Corrigendum to the paper: A way to model stochastic perturbations in population dynamics models with bounded realizations

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TOMÁS CARABALLO

Departamento de Ecuaciones Diferenciales y Análisis Numérico
C/ Tarfia s/n, Facultad de Matemáticas, Universidad de Sevilla,
41012 Sevilla, Spain

RENATO COLUCCI

Dipartimento di Ingegneria Industriale e Scienze Matematiche,
Università Politecnica delle Marche,
Via Brecce Bianche 12, 60131, Ancona, Italy.

JAVIER LÓPEZ-DE-LA-CRUZ*

Departamento de Ecuaciones Diferenciales y Análisis Numérico
C/ Tarfia s/n, Facultad de Matemáticas, Universidad de Sevilla
41012 Sevilla, Spain

AND ALAIN RAPAPORT

MISTEA, Univ. Montpellier, Inra, Montpellier SupAgro
2, place Pierre Viala
34060 Montpellier, France

ABSTRACT. In this corrigendum we correct an error in our paper [T. Caraballo, R. Colucci, J. López-de-la-Cruz and A. Rapaport. A way to model stochastic perturbations in population dynamics models with bounded realizations, *Commun Nonlinear Sci Numer Simulat*, **77** (2019) 239–257]. We present a correct way to model real noisy perturbations by considering a slightly different stochastic process based, as in the original paper, on the Ornstein-Uhlenbeck process. Namely, we correct the formulae that generates the noisy realizations to ensure the boundedness property to be satisfied with probability one (which turns out not to be true in our original paper even though it was observed in all the simulations). Once this modification is done, every result and every example in the initial work remain valid, in fact, the same goal is achieved.

1. INTRODUCTION

In our paper [3] we presented a new way to model real noisy perturbations by making use of the well-known Orstein-Uhlenbeck (OU) process. In addition, we illustrated this idea with different examples coming from population dynamics and several numerical simulations.

Let us describe firstly the explanations given in [3] in order to let the reader understand the error in [3] and the changes needed to preserve the spirit and the

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*Corresponding author: jlopez78@us.es

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goal of our original work.

It is well known that most real phenomena are subject to suffer the effects of some kind of noise. Even though there are many works in the literature leading with stochastic models, they mainly focus on making use of the standard Wiener process, in spite of the fact that it is unbounded with probability one. This last property is completely unrealistic from the point of view of applications, since random perturbations in real life are bounded. In order to understand better the drawback caused by the standard Wiener process when modeling noise, we refer the readers to [4, 5, 6], where the authors investigate chemostat models by means of the Brownian motion.

Motivated by this fact, we presented in [3] a way to model bounded real perturbations involving the OU process, which has proved to fit in a very loyal way the random disturbances observed in real life.

More precisely, we explained that we could consider perturbations of some parameter in a random system of the following kind

$$\dot{x} = f(x, z^*_{\beta,\gamma}(\theta_t\omega)), \tag{1}$$

where $z^*_{\beta,\gamma}(\theta_t\omega)$ denotes the OU process. This type of systems arises when replacing, for instance, some deterministic parameter, namely a, by $a+\alpha z^*_{\beta,\gamma}(\theta_t\omega)$, where $\alpha > 0$ denotes the intensity of the noise.

In order to deal with system (1), we introduced in [3] the OU process which is defined by the random variable

$$z^*_{\beta,\gamma}(\theta_t\omega) = -\beta\gamma \int_{-\infty}^{0} e^{\beta s}\theta_t\omega(s)ds, \quad t \in \mathbb{R},\ \omega \in \Omega,\ \beta,\ \gamma > 0, \tag{2}$$

and solves the Langevin equation

$$dz + \beta z dt = \gamma d\omega, \tag{3}$$

and then we provided some properties, as the ones in Proposition 2 in [3] that we recall below.

Proposition 1.1. *There exists a θ_t-invariant set $\widetilde{\Omega} \in \mathcal{F}$ of Ω of full $\mathbb{P}-$measure such that for $\omega \in \widetilde{\Omega}$ and $\beta, \gamma > 0$, we have*

(i) *the random variable $|z^*_{\beta,\gamma}(\omega)|$ is tempered.*

(ii) *the mapping*

$$(t,\omega) \to z^*_{\beta,\gamma}(\theta_t\omega) = -\beta\gamma \int_{-\infty}^{0} e^{\beta s}\omega(t+s)\mathrm{d}s + \omega(t)$$

is a stationary solution of (2) with continuous trajectories;

(iii) *for any* $\omega \in \tilde{\Omega}$ *one has*

$$\begin{aligned} \lim_{t\to\pm\infty} \frac{|z^*_{\beta,\gamma}(\theta_t\omega)|}{t} &= 0; \\ \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z^*_{\beta,\gamma}(\theta_s\omega)ds &= 0; \\ \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t |z^*_{\beta,\gamma}(\theta_s\omega)|ds &= \mathbb{E}[z^*_{\beta,\gamma}] < \infty; \end{aligned}$$

(iv) *finally, for any* $\omega \in \widetilde{\Omega}$,

$$\lim_{\beta\to\infty} z^*_{\beta,\gamma}(\theta_t\omega) = 0, \quad \textit{for all } t \in \mathbb{R}.$$

Hence, the idea was the following: (1) we are given an interval, namely $[a_1, a_2]$, (2) we fix any event $\omega \in \widetilde{\Omega}$ and (3) for every fixed $\omega \in \widetilde{\Omega}$, thanks to Proposition 1.1 (iv), which can be found in [1], Lemma 4.1, we can choose β large enough such that $a + \alpha z^*_{\beta,\gamma}(\theta_t\omega) \in [a_1, a_2]$ for all $t \in \mathbb{R}$.

Then, Proposition 1.1 (iv) was the key of our paper since it allowed us to ensure that the OU process $z^*_{\beta,\gamma}(\theta_t\omega)$ was bounded in some desired interval (typically fixed by practitioners) for every fixed event $\omega \in \widetilde{\Omega}$ and β large enough.

As we explained in [3], since β could depend on ω, the resulting random system does not generate a random dynamical system (RDS) and then the theory of RDSs and pullback attractors could not be applied. But this was not a problem at all, since the random system could be analyzed for every fixed event ω. In fact, this allowed us to prove every result of the paper forwards in time, which is a more natural way than the pullback one.

Nevertheless, we noticed that property (iv) in Proposition 1.1 (Proposition 2 (iv) in [3]) is not true. By having a deeper look at the proof of the result in [1] we realized that such a result cannot be proved uniformly in time (as we first understood in [1]), but for every fixed t.

After this, we found a slightly different way to model bounded noises which allows us to prove every result and present every example in the original paper [3] in a very similar manner. Let us present now the new correct idea.

Assume that we want to consider some random perturbations in a certain parameter a of a deterministic differential system. In addition, as in real life, those disturbances must be bounded (for all the time) in some interval around a, namely $[a - d, a + d]$, where $d > 0$.

Instead of replacing a by $a + \alpha z^*_{\beta,\gamma}(\theta_t\omega)$, as we did in [3], we replace a by $a + \Phi(z^*(\theta_t\omega))$, where $\Phi(z) = \frac{2d}{\pi}\arctan(z)$ and $z^*(\theta_t\omega)$ denotes the OU process (2). In this case, β and γ are fixed through the whole work. Hence, instead of having to deal with system (1), the resulting random system is of the kind

$$\dot{x} = f(x, \Phi(z^*(\theta_t\omega))), \tag{4}$$

and it will be investigated for every fixed event $\omega \in \Omega$, as in the original work [3].

We notice that, in this case, as β and γ are fixed, the resulting random system (4) generates a RDS and then the theory of RDSs and pullback attractors could be applied (which was not the case in the original paper). Nevertheless, instead of using that theory and pullback attraction, we study the random system (4) for every fixed $\omega \in \Omega$ since it allows us to prove every result forwards in time (which is, as explained before, more natural than the pullback one).

It is easy to check that the random perturbations $a + \Phi(z^*(\theta_t\omega))$ are bounded for every $t \in \mathbb{R}$ and any $\omega \in \Omega$, since Φ is bounded. In fact, for every $\omega \in \Omega$, we have

$$a - d \leq a + \Phi(z^*(\theta_t\omega)) \leq a + d, \quad \text{for all}\, t \in \mathbb{R}.$$

In addition, we can prove the following property.

Proposition 1.2. *Let* $\Phi(z) = \frac{2d}{\pi}\arctan(z)$. *Then*

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t \Phi(z^*(\theta_s\omega))ds = 0, \quad a.s.\ in\ \Omega. \tag{5}$$

Proof. Since $\Phi(z) = \frac{2d}{\pi}\arctan(z)$, then we have

$$\int_\Omega \left|\frac{2d}{\pi}\arctan(z^*(\omega))\right| d\mathbb{P}(\omega) \leq d|\Omega| = d$$

whence $\Phi(z^*(\cdot)) \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Then, since $\mathbb{P}$ is invariant by θ (see [2, 7]), from the Birkhoff ergodic theorem we deduce that

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t \Phi(z^*(\theta_s\omega))ds = \mathbb{E}[\Phi(z^*(\omega))], \quad \text{a.s. in } \Omega.$$

Hence, it is enough to show that $\mathbb{E}[\Phi(z^*(\omega))] = 0$. In fact, we have

$$\mathbb{E}[\Phi(z^*(\omega))] = \int_\mathbb{R} \Phi(x) f_{\text{OU}}(x)dx = 0,$$

where f_{OU} denotes the density function of the random variable $z^*(\cdot)$, which is an even function since it is Gaussian, and Φ is an odd function. □

Hence, the stochastic process $\Phi(z^*(\omega))$ satisfies the desired ergodic property (which was false in Proposition 1.1 (iv)) and it is bounded. Then, every result in our original work [3] is true when considering this little modification and every example there can be given to illustrate this new way to model bounded random perturbations.

Concerning the numerical simulations, they can be done again by considering the new function $\Phi(z) = \frac{2d}{\pi}\arctan(z)$ and they are analogous to the ones in the original work [3].

To finish, we would like to remark that we could also use any odd measurable function Φ such that

$$\lim_{z\to+\infty} \Phi(z) = d < +\infty.$$

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E-mail address: caraball@us.es
E-mail address: renatocolucci@hotmail.com
E-mail address: jlopez78@us.es
E-mail address: alain.rapaport@inra.fr