14,21] (for the case $H_n\mathbb{C}$), we can see that M is locally congruent to a non-homogeneous real hypersurface with $A\xi = 0$ in $P_2\mathbb{C}$ or $H_2\mathbb{C}$. Indeed, the Ricci operator S of such spaces is represented by $S = \frac{3}{2}cI - c\eta \otimes \xi$, and then they are generalized Sasakian space forms with $f_1 = \frac{5}{4}c$, $f_2 = 0$, $f_3 = c$.

(II) Let $M(f_1, f_2, f_3)$ be a real hypersurface generalized Sasakian space form in $E_n\mathbb{C}$. Then, since $S = aI + b\eta \otimes \xi$, where a, b are smooth functions on M, using the arguments in the proof of Theorem 5, we find that M is either an Einstein manifold or a Hopf hypersurface ($A\xi = \alpha\xi$) with a constant α.

If M is an Einstein real hypersurface in $E_n\mathbb{C}$, then M is locally congruent to a sphere, a hyperplane, or a cylinder over a plane curve (cf. [22]).

Next, we consider the case M is a Hopf hypersurface with constant α. Then, from Equation (13), we have $R_\xi X = \alpha AX - \alpha^2\eta(X)\xi$. However, since $R_\xi X = (f_1 - f_3)(X - \eta(X)\xi)$, we have

$$\alpha AX = (f_1 - f_3)X + (\alpha^2 - f_1 + f_3)\eta(X)\xi.$$

Therefore, either M is totally η-umbilical, that is, $A = \lambda I + (\alpha - \lambda)\eta \otimes \xi$, where $\lambda = \frac{f_1 - f_3}{\alpha}$ or $A\xi = 0$ (and $f_1 = f_3$). We first look at the case $A = \lambda I + (\alpha - \lambda)\eta \otimes \xi$, where $\lambda = \frac{f_1 - f_3}{\alpha}$. Since $A\xi = \alpha\xi$ and α is constant, we have that $\nabla_\xi A = -A\phi A + \alpha\phi A$, where we use the second equation of Equation (9) and Equation (11). From the fact that $\nabla_\xi A$ is self-adjoint, it follows that $2A\phi A = \alpha(\phi A + A\phi)$. Thus, we have $\lambda(\lambda - \alpha) = 0$. Then, we have that M is totally umbilical and then Einstein or $A = 0 \oplus \alpha I_1$. Due to the theorem of Hartman–Nirenberg [23] (see also [24]), the latter case yields that M is locally a cylinder $\gamma \times \mathbb{R}^{2n-2}$ over a plane curve γ. Finally, we should treat the case $A\xi = 0$. Then, we have $f_1 = f_3$ and, by Equation (6), the Ricci operator $S = ((2n-3)f_1 + f_2)(I - \eta \otimes \xi)$. From Equation (12), we get $S = HA - A^2$. Hence, we have $A^2X - HAX + ((2n-3)f_1 + f_2)(X - \eta(X)\xi) = 0$ for any vector field X tangent to M. However, since $\alpha = 0$, we also find that $A\phi A = 0$. Thus, we obtain that $((2n-3)f_1 + f_2)A = 0$, so every point is totally geodesic or Ricci flat. Then, we have that M is a hyperplane $\mathbb{R}^{2n-1}$ or a cylinder $\gamma \times \mathbb{R}^{2n-2}$ over a plane curve γ. Therefore, the proof of our main theorem is completed. $\square$

Author Contributions: Writing—original draft, A.C., J.T.C. and V.M.-M. All authors have read and agreed to the published version of the manuscript.

Funding: A. Carriazo was partially supported by the group FQM-327 of the Junta de Andalucía (Spain) and by the H2020-MSCA-RISE-2017 grant agreement No. 778035 PDE-GIR (European Commission). J.T. Cho was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2019R1F1A1040829). V. Martín-Molina was partially supported by the group FQM-226 of the Junta de Andalucía (Spain) and by the grant VIPPIT-IV.4 of the Universidad de Sevilla (Spain).

Conflicts of Interest: The authors declare no conflict of interest.

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