THE JOURNAL OF SYMBOLIC LOGIC
Volume 72, Number 3, Sept. 2007

# A NOTE ON $\Sigma_1$-MAXIMAL MODELS

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**Abstract.** Let $T$ be a recursive theory in the language of first order Arithmetic. We prove that if $T$ extends: (a) the scheme of parameter free $\Delta_1$-minimization (plus $exp$), or (b) the scheme of parameter free $\Pi_1$-induction, then there are no $\Sigma_1$-maximal models with respect to $T$. As a consequence, we obtain a new proof of an unpublished theorem of Jeff Paris stating that $\Sigma_1$-maximal models with respect to $\mathbf{I}\Delta_0 + exp$ do not satisfy the scheme of $\Sigma_1$-collection $\mathbf{B}\Sigma_1$.

**§1. The main result.** We work in the usual language of first order Arithmetic $\mathscr{L} = \{0, 1, +, \cdot, \leq\}$. We assume that the reader is familiar with the basic notions of first order Arithmetic (we recommend the texts [4] and [6] for a detailed introduction to the subject). This note is motivated by the following question posed by Zofia Adamowicz:

PROBLEM 1. *Do $\Sigma_1$-maximal models with respect to $\mathbf{I}\Delta_0 + exp$ satisfy $\Sigma_1$-collection?*

We say that $\mathfrak{A}$ is $\Sigma_1$-maximal with respect to (w.r.t.) a theory $T$ if $\mathfrak{A} \models T$, and, for every $\mathfrak{B} \models T$,

$$\mathfrak{A} \prec_0 \mathfrak{B} \Longrightarrow \mathfrak{A} \prec_1 \mathfrak{B}.$$

The notion of a $\Sigma_1$-maximal model is the arithmetic counterpart of the classic model-theoretic concept of an *existentially closed model.* Observe that $\Sigma_1$-maximal models w.r.t. $\mathbf{I}\Delta_0 + exp$ do exist. In fact, it is well-known (see, e.g., remark 1.2 of [2]) that if $T \subseteq \Pi_2$ then every model of $T$ can be 0-elementary extended to a $\Sigma_1$-maximal model w.r.t. $T$. Concerning Problem 1, Adamowicz first observed that *there are* $\Sigma_1$-maximal models w.r.t. $\mathbf{I}\Delta_0 + exp$ in which $\Sigma_1$-collection fails; and Paris obtained a complete answer to the problem (unpublished) by proving that $\Sigma_1$-collection fails in *every* $\Sigma_1$-maximal model w.r.t. $\mathbf{I}\Delta_0 + exp$. In this note we present a new proof of Paris' result. Indeed, we shall obtain a more general result: $\Sigma_1$-maximal models w.r.t. $\mathbf{I}\Delta_0 + exp$ do not satisfy the scheme of minimization for parameter free $\Delta_1$-formulas $\mathbf{L}\Delta_1^-$.

MAIN THEOREM. *Let $T$ be a recursive theory and let $\mathfrak{A}$ be a $\Sigma_1$-maximal model w.r.t. $T$. Then $\mathfrak{A}$ is not a model of $\mathbf{L}\Delta_1^- + exp$.*

Taking $T = \mathbf{I}\Delta_0 + exp$, we obtain Paris' result since $\mathbf{B}\Sigma_1 \equiv \mathbf{L}\Delta_1 \vdash \mathbf{L}\Delta_1^-$.

Received September 13, 2006.
Research partially supported by grants MTM2005-08658 (Ministerio de Educación y Ciencia, Spain) and TIC-137 (Junta de Andalucía, Spain)


0022-4812/07/7203-0020/$1.70

PROOF OF THE MAIN THEOREM. By contradiction, assume that $\mathfrak{A}$ is a model of $\mathbf{L}\Delta_1^- + exp$. We first need the following two general properties of $\Sigma_1$-maximal models.

LEMMA 1. *If $\mathfrak{A}$ is $\Sigma_1$-maximal w.r.t. $T$, then $\mathfrak{A}$ is also $\Sigma_1$-maximal w.r.t. the $\Pi_1$-consequences of $T$, $Th_{\Pi_1}(T)$.*

PROOF OF LEMMA 1. Let $\mathfrak{B}$ be a model of $Th_{\Pi_1}(T)$ satisfying $\mathfrak{A} \prec_0 \mathfrak{B}$. We must show that $\mathfrak{A} \prec_1 \mathfrak{B}$. Since $\mathfrak{B} \models Th_{\Pi_1}(T)$, there is $\mathfrak{C} \models T$ such that $\mathfrak{B} \prec_0 \mathfrak{C}$ and hence $\mathfrak{A} \prec_0 \mathfrak{C}$. Since $\mathfrak{A}$ is $\Sigma_1$-maximal w.r.t. $T$, $\mathfrak{A} \prec_1 \mathfrak{C}$. But it is immediate to verify that it follows from $\mathfrak{A} \prec_0 \mathfrak{B} \prec_0 \mathfrak{C}$ and $\mathfrak{A} \prec_1 \mathfrak{C}$ that $\mathfrak{A} \prec_1 \mathfrak{B}$. ⊣

LEMMA 2. *If $\mathfrak{A}$ is $\Sigma_1$-maximal w.r.t. $T$, then $T + D_{\Pi_0}(\mathfrak{A}) \vdash Th_{\Sigma_2}(\mathfrak{A})$.*

PROOF OF LEMMA 2. Recall that $D_{\Pi_0}(\mathfrak{A})$ stands for the $\Pi_0$-diagram of $\mathfrak{A}$, and $Th_{\Sigma_2}(\mathfrak{A})$ stands for the set of all $\Sigma_2$-sentences valid in $\mathfrak{A}$. Let $\mathfrak{B}$ be a model of $T + D_{\Pi_0}(\mathfrak{A})$. Then, there exists $\mathfrak{C} \cong \mathfrak{B}$ satisfying $\mathfrak{A} \prec_0 \mathfrak{C}$ and $\mathfrak{C} \models T$. Since $\mathfrak{A}$ is $\Sigma_1$-maximal w.r.t. $T$, $\mathfrak{A} \prec_1 \mathfrak{C}$, and, hence, $\mathfrak{C} \models Th_{\Sigma_2}(\mathfrak{A})$. Consequently, $\mathfrak{B} \models Th_{\Sigma_2}(\mathfrak{A})$ and the result follows. ⊣

We now return to the proof of the Main Theorem.

Since $\mathfrak{A}$ satisfies $\mathbf{L}\Delta_1^-$ and $\mathbf{L}\Delta_1^-$ is axiomatized by a set of $\Sigma_2$-sentences, from Lemmas 1 and 2 it follows that $Th_{\Pi_1}(T) + D_{\Pi_0}(\mathfrak{A}) \vdash \mathbf{L}\Delta_1^-$. Hence, there are a sequence of elements of $\mathfrak{A}$, $\{a_i : i \in \omega\}$, and a sequence of $\Delta_0$-formulas of $\mathscr{L}$, $\{\delta_i(x_0, \dots, x_i) : i \in \omega\}$, satisfying

(i) $\mathfrak{A} \models \delta_i(a_0, \dots, a_i)$, for all $i \in \omega$; and

(ii) $Th_{\Pi_1}(T) + \{\delta_i(a_0, \dots, a_i) : i \in \omega\} \vdash \mathbf{L}\Delta_1^-$.

Define $\Theta$ to be the following set of $\Sigma_1$-sentences:

$$\{\exists x_0 \dots \exists x_i \; (\bigwedge_{k=0}^{i} \delta_k(x_0, \dots, x_k)) : i \in \omega\}.$$

By (i), $\mathfrak{A} \models \Theta$; so $Th_{\Pi_1}(T) + exp + \Theta$ is consistent. In addition, we have

LEMMA 3. $Th_{\Pi_1}(T) + \Theta \vdash \mathbf{L}\Delta_1^-$.

PROOF OF LEMMA 3. Assume $\mathfrak{B} \models Th_{\Pi_1}(T) + \Theta$. We must show that $\mathfrak{B}$ is a model of $\mathbf{L}\Delta_1^-$. Let $\varphi$ be an axiom of $\mathbf{L}\Delta_1^-$. By compactness, from (ii) it follows that there is $m \in \omega$ such that

$$Th_{\Pi_1}(T) + \delta_0(a_0) + \delta_1(a_0, a_1) + \dots + \delta_m(a_0, \dots, a_m) \vdash \varphi. \qquad (\bullet)$$

Since $\mathfrak{B} \models \Theta$, $\mathfrak{B} \models \exists x_0 \dots \exists x_m \; (\bigwedge_{k=0}^{m} \delta_k(x_0, \dots, x_k))$ and hence there exist $b_0, \dots, b_m \in \mathfrak{B}$ such that $\mathfrak{B} \models \bigwedge_{k=0}^{m} \delta_k(b_0, \dots, b_k)$. Let $\mathfrak{B}'$ be the expansion of $\mathfrak{B}$ obtained by interpreting the constant symbols $a_0, \dots, a_m$ as the elements $b_0, \dots, b_m$, respectively. By $(\bullet)$, $\mathfrak{B}' \models \varphi$ and consequently $\mathfrak{B} \models \varphi$. ⊣

So, $\Theta$ is a set of $\Sigma_1$-sentences satisfying that $Th_{\Pi_1}(T) + exp + \Theta$ is consistent and implies $\mathbf{L}\Delta_1^-$. However, by lemma 4.2 and theorem 4.14 of [3], this is impossible. For the sake of completeness, we include here a proof of this fact.

LEMMA 4. [3] *Let $T$ be a recursive theory. There is no set of $\Sigma_1$-sentences of $\mathscr{L}$, $\Gamma$, satisfying that $Th_{\Pi_1}(T) + exp + \Gamma$ is a consistent extension of $\mathbf{L}\Delta_1^-$.*

PROOF OF LEMMA 4. We may assume that $T$ implies $\mathbf{I}\Delta_0$. Suppose that $\Gamma \subseteq \Sigma_1$ and $Th_{\Pi_1}(T) + exp + \Gamma$ is consistent. Firstly, observe that

CLAIM. *$Th_{\Pi_1}(T) + exp + \Gamma$ does not imply the set of all true $\Pi_1$-sentences, $Th_{\Pi_1}(\mathcal{N})$.*

PROOF is by contradiction. Assume $Th_{\Pi_1}(T) + exp + \Gamma \vdash Th_{\Pi_1}(\mathcal{N})$ and let $\mathfrak{B}$ be a model of $Th_{\Pi_1}(T) + exp + \Gamma$. Then, $\mathcal{N} \prec_1 \mathfrak{B}$ since $\mathfrak{B} \models Th_{\Pi_1}(\mathcal{N})$. Consequently, $\Gamma$ is a set of true $\Sigma_1$-sentences and hence $\mathbf{I}\Delta_0 \vdash \Gamma$. So, $Th_{\Pi_1}(T) + exp \vdash Th_{\Pi_1}(\mathcal{N})$, which is impossible since $Th_{\Pi_1}(\mathcal{N})$ is a $\Pi^0_1$-complete set and $T$ is a recursive theory. ⊣

By the Claim, there is $\mathfrak{B} \models Th_{\Pi_1}(T) + exp + \Gamma$ satisfying $\mathfrak{B} \not\models Th_{\Pi_1}(\mathcal{N})$. Hence, the submodel of the $\Sigma_1$-definable elements of $\mathfrak{B}$, $\mathcal{K}_1(\mathfrak{B})$, is nonstandard. Moreover, it holds that

(i) $\mathcal{K}_1(\mathfrak{B}) \models Th_{\Pi_1}(T) + exp + \Gamma$ since $\mathcal{K}_1(\mathfrak{B}) \prec_1 \mathfrak{B}$.

(ii) $\mathcal{K}_1(\mathfrak{B}) \not\models \mathbf{L}\Delta_1^-$. Assume that $\mathcal{K}_1(\mathfrak{B})$ satisfies $\mathbf{L}\Delta_1^-$. It easily follows from the fact every element of $\mathcal{K}_1(\mathfrak{B})$ is $\Sigma_1$-definable that $\mathcal{K}_1(\mathfrak{B})$ also satisfies $\mathbf{L}\Delta_1$. However, by a well-known theorem of [8], if $\mathfrak{B} \models \mathbf{I}\Delta_0 + exp$ and $\mathcal{K}_1(\mathfrak{B})$ is nonstandard, then $\mathbf{B}\Sigma_1 (\equiv \mathbf{L}\Delta_1)$ fails in $\mathcal{K}_1(\mathfrak{B})$. ⊣

This completes the proof of the Main Theorem. ⊣

We conclude this section with some remarks.

(a) Observe that the totality of the exponentiation is only needed for the proof of Lemma 4; namely, in order to ensure that the submodel of the $\Sigma_1$-definable elements $\mathcal{K}_1(\mathfrak{B})$ does not satisfy $\mathbf{B}\Sigma_1$. Moreover, by a result of Kaye–Paris–Dimitracopoulos (see theorem 2.9 of [7]), if $\mathfrak{B}$ is a model of the scheme of induction for parameter free $\Pi_1$-formulas $\mathbf{I}\Pi_1^-$ then $\mathcal{K}_1(\mathfrak{B}) \models exp$ and, so, $\mathcal{K}_1(\mathfrak{B})$ does not satisfy $\mathbf{B}\Sigma_1$. Consequently, the proof of the Main Theorem also gives us (recall that $\mathbf{I}\Pi_1^- \vdash \mathbf{L}\Delta_1^-$ and $\mathbf{I}\Pi_1^-$ is $\Sigma_2$-axiomatized):

COROLLARY 1. *If $T$ is recursive, then $\Sigma_1$-maximal models w.r.t. $T$ do not satisfy $\mathbf{I}\Pi_1^-$.*

In particular, $\mathbf{I}\Pi_1^-$ fails in every $\Sigma_1$-maximal model w.r.t. $\mathbf{I}\Delta_0$.

(b) From the Main Theorem and Corollary 1 we can derive the following improvement of corollary 1.6 in [2], where the authors proved that there are no $\Sigma_1$-maximal models w.r.t. $T$ if $T$ is recursive and $\mathbf{I}\Pi_1^- + exp \subseteq T$.

THEOREM 1. *Let $T$ be a recursive theory.*

1. *If $\mathbf{L}\Delta_1^- + exp \subseteq T$ then there are no $\Sigma_1$-maximal models w.r.t. $T$.*
2. *If $\mathbf{I}\Pi_1^- \subseteq T$ then there are no $\Sigma_1$-maximal models w.r.t. $T$.*

(c) The arguments in the proof rapidly generalize to any $n > 0$. Namely, we have (observe that $\mathbf{L}\Delta_{n+1}^- \vdash \mathbf{I}\Sigma_1^- \vdash exp$ for any $n > 0$):

THEOREM 2 ($n > 0$). *Let $T$ be a recursive theory. Then $\Sigma_{n+1}$-maximal models w.r.t. $T$ do not satisfy $\mathbf{L}\Delta_{n+1}^-$.*

As a corollary we obtain that $\Sigma_{n+1}$-maximal models w.r.t. $\mathbf{I}\Sigma_n$ do not satisfy the $\Sigma_{n+1}$-collection scheme $\mathbf{B}\Sigma_{n+1}$ for any $n > 0$.

(d) Observe that we only make use of particular properties of $\mathbf{L}\Delta_1^-$ in the proof of Lemma 4. For the rest of the proof of the Main Theorem, the only property of this theory that we use is the fact that it is $\Sigma_2$-axiomatized. Hence, we have:

PROPOSITION 1. *If $T' \subseteq \Sigma_2$ and there is no set of $\Sigma_1$-sentences of $\mathscr{L}$, $\Gamma$, such that $Th_{\Pi_1}(T) + \Gamma$ is a consistent extension of $T'$, then $T'$ fails in every $\Sigma_1$-maximal model w.r.t. $T$.*

The use of the set $\Gamma \subseteq \Sigma_1$ cannot be dropped in the result above. For instance, if we put $T = \mathbf{I}\Delta_0$ and $T' = \mathbf{I}\Delta_0 + \neg\, exp$, then $T' \subseteq \Sigma_2$, $T'$ has no consistent $\Pi_1$-extensions and every $\Sigma_1$-maximal model w.r.t. $T$ *does* satisfy $T'$, see Theorem 4 below (another counterexample can be obtained defining $T = T' = Th_{\mathscr{B}(\Sigma_1)}(\mathbf{I}\Delta_0 + exp)$, where $\mathscr{B}(\Sigma_1)$ denotes the set of boolean combinations of $\Sigma_1$-sentences). Even more, it is easy to check that Proposition 1 is best possible in the following sense:

PROPOSITION 2. *Suppose $T' \subseteq \Sigma_2$. The following properties are equivalent:*

(1) *There is a $\Sigma_1$-maximal model w.r.t. $Th_{\Pi_2}(T)$ satisfying $T'$.*
(2) *There is $\Gamma \subseteq \Sigma_1$ such that $Th_{\Pi_1}(T) + \Gamma$ is a consistent extension of $T'$.*
(3) *There is $\Gamma \subseteq \Sigma_1$ such that $Th_{\Pi_2}(T) + \Gamma$ is a consistent extension of $T'$.*

Notice that Lemma 4 says that property (3) fails for $T' = \mathbf{L}\Delta_1^- + exp$ and $T$ any recursive theory. Interestingly, in [3] we introduced the general notion of the *type* of a theory that constitutes a sufficient condition for property (3) to fail. If $k, m \geq 1$ and $S$ has consistent $\Pi_k$-extensions, then we say that $S$ is of *type* $k \rightarrow m$ if for every $\Pi_k$-extension of $S$, $S'$, the set of all true $\Pi_m$-sentences is contained in $S'$. Reasoning as in the Claim in Lemma 4, it is easy to show that if $T$ is recursive and $T'$ is of type $2 \rightarrow 1$ then property (3) fails for $T$ and $T'$. Consequently, we can now formulate the Main Theorem in a more general form as follows:

THEOREM 3. *If $T$ is recursive, $T' \subseteq \Sigma_2$ and $T'$ is of type $2 \rightarrow 1$, then $T'$ fails in every $\Sigma_1$-maximal model w.r.t. $T$.*

(e) It is open whether the totality of the exponentiation can be dropped in the Main Theorem, that is to say,

PROBLEM 2. *Do $\Sigma_1$-maximal models w.r.t.* $\mathbf{I}\Delta_0$ *satisfy $\Sigma_1$-collection*?

Observe that by Lemma 1 and $\Pi_2$-conservativity between $\mathbf{B}\Sigma_1$ and $\mathbf{I}\Delta_0$ Problem 2 can be restated as: *Do there exist $\Sigma_1$-maximal models w.r.t. $\boldsymbol{B}\Sigma_1$*? This seems to be a hard question since it is connected with the End Extension Problem asking if every countable model of $\mathbf{B}\Sigma_1$ has a proper end extension to a model of $\mathbf{I}\Delta_0$. Concretely, it holds that

PROPOSITION 3. *Suppose that $\mathfrak{A}$ is $\Sigma_1$-maximal w.r.t. $\mathbf{I}\Delta_0$. Then $\mathfrak{A}$ does not have proper end extensions to a model of* $\mathbf{I}\Delta_0$.

PROOF. By contradiction, assume that there is $\mathfrak{B} \models \mathbf{I}\Delta_0$ such that $\mathfrak{A} \prec_0^e \mathfrak{B}$ and $\mathfrak{A} \neq \mathfrak{B}$. Since $\mathfrak{A}$ is $\Sigma_1$-maximal w.r.t. $\mathbf{I}\Delta_0$, $\mathfrak{A} \prec_1^e \mathfrak{B}$. By theorem B of [8] it follows from $\mathfrak{A} \prec_1^e \mathfrak{B} \models \mathbf{I}\Delta_0$ and $\mathfrak{A} \neq \mathfrak{B}$ that $\mathfrak{A}$ is a model of $\mathbf{B}\Sigma_2$. Hence, $\mathfrak{A}$ is a $\Sigma_1$-maximal model w.r.t. $\mathbf{I}\Delta_0$ satisfying $\mathbf{L}\Delta_1^- + exp$, which is impossible. ⊣

Therefore, if there exists *some* countable $\Sigma_1$-maximal model w.r.t. $\mathbf{I}\Delta_0$ satisfying $\mathbf{B}\Sigma_1$, then the End Extension Problem has the negative answer. In addition, proving that $\mathbf{B}\Sigma_1$ fails in *every* (countable) $\Sigma_1$-maximal model w.r.t. $\mathbf{I}\Delta_0$ gives the negative answer to a related question raised by Wilkie and Paris in [9] asking if $\mathbf{I}\Delta_0 + \neg\, exp$ implies $\mathbf{B}\Sigma_1$. Concretely,

PROPOSITION 4. *Suppose that* $\mathbf{I}\Delta_0 + \neg\, exp$ *implies* $\mathbf{B}\Sigma_1$. *Then there is a $\Sigma_1$-maximal model w.r.t.* $\mathbf{I}\Delta_0$ *satisfying* $\mathbf{B}\Sigma_1$.

PROOF. Let $\varphi$ be a $\Pi_1$-sentence provable in $\mathbf{I}\Delta_0 + exp$ but not in $\mathbf{I}\Delta_0$. Let $\mathfrak{A}$ be a model of $\mathbf{I}\Delta_0 + \neg\varphi$. Then there is $\mathfrak{B}$ such that $\mathfrak{A} \prec_0 \mathfrak{B}$ and $\mathfrak{B}$ is $\Sigma_1$-maximal w.r.t. $\mathbf{I}\Delta_0$. Since $\mathfrak{A} \prec_0 \mathfrak{B}$, $\mathfrak{B} \models \neg\varphi$. Hence, $\mathfrak{B}$ is a $\Sigma_1$-maximal model w.r.t. $\mathbf{I}\Delta_0$ satisfying $\mathbf{B}\Sigma_1$ since $\mathbf{I}\Delta_0 + \neg\varphi \vdash \mathbf{I}\Delta_0 + \neg\, exp \vdash \mathbf{B}\Sigma_1$. ⊣

Observe that combining Propositions 3 and 4 we obtain another proof of a concluding remark in [9] stating that either the End Extension Problem has the negative answer or $\mathbf{I}\Delta_0 + \neg\, exp$ does not imply $\mathbf{B}\Sigma_1$.

**§2. $\Sigma_1$-maximal models w.r.t. $\mathbf{I}\Delta_0$.** In Section 1 we have obtained that $\Sigma_1$-maximal models w.r.t. $\mathbf{I}\Delta_0$ do not satisfy $\mathbf{I}\Pi_1^-$ (see Corollary 1). In this section we shall prove that $\Sigma_1$-maximal models w.r.t. $\mathbf{I}\Delta_0$ do not satisfy the $\Pi_1$-consequences of $\mathbf{I}\Delta_0 + exp$, $Th_{\Pi_1}(\mathbf{I}\Delta_0 + exp)$. This is a stronger result since $Th_{\Pi_1}(\mathbf{I}\Delta_0 + exp) \subset \mathbf{I}\Pi_1^-$ by theorem 2.9 of [7]. We shall derive this result from a theorem of Adamowicz in [1] stating that any *maximal theory* w.r.t. $\mathbf{I}\Delta_0$ is inconsistent with $exp$. Recall that a theory $S$ is said to be maximal w.r.t. $T$ if $S \subseteq \Sigma_1$ and $S$ is maximal consistent with $T$, that is, there is no $\Sigma_1$-sentence consistent with $S + T$ which is not already provable in $S$. The key observation is the following lemma relating $\Sigma_1$-maximal models and maximal theories.

LEMMA 5. *Suppose $T \subseteq \Pi_2$ and $S$ is a set of $\Sigma_1$-sentences. The following are equivalent*:

(a) *$S$ is maximal w.r.t. $T$.*

(b) *There exists $\mathfrak{A} \models T$ such that $\mathfrak{A}$ is $\Sigma_1$-maximal w.r.t. $T$ and $Th_{\Sigma_1}(\mathfrak{A}) = S$.*

PROOF. (a $\Rightarrow$ b): Let $\mathfrak{B}$ be a model of $S + T$. Since $T \subseteq \Pi_2$, there is $\mathfrak{A} \models T$ satisfying $\mathfrak{B} \prec_0 \mathfrak{A}$ and $\mathfrak{A}$ is $\Sigma_1$-maximal w.r.t. $T$. Then $\mathfrak{A} \models S + T$ and hence $S \subseteq Th_{\Sigma_1}(\mathfrak{A})$. Moreover, since $Th_{\Sigma_1}(\mathfrak{A})$ is consistent with $T + S$ and $S$ is maximal w.r.t. $T$, $S = Th_{\Sigma_1}(\mathfrak{A})$.

(b $\Rightarrow$ a): Let $\mathfrak{A}$ be a model of $T$ satisfying that $\mathfrak{A}$ is $\Sigma_1$-maximal w.r.t. $T$ and $Th_{\Sigma_1}(\mathfrak{A}) = S$. Clearly, $S + T$ is consistent. To see that $S$ is maximal consistent with $T$, it is enough to prove that $Th_{\Sigma_1}(\mathfrak{B}) \subseteq S$ for every $\mathfrak{B} \models S + T$. Assume that $\mathfrak{B}$ is a model of $S + T$. Then

CLAIM. *$T + D_{\Pi_0}(\mathfrak{A}) + D_{\Pi_0}(\mathfrak{B})$ is consistent.*

PROOF is by contradiction. If not, there exist $\vec{a} \in \mathfrak{A}$ and $\varphi(\vec{x}) \in \Pi_0$ such that $\mathfrak{A} \models \varphi(\vec{a})$ and $T + D_{\Pi_0}(\mathfrak{B}) \vdash \neg\varphi(\vec{a})$. Then $T + D_{\Pi_0}(\mathfrak{B}) \vdash \forall\vec{x}\,\neg\varphi(\vec{x})$ and, as a consequence, $\mathfrak{B} \models \forall\vec{x}\,\neg\varphi(\vec{x})$. But $\mathfrak{A} \models \exists\vec{x}\,\varphi(\vec{x})$ and $S = Th_{\Sigma_1}(\mathfrak{A})$; so, $\exists\vec{x}\,\varphi(\vec{x}) \in S$. Since $\mathfrak{B} \models S$, $\mathfrak{B} \models \exists\vec{x}\,\varphi(\vec{x})$, and this is a contradiction. ⊣

By the Claim, there exists $\mathfrak{C} \models T$ such that $\mathfrak{A} \prec_0 \mathfrak{C}$ and $\mathfrak{B} \prec_0 \mathfrak{C}$. Let $\psi$ be a $\Sigma_1$-sentence such that $\mathfrak{B} \models \psi$. Then, $\mathfrak{C} \models \psi$ since $\mathfrak{B} \prec_0 \mathfrak{C}$. But $\mathfrak{C} \models T$ and $\mathfrak{A}$ is $\Sigma_1$-maximal w.r.t. $T$, so $\mathfrak{A} \prec_1 \mathfrak{C}$ and we get that $\mathfrak{A} \models \psi$. This proves that $Th_{\Sigma_1}(\mathfrak{B}) \subseteq S$, as required. ⊣

It is easy to verify that $S$ is maximal w.r.t. $T$ if and only if $S$ is maximal w.r.t. the $\Pi_2$-consequences of $T$, $Th_{\Pi_2}(T)$. Hence, in Lemma 5 we can drop the assumption that $T \subseteq \Pi_2$ if we replace $T$ by $Th_{\Pi_2}(T)$ in the statement (b).

THEOREM 4. *If $\mathfrak{A}$ is a $\Sigma_1$-maximal model w.r.t. $\mathbf{I}\Delta_0$, then $\mathfrak{A}$ does not satisfy $Th_{\Pi_1}(\mathbf{I}\Delta_0 + exp)$.*

PROOF. It suffices to show that $\mathfrak{A} \not\models exp$. To see this, assume that $\mathfrak{A}$ satisfies $Th_{\Pi_1}(\mathbf{I}\Delta_0 + exp)$. Then there is $\mathfrak{B} \models \mathbf{I}\Delta_0 + exp$ such that $\mathfrak{A} \prec_0 \mathfrak{B}$. Since $\mathfrak{A}$ is $\Sigma_1$-maximal w.r.t. $\mathbf{I}\Delta_0$, $\mathfrak{A} \prec_1 \mathfrak{B}$ and hence $\mathfrak{A}$ also satisfies $exp$.

Let us prove that $\mathfrak{A} \not\models exp$. Since $\mathfrak{A}$ is $\Sigma_1$-maximal w.r.t. $\mathbf{I}\Delta_0$, from Lemma 5 it follows that $Th_{\Sigma_1}(\mathfrak{A})$ is a maximal theory w.r.t. $\mathbf{I}\Delta_0$. So, by theorem 2 of [1] $\mathbf{I}\Delta_0 + Th_{\Sigma_1}(\mathfrak{A})$ is inconsistent with $exp$. Consequently, $\mathfrak{A} \not\models exp$. ⊣

In view of Theorem 4 we can strengthen Proposition 4 replacing "there is a $\Sigma_1$-maximal model w.r.t. $\mathbf{I}\Delta_0$ satisfying $\mathbf{B}\Sigma_1$" by "*every* $\Sigma_1$-maximal model w.r.t. $\mathbf{I}\Delta_0$ satisfies $\mathbf{B}\Sigma_1$". As a consequence, in order to answer negatively the open question whether $\mathbf{I}\Delta_0 + \neg\, exp$ implies $\mathbf{B}\Sigma_1$ it suffices to show that there is *some* $\Sigma_1$-maximal model w.r.t. $\mathbf{I}\Delta_0$ in which $\mathbf{B}\Sigma_1$ fails.

§3. **Concluding remarks.** We finish with two general properties of $\Sigma_1$-maximal models that constitute the arithmetic counterparts of old results on existentially closed models contained in [5]. These properties will allow us to obtain a slightly different, somehow more *structural*, proof of the Main Theorem. We think, however, that stating them explicitly is of independent interest.

PROPOSITION 5 (essentially, lemmas 1.8 and 1.10 of [5]). *Let $\mathfrak{A}$, $\mathfrak{B}$ be models of $T$. If $\mathfrak{B}$ is $\Sigma_1$-maximal w.r.t. $T$ and $\mathfrak{A} \prec_1 \mathfrak{B}$, then $\mathfrak{A}$ is also $\Sigma_1$-maximal w.r.t. $T$.*

PROOF. Let $\mathfrak{C}$ be a model of $T$ satisfying $\mathfrak{A} \prec_0 \mathfrak{C}$.

CLAIM. *There is $\mathfrak{D} \models T$ such that $\mathfrak{A} \prec_0 \mathfrak{C} \prec_0 \mathfrak{D}$ and $\mathfrak{A} \prec_0 \mathfrak{B} \prec_0 \mathfrak{D}$.*

PROOF OF THE CLAIM. It is enough to prove that $T + D_{\Pi_0}(\mathfrak{B}) + D_{\Pi_0}(\mathfrak{C})$ is consistent. By contradiction, assume that $T + D_{\Pi_0}(\mathfrak{B}) + D_{\Pi_0}(\mathfrak{C})$ is inconsistent (we suppose that the elements of $\mathfrak{A}$ are denoted in both diagrams by the same constants). Then there exist $\vec{a} \in \mathfrak{A}$, $\vec{b} \in \mathfrak{B} - \mathfrak{A}$ and $\varphi(\vec{x}, \vec{y}) \in \Pi_0$ such that $\mathfrak{B} \models \varphi(\vec{a}, \vec{b})$ and

$$T + D_{\Pi_0}(\mathfrak{C}) \vdash \neg\varphi(\vec{a}, \vec{b}).$$

Then $T + D_{\Pi_0}(\mathfrak{C}) \vdash \forall\vec{y}\,\neg\varphi(\vec{a}, \vec{y})$ and, therefore, $\mathfrak{C} \models \forall\vec{y}\,\neg\varphi(\vec{a}, \vec{y})$. Hence, $\mathfrak{A} \models \forall\vec{y}\,\neg\varphi(\vec{a}, \vec{y})$ since $\mathfrak{A} \prec_0 \mathfrak{C}$. But it follows from $\mathfrak{A} \prec_1 \mathfrak{B}$ that $\mathfrak{B} \models \forall\vec{y}\,\neg\varphi(\vec{a}, \vec{y})$. In particular, $\mathfrak{B} \models \neg\varphi(\vec{a}, \vec{b})$ and this contradicts $\varphi(\vec{a}, \vec{b}) \in D_{\Pi_0}(\mathfrak{B})$. ⊣

Let $\mathfrak{D}$ be as in the Claim. To show that $\mathfrak{A} \prec_1 \mathfrak{C}$, assume $\mathfrak{C} \models \varphi(\vec{a})$, where $\vec{a} \in \mathfrak{A}$ and $\varphi(\vec{x}) \in \Sigma_1$. Then, $\mathfrak{D} \models \varphi(\vec{a})$. Since $\mathfrak{B}$ is $\Sigma_1$-maximal w.r.t. $T$, $\mathfrak{B} \prec_1 \mathfrak{D}$. So, $\mathfrak{B} \models \varphi(\vec{a})$ and hence $\mathfrak{A} \models \varphi(\vec{a})$ since $\mathfrak{A} \prec_1 \mathfrak{B}$. ⊣

COROLLARY 2. *Let $T$ be a $\Pi_2$-extension of $\mathbf{I}\Delta_0$. If $\mathfrak{A}$ is $\Sigma_1$-maximal w.r.t. $T$, then $\mathscr{K}_1(\mathfrak{A})$ is also $\Sigma_1$-maximal w.r.t. $T$.*

PROPOSITION 6 (essentially, proposition 1.14 of [5]). *Let $\mathfrak{A}$ be a $\Sigma_1$-maximal model w.r.t. $T$. Then,*

$$\mathfrak{B} \prec_1 \mathfrak{A} \Longrightarrow \mathfrak{B} \prec_2 \mathfrak{A}.$$

PROOF. Suppose $\mathfrak{B} \prec_1 \mathfrak{A}$. Firstly, observe that

CLAIM. *There is $\mathfrak{C} \models Th_{\Pi_1}(T)$ such that $\mathfrak{B} \prec_1 \mathfrak{A} \prec_0 \mathfrak{C}$ and $\mathfrak{B} \prec \mathfrak{C}$.*

PROOF OF THE CLAIM. It suffices to show that $Th_{\Pi_1}(T) + ED(\mathfrak{B}) + D_{\Pi_0}(\mathfrak{A})$ is consistent (again the elements of $\mathfrak{B}$ are denoted in the elementary diagram of $\mathfrak{B}$, $ED(\mathfrak{B})$, and in $D_{\Pi_0}(\mathfrak{A})$ by the same constants). By contradiction, assume that it is inconsistent. Then there exist $\vec{b} \in \mathfrak{B}$, $\vec{a} \in \mathfrak{A} - \mathfrak{B}$ and $\varphi(\vec{x}, \vec{y}) \in \Pi_0$ such that $\mathfrak{A} \models$

$\varphi(\vec{b}, \vec{a})$ and $Th_{\Pi_1}(T) + ED(\mathfrak{B}) \vdash \neg\varphi(\vec{b}, \vec{a})$. Then $Th_{\Pi_1}(T) + ED(\mathfrak{B}) \vdash \forall \vec{y} \neg\varphi(\vec{b}, \vec{y})$ and, therefore, $\mathfrak{B} \models \forall \vec{y} \neg\varphi(\vec{b}, \vec{y})$. Since $\mathfrak{B} \prec_1 \mathfrak{A}$, $\mathfrak{A} \models \forall \vec{y} \neg\varphi(\vec{b}, \vec{y})$. In particular, $\mathfrak{A} \models \neg\varphi(\vec{b}, \vec{a})$, contradicting $\varphi(\vec{b}, \vec{a}) \in D_{\Pi_0}(\mathfrak{A})$. ⊣

Let $\mathfrak{C}$ be as in the Claim. By Lemma 1, $\mathfrak{A}$ is $\Sigma_1$-maximal w.r.t. $Th_{\Pi_1}(T)$ and so $\mathfrak{A} \prec_1 \mathfrak{C}$. To see that $\mathfrak{B} \prec_2 \mathfrak{A}$, assume $\mathfrak{A} \models \exists x\, \varphi(x, \vec{b})$, where $\vec{b} \in \mathfrak{B}$ and $\varphi \in \Pi_1$. Since $\mathfrak{A} \prec_1 \mathfrak{C}$, $\mathfrak{C} \models \exists x\, \varphi(x, \vec{b})$. But $\mathfrak{B} \prec \mathfrak{C}$ and hence $\mathfrak{B} \models \exists x\, \varphi(x, \vec{b})$, as required. ⊣

Corollary 3. *Suppose* $I\Delta_0 \subseteq T$. *If* $\mathfrak{A}$ *is* $\Sigma_1$*-maximal w.r.t.* $T$, $\mathscr{K}_1(\mathfrak{A}) \prec_2 \mathfrak{A}$.

We can now derive a new version of our proof of the Main Theorem.

Proof of the Main Theorem (revisited): Let $T$ be a recursive theory and let $\mathfrak{A}$ be $\Sigma_1$-maximal w.r.t. $T$ (we may assume that $T$ implies $\mathbf{I}\Delta_0$). Then

(i) $\mathscr{K}_1(\mathfrak{A}) \prec_2 \mathfrak{A}$ by Corollary 3.

(ii) $\mathscr{K}_1(\mathfrak{A})$ is nonstandard. If not, from Lemma 1 and Corollary 2 it follows that $\mathscr{N} = \mathscr{K}_1(\mathfrak{A})$ is $\Sigma_1$-maximal w.r.t. $Th_{\Pi_1}(T)$. Hence, $\mathscr{N} \prec_1 \mathfrak{B}$ for each $\mathfrak{B} \models T$. So, $T$ implies $Th_{\Pi_1}(\mathscr{N})$, which is impossible since $T$ is recursive.

By a well-known theorem of [8], $\mathscr{K}_1(\mathfrak{A}) \not\models \mathbf{B}\Sigma_1 + exp$. So, $\mathscr{K}_1(\mathfrak{A}) \not\models \mathbf{L}\Delta_1^- + exp$ since $\mathbf{B}\Sigma_1 \equiv \mathbf{L}\Delta_1$ and all elements of $\mathscr{K}_1(\mathfrak{A})$ are $\Sigma_1$-definable. Hence, $\mathbf{L}\Delta_1^- + exp$ also fails in $\mathfrak{A}$ (recall that $\mathscr{K}_1(\mathfrak{A}) \prec_2 \mathfrak{A}$ and $\mathbf{L}\Delta_1^-$ is axiomatized by a set of $\Sigma_2$-sentences).

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